The Cauchy integral theorem

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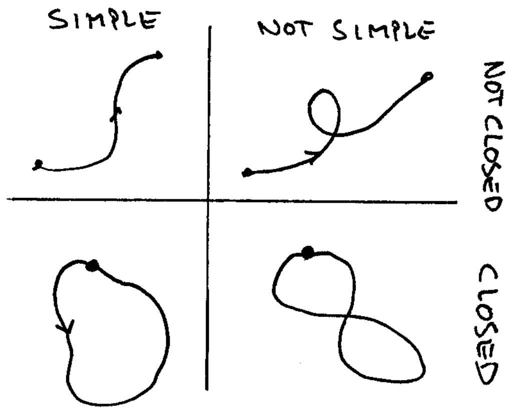
Curves and paths

A (parametrized) *curve* in the complex plane is a continuous map γ from a compact[1] interval $[a, b]$ into $\mathbb{C}$. We call the curve *closed* if its starting point and endpoint coincide, that is if $\gamma(a) = \gamma(b)$. We call it *simple* if it does not cross itself, that is if $\gamma(s) \neq \gamma(t)$ when $s < t$. Exception: We allow the curve to be closed, so a better way to say it is that $\gamma(s) = \gamma(t)$ and $s < t$ imply $s = a$ and $t = b$.



A simple, closed curve is often called a *Jordan* curve, because it was Camille Jordan (1838–1922) who first realized that the seemingly obvious fact that such a curve divides the plane into two components – an inside and an outside – was far from obvious, and needed a proof.

Curves are in general quite nontrivial objects. Giuseppe Peano (1858–1932) discovered a curve that covers an entire square in the plane, and William Os-

[1] If you don't know what *compact* means, just treat it as a synonym for *closed and bounded*. Compact sets have the property that from any sequence of points in the set, you can extract a subsequence that converges to a point within the set.

good (1864–1943) found that even a simple curve can have a positive area! (Though it cannot fill a square.)

General curves are to too general for our purpose, which is to use them as integration paths.

Definition of the path integral

Recall the definition of the *Riemann integral*, as the limit of what is known as Riemann sums:

$$\int_a^b f(x)\,dx = \lim \sum_{j=1}^{n} f(x_j^*)(x_j - x_{j-1})$$

where the sum involves a *partition* of $[a, b]$: That is, a set of points $a = x_0 < x_1 < \cdots < x_n = b$. There are also arbitrary points $x_j^* \in [x_{j-1}, x_j]$. Finally, the limit is to be taken as the partition gets finer and finer, which we may take to mean that its *mesh size*, which is just the maximal value of $x_j - x_{j-1}$, goes to zero.

A classic existence theorem on the Riemann integral states that it exists (which means the limit exists) whenever f is continuous on $[a, b]$.[2]

The Riemann integral as defined here works just as well if f is a complex valued function. If you wish, you can integrate the real part and imaginary parts separately and combine the results, but the definition and all the rules of calculating with it works just fine as they are, even in the complex case.

Since we shall use the path integral as a tool to discover interesting things about the Riemann integral, it is an absolute requirement that the path integral *generalizes* the Riemann integral. We just wish to replace $[a, b]$ by a curve.

In some sense, $[a, b]$ *is* a curve, parametrized by itself: Just put $\gamma(t) = t$ for $t \in [a, b]$.

To spare the suspense, here then is the definition of the path integral.

1 Definition. Assume $\gamma\colon [a, b] \to \mathbb{C}$ is a curve, and f is a function defined on the curve, by which we just mean that whenever $z = \gamma(t)$ then $f(z)$ is defined. Then

$$\int_\gamma f(z)\,dz = \lim \sum_{j=1}^{n} f(z_j^*)(z_j - z_{j-1}) \tag{1}$$

[2] More precisely, it is known to exist if and only if f is bounded, and the set of points where it is discontinuous has measure zero. Whatever *that* means – I will not define it here.

where the points $z_0, z_1, \dots, z_n$ are points in order along the curve, with the z_j^* in between – or more precisely, we start with a partition $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$, pick $t_j^* \in [t_{j-1}, t_j]$ and put $z_j = \gamma(t_j)$ and $z_j^* = \gamma(t_j^*)$. Then the limit is taken as the partition gets finer, just as in the definition of the Riemann integral. We say the integral *exists* if the limit exists.

