SEMI-LAGRANGIAN SCHEMES FOR LINEAR AND FULLY NON-LINEAR HAMILTON-JACOBI-BELLMAN EQUATIONS

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Abstract. We consider the numerical solution of Hamilton-Jacobi-Bellman equations arising in stochastic control theory. We introduce a class of monotone approximation schemes relying on monotone interpolation. These schemes converge under very weak assumptions, including the case of arbitrary degenerate diffusions. Besides providing a unifying framework that includes several known first order accurate schemes, stability and convergence results are given, along with two different robust error estimates. Finally, the method is applied to a super-replication problem from finance.

1. **Introduction.** In this paper we consider the numerical solution of partial differential equations of Hamilton-Jacobi-Bellman type,

$$u_t - \inf_{\alpha \in \mathcal{A}} \left\{ L^\alpha[u](t,x) + c^\alpha(t,x)u + f^\alpha(t,x) \right\} = 0 \qquad \text{in} \quad Q_T, \qquad (1)$$

$$u(0,x) = g(x) \qquad \text{in} \quad \mathbb{R}^N, \qquad (2)$$

where

$$L^\alpha[u](t,x) = \text{tr}[a^\alpha(t,x)D^2 u(t,x)] + b^\alpha(t,x)Du(t,x),$$

$Q_T := (0,T] \times \mathbb{R}^N$, and $\mathcal{A}$ is a complete metric space. The coefficients $a^\alpha = \frac{1}{2}\sigma^\alpha \sigma^{\alpha \top}$, b^α, c^α, f^α and the initial data g take values respectively in $\mathbb{S}^N$, the space of $N \times N$ symmetric matrices, $\mathbb{R}^N$, $\mathbb{R}$, $\mathbb{R}$, and $\mathbb{R}$. We will only assume that a^α is positive semi-definite, thus the equation is allowed to degenerate and hence not have smooth solutions in general. By solutions in this paper we will therefore always mean generalized solutions in the viscosity sense, see e.g. [6, 12]. Then the solution coincides with the value function of a finite horizon, optimal stochastic control problem [12].

To ensure comparison and well-posedness of (1)–(2) in the class of bounded x-Lipschitz functions, we will use the following standard assumptions on its data:

(A1) For any $\alpha \in \mathcal{A}$, $a^\alpha = \frac{1}{2}\sigma^\alpha \sigma^{\alpha\top}$ for some $N \times P$ matrix σ^α. Moreover, there is a constant K independent of α such that

$$|g|_1 + |\sigma^\alpha|_1 + |b^\alpha|_1 + |c^\alpha|_1 + |f^\alpha|_1 \leq K,$$

where $|\phi|_1 = \sup_{(t,x)\in Q_T} |\phi(x,t)| + \sup_{(x,t)\neq(y,s)} \frac{|\phi(x,t)-\phi(y,s)|}{|x-y|+|t-s|^{1/2}}$ is a space-time Lipschitz/Hölder-norm.

The following result is standard.

Proposition 1. *Assume that (A1) holds. Then there exist a unique solution u of (1)–(2) and a constant C only depending on T and K from (A1) such that*

$$|u|_1 \leq C.$$

Furthermore, if u_1 and u_2 are sub- and supersolutions of (1) satisfying $u_1(0,\cdot) \leq u_2(0,\cdot)$, then $u_1 \leq u_2$.

2. **Semi-Lagrangian schemes.** Following [8] we propose a class of approximation schemes for (1)–(2) which we call Semi-Lagrangian or SL schemes. These schemes converge under very weak assumptions, including the case of arbitrary degenerate diffusions. In particular, these schemes are L^∞-stable and convergent for problems involving diffusion matrices that are not diagonally dominant. This class includes (parabolic versions of) the "control schemes" of Menaldi [11] and Camilli and Falcone [4] and some of the monotone schemes of Crandall and Lions [7]. It also includes SL schemes for first order Bellman equations [5, 9] and some new versions as discussed in the following section.