Notice that the details of the parametrization plays a very minor role here: It is only used to keep track of, and using, the points along the curve in some prescribed order. Therefore it is immediately obvious that the path integral is independent of the particular parametrization. So, if $[c, d]$ is another interval and $h\colon [c, d] \to [a, b]$ is a continuous, strictly increasing function with $h(c) = a$ and $h(d) = b$, then we can put $\gamma_1(s) = \gamma(h(s))$ and think of γ_1 as a reparametrization of γ. It follows directly from the definition that $\int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz$.

Arc length

Under what circumstances can we expect the arc integral to exist? At the very least, we must have some reassurance that the sums used to define it don't go to infinity as the partition becomes finer. So first, we must assume that the integrand $f(z)$ is bounded along γ, say $|f(\gamma(t))| \le M$ for some finite number M. Even so, the best upper estimate we can think of is

$$\Bigl|\sum_{j=1}^n f(z_j^*)(z_j - z_{j-i})\Bigr| \le M \sum_{k=1}^n |z_j - z_{j-i}| \le M\ell(\gamma)$$

where $\ell(\gamma)$ is the length of the curve.

That is, if we believe that such a thing as the length of a curve can be defined, and if we further believe that a straight line is the shortest distance between two points: For surely then $|z_j - z_{j-i}|$ is no greater than the length of the curve γ between the two points, and adding up this inequality for $j = 1, 2, \dots, n$ we get the inequality above.

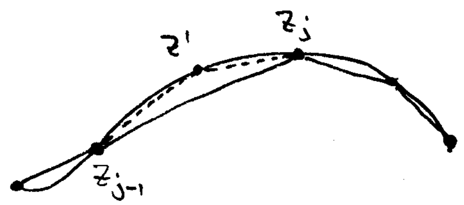
Basically, we define the length of a curve precisely so that this is so.

2 Definition. The *length* of a curve $\gamma\colon [a, b] \to \mathbb{C}$ is

$$\ell(\gamma) = \sup \sum_{j=1}^n |z_j - z_{j-1}|, \qquad z_j = \gamma(t_j)$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \dots < t_n = b$ of $[a, b]$. (More precisely, it is the supremum of the set of numbers obtained from the above sum as we consider every partition.)

The curve γ is said to have *finite length* if $\ell(\gamma) < \infty$. In that case, we shall call the curve a *path*.



If we add new points to a partition, we obtain a new partition which is said to be a *refinement* of the original. Clearly, any refinement can be obtained by adding just one point at the time. Now, if we start with a partition $a = t_0 < t_1 < \dots < t_n = b$ and add to it a single point t' between t_{j-1} and t_j, then one term $z_j - z_{j-1}$ in the sum above will be replaced by two terms, namely

$$|z' - z_{j-1}| + |z_j - z'| \ge |z_j - z_{j-1}|$$

so that the whole sum becomes larger (or at least not smaller). We have shown that the sum in the definition above is non-decreasing when the partition is refined. Therefore, we could also have defined the length as the *limit* of the above sum as the partition is refined – in analogy with the path integral.

3 Proposition. *If the curve γ is (piecewise) smooth, then*

$$\ell(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

Proof: We begin by estimating $|\gamma(t_j) - \gamma(t_{j-1})|$: Clearly

$$\gamma(t_j) - \gamma(t_{j-1}) = \int_{t_{j-1}}^{t_j} \gamma'(t)\,dt.$$

We wish to compare this to $\gamma'(t_j)(t_j - t_{j-1})$:

$$\gamma(t_j) - \gamma(t_{j-1}) - \gamma'(t_j)(t_j - t_{j-1}) = \int_{t_{j-1}}^{t_j} \bigl(\gamma'(t) - \gamma'(t_j)\bigr)\,dt.$$

If γ is smooth then by definition γ' is continuous, and hence uniformly continuous. So given $\varepsilon > 0$, we can choose $\delta > 0$ so that $|t-s| < \delta$ implies $|\gamma(t) - \gamma(s)| < \varepsilon$. If we have chosen a partition with $|t_j - t_{j-1}| < \delta$ for all j, we get $|\gamma'(t) - \gamma'(t_j)| < \varepsilon$ in the above integral, so that

$$\bigl|\gamma(t_j) - \gamma(t_{j-1}) - \gamma'(t_j)(t_j - t_{j-1})\bigr| \le (t_j - t_{j-1})\varepsilon$$

and therefore

$$\Bigl|\bigl|\gamma(t_j) - \gamma(t_{j-1})\bigr| - |\gamma'(t_j)|(t_j - t_{j-1})\Bigr| \le (t_j - t_{j-1})\varepsilon.$$

We may now sum this:

$$\begin{aligned}\Bigl|\sum_{j=1}^{n}\bigl|\gamma(t_j) - \gamma(t_{j-1})\bigr| - \sum_{j=1}^{n}|\gamma'(t_j)|(t_j - t_{j-1})\Bigr|\\ \le \sum_{j=1}^{n}\Bigl|\bigl|\gamma(t_j) - \gamma(t_{j-1})\bigr| - |\gamma'(t_j)|(t_j - t_{j-1})\Bigr| \le \sum_{j=1}^{n}(t_j - t_{j-1})\varepsilon = (b-a)\varepsilon.\end{aligned}$$

As the partition is refined then we get in the limit

$$\Bigl|\ell(\gamma) - \int_a^b |\gamma'(t)|\,dt\Bigr| \le (b-a)\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we are done. ∎

The existence of the integral

Let us consider what changes happen in the sum as the partition is refined. If $f(\gamma(t))$ varies only slightly in each interval, we expect the sum not to change much with refinement.

So we assume that $\varepsilon > 0$ is given, and choose $\delta > 0$ so that $|s-t| < \delta$ implies $|f(\gamma(s)) - f(\gamma(t))| < \varepsilon$, and we then assume that the partition is chosen so that $0 < t_j - t_{j-1} < \delta$.

We concentrate on one term $f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j) - \gamma(t_{j-1})\bigr)$ and what happens when we add new points to the partition, say $t_{j-1} = s_0 < s_1 < \cdots < s_m = t_j$: Then this term is replaced by a sum, and the difference between the original term and the new sum is

$$\begin{aligned}f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j) - \gamma(t_{j-1})\bigr) - \sum_{j=1}^{m} f\bigl(\gamma(s_j^*)\bigr)\bigl(\gamma(s_j) - \gamma(s_{j-1})\bigr)\\ = \sum_{j=1}^{m}\Bigl(f\bigl(\gamma(t_j^*)\bigr) - f\bigl(\gamma(s_j^*)\bigr)\Bigr)\bigl(\gamma(s_j) - \gamma(s_{j-1})\bigr)\end{aligned}$$

whose absolute value is not greater than

$$\varepsilon\sum_{j=1}^{m}\bigl|\gamma(s_j) - \gamma(s_{j-1})\bigr| \le \varepsilon\ell\bigl(\gamma\big|_{[t_{j-1},t_j]}\bigr)$$

(where $\gamma\big|_{[t_{j-1},t_j]}$ is that part of γ corresponding to parameter values t lying in $[t_{j-1}, t_j]$).

Adding all the terms together, we conclude that when the partition is chosen as explained above, and is then replaced by a refinement, the sum in (1) changes by not more than $\varepsilon\ell(\gamma)$.

By a sort of generalizing of Cauchy's convergence criterion (the fact that Cauchy sequences converge) this is enough to guarantee the existence of the integral $\int_\gamma f(z)\,dz$ as a limit of the sum in (1) when the partition is refined, and it also yields the estimate

$$\Bigl|\int_\gamma f(z)\,dz - \sum_{j=1}^{n} f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j) - \gamma(t_{j-1})\bigr)\Bigr| \le \varepsilon\ell(\gamma) \tag{2}$$

assuming that the partition is chosen so that $|f(\gamma(s)) - f(\gamma(t))| < \varepsilon$ whenever s and t lie in the same interval $[t_{j-1}, t_j]$ defined by the partition.