The schemes are defined on a possibly unstructured family of grids $\{G_{\Delta t,\Delta x}\}$,

$$G = G_{\Delta t,\Delta x} = \{(t_n, x_i)\}_{n\in\mathbb{N}_0, i\in\mathbb{N}} = \{t_n\}_{n\in\mathbb{N}_0} \times X_{\Delta x},$$

for $\Delta t, \Delta x > 0$. Here $0 = t_0 < t_1 < \cdots < t_n < t_{n+1}$ satisfy

$$\max_n \Delta t_n \leq \Delta t \qquad \text{where} \qquad \Delta t_n = t_n - t_{n-1},$$

and $X_{\Delta x} = \{x_i\}_{i\in\mathbb{N}}$ is the set of vertices or nodes for a non-degenerate polyhedral subdivision of $\mathbb{R}^N$.

We consider the following general finite difference approximations of the differential operator $L^\alpha[\phi]$ in (1):

$$L_k^\alpha[\phi](t,x) := \sum_{i=1}^M \frac{\phi(t, x + y_{k,i}^{\alpha,+}(t,x)) - 2\phi(t,x) + \phi(t, x + y_{k,i}^{\alpha,-}(t,x))}{2k^2}, \qquad (3)$$

for $k > 0$ and some $M \geq 1$. For this approximation we will assume

$$(Y1) \begin{cases} \sum_{i=1}^{M} [y_{k,i}^{\alpha,+} + y_{k,i}^{\alpha,-}] = 2k^2 b^\alpha + \mathcal{O}(k^4), \\ \sum_{i=1}^{M} [y_{k,i}^{\alpha,+} y_{k,i}^{\alpha,+\top} + y_{k,i}^{\alpha,-} y_{k,i}^{\alpha,-\top}] = 2k^2 \sigma^\alpha \sigma^{\alpha\top} + \mathcal{O}(k^4), \\ \sum_{i=1}^{M} [y_{k,i,j_1}^{\alpha,+} y_{k,i,j_2}^{\alpha,+} y_{k,i,j_3}^{\alpha,+} + y_{k,i,j_1}^{\alpha,-} y_{k,i,j_2}^{\alpha,-} y_{k,i,j_3}^{\alpha,-}] = \mathcal{O}(k^4), \\ \sum_{i=1}^{M} [y_{k,i,j_1}^{\alpha,+} y_{k,i,j_2}^{\alpha,+} y_{k,i,j_3}^{\alpha,+} y_{k,i,j_4}^{\alpha,+} + y_{k,i,j_1}^{\alpha,-} y_{k,i,j_2}^{\alpha,-} y_{k,i,j_3}^{\alpha,-} y_{k,i,j_4}^{\alpha,-}] = \mathcal{O}(k^4), \end{cases}$$

for all $j_1, j_2, j_3, j_4 = 1, 2, \ldots, N$ indicating components of the y-vectors.

Under assumption (Y1), a Taylor expansion shows that L_k^α is a second order consistent approximation satisfying

$$|L_k^\alpha[\phi] - L^\alpha[\phi]| \leq C(|D\phi|_0 + \cdots + |D^4\phi|_0)k^2 \tag{4}$$

for all smooth functions ϕ, where $|\phi|_0 = \sup_{(t,x)\in Q_T} |\phi(x,t)|$.

To relate this approximation to the spatial grid $X_{\Delta x}$, we replace ϕ by its interpolant $\mathcal{I}\phi$, yielding overall a semi-discrete approximation of (1),

$$U_t - \inf_{\alpha \in \mathcal{A}} \left\{ L_k^\alpha[\mathcal{I}U](t,x) + c^\alpha(t,x)U + f^\alpha(t,x) \right\} = 0 \quad \text{in} \quad (0,T) \times X_{\Delta x}.$$

We require the interpolation operator $\mathcal{I}$ to fulfill the following two conditions:

(I1) There are $K \geq 0, r \in \mathbb{N}$ such that for all smooth functions ϕ

$$|(\mathcal{I}\phi) - \phi|_0 \leq K|D^r\phi|_0 \Delta x^r.$$

(I2) There is a set of non-negative functions $\{w_j(x)\}_j$ such that

$$(\mathcal{I}\phi)(x) = \sum_j \phi(x_j)w_j(x),$$

and

$$w_j(x) \geq 0, \qquad w_i(x_j) = \delta_{ij}$$

for all $i, j \in \mathbb{N}$.

(I1) implies together with (4) that $L_k^\alpha[\mathcal{I}\phi]$ is a consistent approximation of $L^\alpha[\phi]$ if $\frac{\Delta x^r}{k^2} \to 0$. An interpolation satisfying (I2) is said to be *positive* and is *monotone* in the sense that $U \leq V$ implies that $\mathcal{I}U \leq \mathcal{I}V$. Typically $\mathcal{I}$ will be constant, linear, or multi-linear interpolation (i.e. $r \leq 2$ in (I1)), because higher order interpolation is not monotone in general.

The final scheme can now be found by discretizing in time using a parameter $\theta \in [0,1]$,

$$\delta_{\Delta t_n} U_i^n = \inf_{\alpha \in \mathcal{A}} \left\{ L_k^\alpha[\mathcal{I}\bar{U}_\cdot^{\theta,n}]_i^{n-1+\theta} + c_i^{\alpha,n-1+\theta}\bar{U}_i^{\theta,n} + f_i^{\alpha,n-1+\theta} \right\} \tag{5}$$

in G, where $U_i^n = U(t_n, x_i)$, $f_i^{\alpha,n-1+\theta} = f^\alpha(t_{n-1} + \theta\Delta t_n, x_i), \ldots$ for $(t_n, x_i) \in G$,

$$\delta_{\Delta t}\phi(t,x) = \frac{\phi(t,x) - \phi(t - \Delta t, x)}{\Delta t}, \qquad \text{and} \qquad \bar{\phi}_\cdot^{\theta,n} = (1-\theta)\phi_\cdot^{n-1} + \theta\phi_\cdot^n.$$

As initial conditions we take

$$U_i^0 = g(x_i) \quad \text{in} \quad X_{\Delta x}. \tag{6}$$

For the choices $\theta = 0, 1$, and $1/2$ the time discretization corresponds to respectively explicit Euler, implicit Euler, and midpoint rule. For $\theta = 1/2$, the full scheme can be seen as generalized Crank-Nicolson type discretization.

3. **Examples of approximations L_k^α.**

1. The approximation of Falcone [9] (see also [5]),

$$b^\alpha D\phi \approx \frac{\mathcal{I}\phi(x + hb^\alpha) - \mathcal{I}\phi(x)}{h},$$

corresponds to our L_k^α if $k = \sqrt{h}$, $y_k^{\alpha,\pm} = k^2 b^\alpha$.

2. The approximation of Crandall-Lions [7],

$$\frac{1}{2}\mathrm{tr}[\sigma^\alpha \sigma^{\alpha\top} D^2\phi] \approx \sum_{j=1}^{P} \frac{\mathcal{I}\phi(x + k\sigma_j^\alpha) - 2\mathcal{I}\phi(x) + \mathcal{I}\phi(x - k\sigma_j^\alpha)}{2k^2},$$

corresponds to our L_k^α if $y_{k,j}^{\alpha,\pm} = \pm k\sigma_j^\alpha$ and $M = P$.

3. The corrected version of the approximation of Camilli-Falcone [4] (see also [11]),

$$\frac{1}{2}\mathrm{tr}[\sigma^\alpha \sigma^{\alpha\top} D^2\phi] + b^\alpha D\phi$$
$$\approx \sum_{j=1}^{P} \frac{\mathcal{I}\phi(x + \sqrt{h}\sigma_j^\alpha + \frac{h}{P}b^\alpha) - 2\mathcal{I}\phi(x) + \mathcal{I}\phi(x - \sqrt{h}\sigma_j^\alpha + \frac{h}{P}b^\alpha)}{2h},$$

corresponds to our L_k^α if $k = \sqrt{h}$, $y_{k,j}^{\alpha,\pm} = \pm k\sigma_j^\alpha + \frac{k^2}{P}b^\alpha$ and $M = P$.