To be just a bit more rigorous, write $S(P)$ for the sum in (1) associated with a partition P. Let $P_1, P_2, \ldots$ be a sequence of progressively finer partitions, so that the maximal mesh width of P_j goes to zero when $j \to \infty$. The estimates above show that $\bigl(S(P_j)\bigr)$ is a Cauchy sequence, and therefore convergent. So we are tempted to define $\int_\gamma f(z)\,dz = \lim_{j\to\infty} S(P_j)$. But what if we had chosen another sequence (P_j^*) of partitions? Could we have obtained a different limit? The answer is no, for P_j and P_j^* have a common refinement P_j^{**}, and $\bigl|S(P_j) - S(P_j^*)\bigr| \le \bigl|S(P_j) - S(P_j^{**})\bigr| + \bigl|S(P_j^{**}) - S(P_j^*)\bigr| \to 0$ when $j \to \infty$, again thanks to the above estimate.

Let us compute some integrals directly from the definition.

4 Lemma. *For every path $\gamma\colon [a,b] \to \mathbb{C}$ we find*

$$\int_\gamma 1\,dz = \gamma(b) - \gamma(a) \quad \text{and} \quad \int_\gamma z\,dz = \tfrac{1}{2}\bigl(\gamma(b)^2 - \gamma(a)^2\bigr).$$

Proof: The first integral is obvious, since *every* approximating sum has the value $\sum_{j=1}^n \bigl(\gamma(t_i)-\gamma(t_{i-1})\bigr)=\gamma(t_n)-\gamma(t_0)=\gamma(b)-\gamma(a)$.

The other integral certainly exist, since the integrand is continuous. It can be written as a limit of either of the two sums

$$\sum_{j=1}^n \gamma(t_i)\bigl(\gamma(t_i)-\gamma(t_{i-1})\bigr) \quad\text{and}\quad \sum_{j=1}^n \gamma(t_{i-1})\bigl(\gamma(t_i)-\gamma(t_{i-1})\bigr),$$

so the integral is also a limit of the mean of the two, that is

$$\frac{1}{2}\sum_{j=1}^n \bigl(\gamma(t_i)+\gamma(t_{i-1})\bigr)\bigl(\gamma(t_i)-\gamma(t_{i-1})\bigr)=\frac{1}{2}\sum_{j=1}^n \bigl(\gamma(t_i)^2-\gamma(t_{i-1})^2\bigr)=\tfrac{1}{2}\bigl(\gamma(b)^2-\gamma(a)^2\bigr)$$

and the conclusion is once more obvious. ∎

Piecewise smooth paths

5 Proposition. *If γ is (piecewise) smooth then*

$$\int_\gamma f(z)\,dz=\int_a^b f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt.$$

Proof: We start by comparing one term in (1) with the corresponding part of the integral:

$$f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j)-\gamma(t_{j-1})\bigr)-\int_{t_{j-1}}^{t_j} f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt=\int_{t_{j-1}}^{t_j}\Bigl(f\bigl(\gamma(t_j^*)\bigr)-f\bigl(\gamma(t)\bigr)\Bigr)\gamma'(t)\,dt$$

By choosing the partition fine enough we get $\bigl|f\bigl(\gamma(t_j^*)\bigr)-f\bigl(\gamma(t)\bigr)\bigr|<\varepsilon$, and so we have

$$\Bigl|f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j)-\gamma(t_{j-1})\bigr)-\int_{t_{j-1}}^{t_j} f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt\Bigr|\le\varepsilon\int_{t_{j-1}}^{t_j}\bigl|\gamma'(t)\bigr|\,dt.$$

Summing this we conclude that

$$\Bigl|\sum_{j=1}^n f\bigl(\gamma(t_j^*)\bigr)\bigl(\gamma(t_j)-\gamma(t_{j-1})\bigr)-\int_a^b f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt\Bigr|\le\varepsilon\int_a^b\bigl|\gamma'(t)\bigr|\,dt,$$

and the proof is complete. ∎

While the formula above is easier to use than the definition of the integral, the next result is even easier, when it can be used:

6 Proposition. *Assume that f is continuous and has an* antiderivative *F, that is a function so that $F'(z)=f(z)$ for all z in the domain of f. For every path $\gamma\colon [a,b]\to\mathbb{C}$ in the domain of f starting in α and ending in β we then find*

$$\int_\gamma f(z)\,dz=F(\beta)-F(\alpha).$$

Proof: If γ is smooth, then this is obvious, for then

$$\begin{aligned}\int_\gamma f(z)\,dz&=\int_a^b f\bigl(\gamma(t)\bigr)\gamma'(t)\,dt=\int_a^b\frac{d}{dt}F\bigl(\gamma(t)\bigr)\,dt\\&=F\bigl(\gamma(b)\bigr)-F\bigl(\gamma(a)\bigr)=F(\beta)-F(\alpha).\end{aligned}$$

The result follows for piecewise smooth paths by adding this result over each smooth part.

But then the result follows by approximating a general path γ by broken lines, which are special cases of piecewise smooth paths. ∎

Approximating a path by a broken line

Consider an integration path $\gamma\colon [a,b]\to\mathbb{C}$. Given a partition $a=t_0<t_1<\cdots<t_n=b$ of $[a,b]$ we can create a new integration path $\gamma^*=[z_0,z_1,\dots,z_n]$ (where $z_j=\gamma(t_j)$) by joining straight line segments $[\gamma(t_{j-1}),\gamma(t_j)]$: More precisely, we can define $\gamma^*\colon [0,n]\to\mathbb{C}$ by setting

$$\gamma^*(j+s)=sz_j+(1-s)z_{j-1},\qquad s\in[0,1],\ j=0,1,\dots,n.$$

7 Proposition. *Assume that f is continuous in a neighborhood of an integration path γ. Then with the above notation,*

$$\int_{[z_0,z_1,\dots,z_n]} f(z)\,dz\to\int_\gamma f(z)\,dz$$

with convergence as the partition is refined.

Proof: We find

$$\int_{\gamma^*} f(z)\,dz=\sum_{j=1}^n\int_0^1 f\bigl(sz_j+(1-s)z_{j-1}\bigr)\,ds\cdot(z_j-z_{j-1})$$

which we can compare directly with the sum in (1):

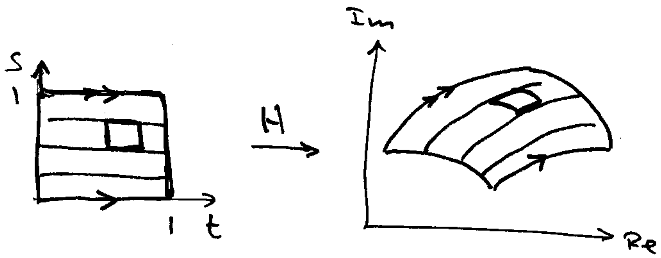
$$\int_{\gamma^*} f(z)\,dz - \sum_{j=1}^{n} f\big(\gamma(t_j^*)\big)\big(\gamma(t_j)-\gamma(t_{j-1})\big)$$
$$= \sum_{j=1}^{n}\int_0^1 \Big(f\big(s\gamma(t_j)+(1-s)\gamma(t_{j-1})\big)-f\big(\gamma(t_j^*)\big)\Big)\,ds\cdot\big(\gamma(t_j)-\gamma(t_{j-1}\big)$$

Here we can ensure that the integrand in the final line has absolute value less than an given $\varepsilon>0$ by choosing a fine enough partition, and then the absolute value of the entire sum is less than $\varepsilon\ell(\gamma)$. ∎

The Cauchy integral theorem

A *homotopy* in a region $\Omega\subseteq\mathbb{C}$ is simply a continuous mapping $H\colon [0,1]\times[0,1]\to\Omega$.

As we keep s fixed, then $t\mapsto H(t,s)$ is a curve in Ω, and similarly if we keep t fixed, then $s\mapsto H(t,s)$ is a curve as well.



So if we consider a small subrectangle $[a,b]\times[c,d]$ of $[0,1]\times[0,1]$ then going around this subrectangle in the positive direction we get a closed path in Ω, which we could parametrize as follows:

$$\gamma(t)=\begin{cases} H\big(tb+(1-t)a,c\big), & 0\le t\le 1,\\ H\big(b,(t-1)d+(2-t)c\big), & 1\le t\le 2,\\ H\big((t-2)a+(3-t)b,d\big), & 2\le t\le 3,\\ H\big(a,(t-3)c+(4-t)d\big), & 3\le t\le 4.\end{cases}$$

In what follows we shall need not only to integrate along these paths, so they have to have finite length, but we need strong estimates on these lengths.