4. The new approximation obtained by combining approximations 1 and 2,

$$\frac{1}{2}\mathrm{tr}[\sigma^\alpha \sigma^{\alpha\top} D^2\phi] + b^\alpha D\phi$$
$$\approx \frac{\mathcal{I}\phi(x + k^2 b^\alpha) - \mathcal{I}\phi(x)}{k^2} + \sum_{j=1}^{P} \frac{\mathcal{I}\phi(x + k\sigma_j^\alpha) - 2\mathcal{I}\phi(x) + \mathcal{I}\phi(x - k\sigma_j^\alpha)}{2k^2},$$

corresponds to our L_k^α if $y_{k,j}^{\alpha,\pm} = \pm k\sigma_j^\alpha$ for $j \leq P$, $y_{k,P+1}^{\alpha,\pm} = k^2 b^\alpha$ and $M = P + 1$.

5. Yet another new approximation,

$$\frac{1}{2}\mathrm{tr}[\sigma^\alpha \sigma^{\alpha\top} D^2\phi] + b^\alpha D\phi \approx \sum_{j=1}^{P-1} \frac{\mathcal{I}\phi(x + k\sigma_j^\alpha) - 2\mathcal{I}\phi(x) + \mathcal{I}\phi(x - k\sigma_j^\alpha)}{2k^2}$$
$$+ \frac{\mathcal{I}\phi(x + k\sigma_P^\alpha + k^2 b^\alpha) - 2\mathcal{I}\phi(x) + \mathcal{I}\phi(x - k\sigma_P^\alpha + k^2 b^\alpha)}{2k^2},$$

corresponds to our L_k^α if $y_{k,j}^{\alpha,\pm} = \pm k\sigma_j^\alpha$ for $j < P$, $y_{k,P}^{\alpha,\pm} = \pm k\sigma_P^\alpha + k^2 b^\alpha$ and $M = P$.

When σ^α does not depend on α but b^α does, approximations 4 and 5 are much more efficient than approximation 3.

4. **Linear interpolation SL scheme (LISL).** To keep the scheme (5) monotone, linear or multi-linear interpolation is the most accurate interpolation one can use in general. In this typical case we call the full scheme (5)–(6) the LISL scheme. In the following, we denote by $c^{\alpha,+}$ the positive part of c^α. Then we have the following result by [8]:

Theorem 4.1. *Assume that (A1), (I1), (I2), and (Y1) hold.*
(a) The LISL scheme is monotone if the following CFL conditions hold:

$$(1-\theta)\Delta t\Big[\frac{M}{k^2} - c_i^{\alpha,n-1+\theta}\Big] \leq 1 \quad and \quad \theta\Delta t\, c_i^{\alpha,n-1+\theta} \leq 1 \; for \; all \; \alpha,n,i. \qquad (7)$$

(b) The truncation error of the LISL scheme is $O(|1-2\theta|\Delta t + \Delta t^2 + k^2 + \frac{\Delta x^2}{k^2})$; it is first order accurate for $k = O(\Delta x^{1/2})$, $\Delta t = O(\Delta x)$ ($\Delta t = O(\Delta x^{1/2})$ if $\theta = \frac{1}{2}$).
(c) If $2\theta\Delta t\sup_\alpha |c^{\alpha,+}|_0 \leq 1$ and (7) holds, then there exists a unique bounded and L^∞-stable solution U of the LISL scheme converging uniformly to the solution u of (1)–(2) as $\Delta t, k, \frac{\Delta x}{k} \to 0$.

From this result it follows that the scheme is at most *first order accurate*, has *wide and increasing stencil* and a *good CFL condition*. From the truncation error and the definition of L_k^α the stencil is wide since the scheme is consistent only if $\Delta x/k \to 0$ as $\Delta x \to 0$ and has stencil length proportional to

$$l := \frac{\max\limits_{t,x,\alpha,i}\{|y_{k,i}^{\alpha,-}|,|y_{k,i}^{\alpha,+}|\}}{\Delta x} \sim \frac{k}{\Delta x} \to \infty \quad as \quad \Delta x \to 0.$$

Here we have used that if (Y1) holds and $\sigma \not\equiv 0$, then typically $y_{k,i}^{\alpha,\pm} \sim k$. Note that if $k = \Delta x^{1/2}$, then $l \sim \Delta x^{-1/2}$. Finally, in the case $\theta \neq 1$ the CFL condition for (5) is $\Delta t \leq Ck^2 \sim \Delta x$ when $k = O(\Delta x^{1/2})$, and it is much less restrictive than the usual parabolic CFL condition, $\Delta t = O(\Delta x^2)$.