The easiest way to get such estimates is to assume that H is *Lipschitz continuous*, which means that there exists a constant L (called the Lipschitz constant) so that

$$\big|H(c,d)-H(a,b)\big|\le L\sqrt{(c-a)^2+(d-b)^2}\qquad\text{for all } a,\ b,\ c,\ d.$$

This is case if H has partial derivatives satisfying $|\partial H/\partial t|\le L$ and $|\partial H/\partial s|\le L$, so being Lipschitz continuous is not at all uncommon.

For fixed s, we estimate the length of the path $t\mapsto H(t,s)$ by noting that

$$\sum_{j=1}^{n}\big|H(t_j,s)-H(t_{j-1},s)\big|\le\sum_{j=1}^{n}L\cdot(t_j-t_{j-1})=L\cdot(t_n-t_0),$$

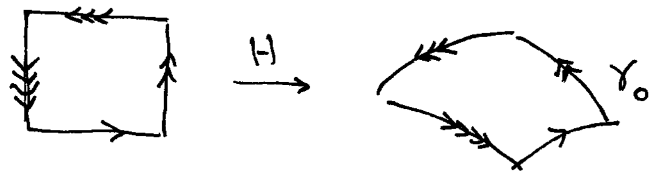
so the path is at most L times the length of the parameter interval. The same argument holds for paths $s\mapsto H(t,s)$, and by combining these result we get the same result for paths created as the boundary of subrectangles such as above.

We shall call a function f *analytic* in a region Ω if its derivative f' exists at all points in Ω. Note that we do *not* require f' to be continuous. That will turn out to be a consequence of analyticity.

8 Theorem. (Cauchy–Goursat) *Assume f is analytic in a region Ω, and let $H\colon[0,1]\times[0,1]\to\Omega$ be a Lipschitz continuous homotopy. Let γ_0 be the path obtained from H by traversing the boundary of the square $[0,1]\times[0,1]$ once in the positive direction, as explained above. Then*

$$\int_{\gamma_0} f(z)\,dz=0.$$

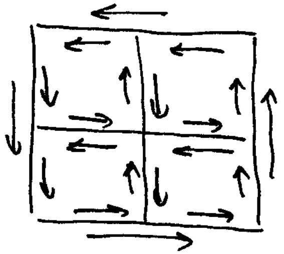
Cauchy and Goursat did not prove the theorem in this form, but their versions of it follows easily from this one, and the main idea of the proof below is due to Goursat.



Proof: Assuming that the integral is not zero, we shall arrive at a contradiction. If the integral is not zero, we can divide f by the value of the integral, so we might as well assume that

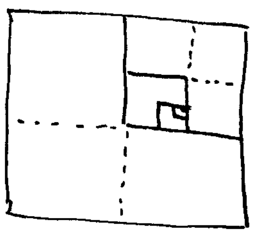
$$\int_{\gamma_0} f(z)\,dz = 1.$$

Now divide $\square_0 = [0,1]\times[0,1]$ into four equal squares. The integral around



the main square is equal to the sum of the four integrals around the for subsquares (the interior parts cancel), so at least one of the four integrals must have absolute value $\ge \frac{1}{4}$. Let $\square_1$ be one such square, and γ_1 the path obtained from H by following the boundary of $\square_1$. So $\left|\int_{\gamma_1} f(z)\,dz\right| \ge \frac{1}{4}$.

Next, divide $\square_1$ into four pieces, and apply the same reasoning. We find one of these, call it $\square_2$, so that the corresponding path γ_2 satisfies $\left|\int_{\gamma_2} f(z)\,dz\right| \ge \frac{1}{16}$.



Now repeat this: We find squares $\square_0 \supset \square_1 \supset \square_2 \supset \cdots$ each with a path γ_k corresponding to the boundary of $\square_k$, so that

$$\left|\int_{\gamma_k} f(z)\,dz\right| \ge 4^{-k} = 2^{-2k}. \tag{3}$$

We shall now turn around and get an upper estimate on the same integral, which contradicts the above.