Remark 1. The LISL scheme is consistent and monotone for arbitrary degenerating diffusions, without requiring that a^α is diagonally dominant or similar conditions. In comparison to other schemes applicable in this situation, like the ones of Bonnans-Zidani [3], it is much easier to analyze and to implement and faster in the sense that the computational cost for approximating the diffusion matrix is for fixed x, t, α independent of the stencil size.

5. **The error estimate of [8].** To simplify the presentation, in the following we restrict to a uniform time-grid, $G = \Delta t\{0,1,\ldots,N_T\} \times X_{\Delta x}$. Let $Q_{\Delta t,T} := \Delta t\{0,1,\ldots,N_T\} \times \mathbb{R}^N$. To apply the regularization method of Krylov [10] we need a regularity and continuous dependence result for the scheme that relies on the following additional (covariance-type) assumptions: Whenever two sets of data σ, b and $\tilde{\sigma}, \tilde{b}$ are given, the corresponding approximations $L_k^\alpha, y_{k,i}^{\alpha,\pm}$ and $\tilde{L}_k^\alpha, \tilde{y}_{k,i}^{\alpha,\pm}$ in (3) satisfy

$$(Y2)\begin{cases} \displaystyle\sum_{i=1}^{M}[y_{k,i}^{\alpha,+} + y_{k,i}^{\alpha,-}] - [\tilde{y}_{k,i}^{\alpha,+} + \tilde{y}_{k,i}^{\alpha,-}] \leq 2k^2(b^\alpha - \tilde{b}^\alpha), \\[2mm] \displaystyle\sum_{i=1}^{M}[y_{k,i}^{\alpha,+}y_{k,i}^{\alpha,+\top} + y_{k,i}^{\alpha,-}y_{k,i}^{\alpha,-\top}] + [\tilde{y}_{k,i}^{\alpha,+}\tilde{y}_{k,i}^{\alpha,+\top} + \tilde{y}_{k,i}^{\alpha,-}\tilde{y}_{k,i}^{\alpha,-\top}] \\[2mm] \qquad -[y_{k,i}^{\alpha,+}\tilde{y}_{k,i}^{\alpha,+\top} + \tilde{y}_{k,i}^{\alpha,+}y_{k,i}^{\alpha,+\top} + y_{k,i}^{\alpha,-}\tilde{y}_{k,i}^{\alpha,-\top} + \tilde{y}_{k,i}^{\alpha,-}y_{k,i}^{\alpha,-\top}] \\[2mm] \qquad \leq 2k^2(\sigma^\alpha - \tilde{\sigma}^\alpha)(\sigma^\alpha - \tilde{\sigma}^\alpha)^\top + 2k^4(b^\alpha - \tilde{b}^\alpha)(b^\alpha - \tilde{b}^\alpha)^\top, \end{cases}$$

when $\sigma, b, y_k^{\pm}$ are evaluated at (t, x) and $\tilde{\sigma}, \tilde{b}, \tilde{y}_k^{\pm}$ are evaluated at (t, y) for all t, x, y.

Then one can prove the following error estimate [8]:

Theorem 5.1 (Error Bound I). *Assume that (A1), (I1), (I2), (Y1), and (Y2) hold, and that $\Delta t, \Delta x > 0$, $k \in (0, 1)$ satisfy the CFL conditions* (7). *If u solves* (1)–(2) *and U solves* (5)–(6), *then there is $c_0 > 0$ such that for any $\Delta t \in (0, c_0)$*

$$|u - U| \leq C(|1 - 2\theta|\Delta t^{1/4} + \Delta t^{1/3} + k^{1/2} + \frac{\Delta x}{k^2}) \quad in \quad G.$$

This error bound holds also for unstructured grids. For more regular solutions it is possible to obtain better error estimates, but general and optimal results are not available. The best estimate in our case is $O(\Delta x^{1/5})$ which is achieved when $k = O(\Delta x^{2/5})$ and $\Delta t = O(k^2)$. Note that the CFL conditions (7) already imply that $\Delta t = O(k^2)$ if $\theta < 1$. Also note that the above bound does not show convergence when k is optimal for the LISL scheme ($k = O(\Delta x^{1/2})$).

6. **A new error estimate.** In the above error estimate, the lower estimate on $u - U$ follows if you can prove regularity and continuous dependence results for the solution of the equation only. The proof of the upper estimate is symmetric and requires such results for the numerical solution. However, it is possible to avoid using such properties of the numerical solution by a clever approximation argument, see e.g. [1]. This allows for error estimates that show convergence for any k such that the scheme is consistent. We need an extra assumption on the coefficients:

(A2) The coefficients σ^{α}, b^{α}, c^{α}, f^{α} are continuous in α for all x, t.

Theorem 6.1 (Error Bound II). *Assume that (A1), (A2), (I1) with $r = 2$ (~linear interpolation), (I2), and (Y1) hold, and that $\Delta t, \Delta x > 0$, $k \in (0, 1)$ satisfy the CFL conditions* (7). *If u solves* (1)–(2) *and U solves* (5)–(6), *then there is $c_0 > 0$ such that for any $\Delta t \in (0, c_0)$*

$$u - U \geq C\Big(|1 - 2\theta|\Delta t^{1/4} + \Delta t^{1/3} + k^{1/2} + \frac{\Delta x}{k}\Big) \quad in \quad G,$$

$$u - U \leq C\Big(|1 - 2\theta|\Delta t^{1/10} + \Delta t^{1/8} + k^{1/5} + \big(\frac{\Delta x}{k}\big)^{1/2}\Big) \quad in \quad G.$$

With optimal k for the LISL scheme, $\Delta t = O(k^2)$ and $k = O(\Delta x^{1/2})$, we find that $u - U = O(\Delta x^{1/10})$.

Proof. By a direct computation the local truncation error of the method is bounded by

$$\frac{|1 - 2\theta|}{2}|\phi_{tt}|_0 \Delta t + C\Big(\Delta t^2 \left(|\phi_{tt}|_0 + |\phi_{ttt}|_0 + |D\phi_{tt}|_0 + |D^2\phi_{tt}|_0\right)$$
$$+ |D^2\phi|_0 \frac{\Delta x^2}{k^2} + (|D\phi|_0 + \cdots + |D^4\phi|_0)k^2\Big)$$

for smooth ϕ (cf. Lemma 4.1 in [8]). Moreover if also $\partial_t^{k_1} D_x^{k_2} \phi = O(\varepsilon^{1 - 2k_1 - k_2})$ for any $k_1, k_2 \in \mathbb{N}_0$, then the truncation error is of order

$$(1 - 2\theta)\Delta t\varepsilon^{-3} + \Delta t^2 \varepsilon^{-5} + k^2 \varepsilon^{-3} + \frac{\Delta x^2}{k^2}\varepsilon^{-1} =: E(\varepsilon).$$

Since the scheme is monotone (under the CFL condition) and condition (A1) holds, it now follows from Theorem 3.1 in [1] that

$$C \inf_{\varepsilon > 0} \big(\varepsilon + E(\varepsilon)\big) \leq u - U \leq C \inf_{\varepsilon > 0} \big(\varepsilon^{1/3} + E(\varepsilon)\big),$$

and we complete the proof optimizing over ε (as e. g. in [1, 8]). □

7. **Convergence test for a super-replication problem.** We consider a test problem from [2] which was used to test convergence rates for numerical approximations of a super-replication problem from finance. The corresponding PDE is