First, we estimate the length of γ_k. The boundary of $\square_k$ has length $4\cdot 2^{-k}$, and so $\ell(\gamma_k) \le 4L2^{-k}$.

Since all the squares $\square_k$ are compact (closed and bounded), there is a point $(t_0, s_0))$ common to them all. And since f is differentiable at $z_0 = H(t_0, s_0)$, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + e(z)(z - z_0), \qquad \lim_{z\to z_0} e(z) = 0.$$

But $\int_{\gamma_k}\bigl(f(z_0) + f'(z_0)(z - z_0)\bigr)\,dz = 0$, so that $\int_{\gamma_k} f(z)\,dz = \int_{\gamma_k} e(z)(z - z_0)\,dz$. Now $|z - z_0| \le 2^{-k}\sqrt{2}L$ for z on γ_k, and given $\varepsilon > 0$, we get $|e(z)| < \varepsilon$ along γ_k if k is large enough. Then

$$\begin{aligned}\left|\int_{\gamma_k} f(z)\,dz\right| &= \left|\int_{\gamma_k}\bigl(e(z)(z - z_0)\bigr)\,dz\right| \\ &\le \ell(\gamma_k)\varepsilon 2^{-k}\sqrt{2} \le 4L2^{-k}\varepsilon 2^{-k}\sqrt{2} = 4\sqrt{2}\varepsilon L 2^{-2k},\end{aligned}$$

which contradicts (3) if $\varepsilon < 1/(4\sqrt{2}L)$. ∎

Homotopies with fixed end points. Consider two paths γ_0 and γ_1 in Ω, with the same starting and ending points:[3]

$$\gamma_0(0) = \gamma_1(0) = \alpha, \quad \gamma_0(1) = \gamma_1(1) = \beta.$$

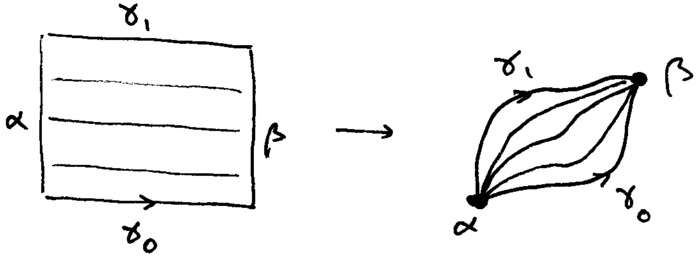
They will be called *homotopic with fixed end points* in Ω if there exists a homotopy H so that

$$H(t,0) = \gamma_0(t), \quad H(t,1) = \gamma_1(t), \quad H(0,s) = \alpha, \quad H(1,s) = \beta, \qquad \text{for all } t \text{ and } s.$$

9 Corollary. *If γ_0 and γ_1 are homotopic with fixed end points in Ω and f is analytic in Ω then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz.$$

[3] There is no loss of generality to assume that all paths are parametrized on the interval $[0,1]$, and it simplifies the notation in many proofs.



Proof sketch: This is quite obvious if the homotopy H is Lipschitz. The integral around the curve corresponding to the boundary of $[0,1]\times[0,1]$ consists of four parts: γ_0 in the forward direction, γ_1 in the reverse direction, and two parts (corresponding to $t=0$ and $t=1$) which are degenerate curves staying still at α and β. So the integrals along the latter two are zero, and what remains is $\int_{\gamma_1} f(z)\,dz - \int_{\gamma_1} f(z)\,dz$.

If H is *not* Lipschitz, then it can easily be approximated by Lipschitz homotopies: Given any partition $0=t_0<t_1<\cdots<t_n=1$, put $\tilde H(t_j,t_k)=H(t_j,t_k)$ for $j,k=0,1,\ldots,n$ and interpolate in each subrectangle $[t_{j-1},t_j]\times[t_{k-1},t_k]$:

$$\begin{aligned}\tilde H\big(ut_j+(1-u)t_{j-1},vt_k+(1-v)t_{k-1}\big)&\\ =uvH(t_j,t_k)+u(1-v)H(t_j,t_{k-1})&\\ +(1-u)vH(t_{j-1},t_k)+(1-u)(1-v)H(t_{j-1},t_{k-1}).&\end{aligned}$$