$$\inf_{\alpha_1^2+\alpha_2^2=1} \left\{ \alpha_1^2 u_t(t,x) - \frac{1}{2}\mathrm{tr}\left(\sigma^\alpha(t,x)\sigma^{\alpha\top}(t,x)D^2u(t,x)\right) \right\} = f(t,x) \qquad (8)$$

with $0 \leq x_1, x_2 \leq 3$, $\sigma^\alpha(t,x) = \begin{pmatrix} \alpha_1 x_1\sqrt{x_2} \\ \alpha_2\eta(x_2) \end{pmatrix}$ and $\eta(x) = x(3-x)$. We take $u(t,x) = 1 + t^2 - e^{-x_1^2-x_2^2}$ as exact solution as in [2], and then f is forced to be

$$f(t,x) = \frac{1}{2}\left(u_t - \frac{1}{2}x_1^2 x_2 u_{x_1 x_1} - \frac{1}{2}x_2^2(3-x_2)^2 u_{x_2 x_2} \right.$$
$$\left. -\sqrt{\left(-u_t + \frac{1}{2}x_1^2 x_2 u_{x_1 x_1} - \frac{1}{2}x_2^2(3-x_2)^2 u_{x_2 x_2}\right)^2 + \left(x_1\sqrt{x_2}^3(3-x_2)u_{x_1 x_2}\right)^2} \right).$$

In [2] $\eta(x) = x$, while we take $\eta(x) = x(3-x)$ to prevent the LISL scheme from overstepping the boundaries. Note that changing η does *not* change the solutions as long as $\eta > 0$ in the interior of the domain, see [2], and hence the above equation is equivalent to the equation used in [2]. The initial values and Dirichlet boundary values at $x_1 = 0$ and $x_2 = 0$ are taken from the exact solution. As in [2], at $x = 3$ and $y = 3$ homogeneous Neumann boundary conditions are implemented. To approximate the values of α_1, α_2, the Howard algorithm is used (see [2]), which requires an implicit time discretization, so we choose $\theta = 1$. We choose $k = \sqrt{\Delta x}$ and a regular triangular grid. The numbers of time steps are chosen as $\frac{1}{\Delta x}$.

The results at $t = 1$ are given in Table 1. The numerical order of convergence is approximately one.

Δx	$|u - U|_0$	rate
1.50e-1	2.01e-1	
7.50e-2	9.49e-2	1.08
3.75e-2	4.29e-2	1.15
1.87e-2	1.94e-2	1.15

TABLE 1. Results for the convergence test for the super-replication problem at $t = 1$

Remark 2. Equation (8) can not be written in a form (1) satisfying the assumptions of this paper, so the results of this paper do not apply to this problem. However, it seems possible to extend them to cover this problem using comparison results from [2] along with L^∞-bounds on the numerical solution that follow from the maximum principle.

8. **A super-replication problem.** We apply our method to solve a problem from finance, the super-replication problem under gamma constraints considered in [2]. It consists of solving equation (8) with $f \equiv 0$, Neumann boundary conditions and σ^α as in Subsection 7, and initial and Dirichlet conditions given by

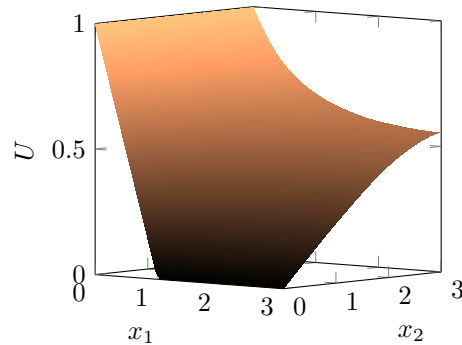$$u(t,x) = \max(0, 1 - x_1), \quad t = 0 \quad or \quad x_1 = 0 \quad or \quad x_2 = 0.$$

FIGURE 1. Numerical solution of super-replication problem at $t = 1$

The solution obtained with the LISL scheme is given in Figure 1 and coincides with the solution found in [2]. It gives the price of a put option of strike and maturity 1, and x_1 and x_2 are respectively the price of the underlying and the price of the forward variance swap on the underlying.

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