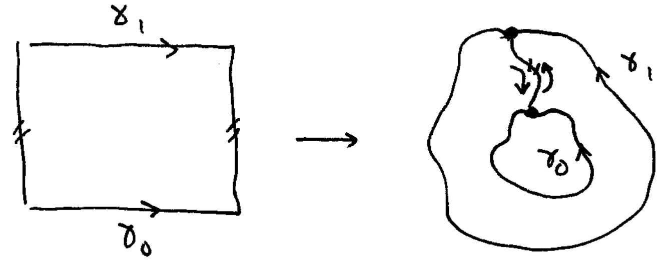
This is Lipschitz, and two of the boundary curves are still stationary at α and β, so the first part applies. The other two boundary curves yield broken line approximations to γ_0 and γ_1, and the result follows by going to the limit as the partition is refined. ∎

Homotopies via closed paths. Now consider two closed paths γ_0 and γ_1 in Ω:

$$\gamma_0(0)=\gamma_0(1),\quad \gamma_1(0)=\gamma_1(1).$$

They will be called *homotopic via closed paths* in Ω if there exists a homotopy H so that

$$H(0,t)=\gamma_0(t),\quad H(1,t)=\gamma_1(t),\quad H(s,0)=H(s,1),\qquad \text{for all } s \text{ and } t.$$



10 Corollary. *If γ_0 and γ_1 are homotopic via closed paths in Ω and f is analytic in Ω then*

$$\int_{\gamma_0} f(z)\,dz=\int_{\gamma_1} f(z)\,dz.$$

Proof sketch: The proof is just like the corresponding proof for fixed end points. The difference is that now, the two paths corresponding to $t=0$ and $t=1$ are not stationary, but one is the reverse of the other, so their integrals cancel. ∎

A region Ω is called *simply connected* if any two closed paths in Ω are homotopic in Ω via closed paths. Equivalently, any two paths with the same starting and ending points are homotopic with fixed endpoints. Then the integral of an analytic function on Ω is independent of the path, so we can write $\int_\gamma f(z)\,dz=\int_\alpha^\beta f(z)\,dz$ where γ starts in α and ends in β. In particular, f has an antiderivative F which can be defined by $F(z)=\int_\alpha^z f(\zeta)\,d\zeta$ for some fixed $\alpha\in\Omega$. Also, $\int_\gamma f(z)\,dz=0$ for any closed path γ in Ω.

The logarithm, revisited. Let Ω be any simply connected region *not* containing 0. Then the function $z\mapsto 1/z$ has an antiderivative in Ω. If $1\in\Omega$, it seems natural to use it as a starting point and define

$$F(z)=\int_1^z\frac{d\zeta}{\zeta},\quad \text{so that } F'(z)=\frac{1}{z}.$$

From this we find using the chain rule:

$$\frac{d}{dz}\big(ze^{-F(z)}\big)=e^{-F(z)}-z\frac{e^{-F(z)}}{z}=0,$$

so that $ze^{-F(z)}$ is a constant. Evaluating at $z=1$ we find that this constant is 1, and so $e^{F(z)}=z$ for all z. In other words, F is a branch of the logarithm, and

we can write
$$\ln z = \int_1^z \frac{d\zeta}{\zeta}.$$
If Ω does not contain 1, we just pick any starting point $\alpha \in \Omega$ and any β so that $e^\beta = \alpha$, and we can then put
$$\ln z = \beta + \int_\alpha^z \frac{d\zeta}{\zeta}.$$
Again, this will define a branch of the logarithm in Ω, and it is completely determined by the requirement that $\ln\alpha = \beta$.

The most common way to create a simply connected region on which to define a branch the logarithm is to introduce a *branch cut*, which is just a simple curve starting at 0 and going off to infinity. (This curve must be parametrized on a half open interval in order to be able to continue to infinity, so it's a little different from curves previously considered. Most commonly a half line is used, but any curve will do, and sometimes unorthodox choices are useful.) One then lets Ω be all of $\mathbb{C}$ with the points on the branch cut removed.

Sometimes the branch defined by $\ln z = \int_1^z \frac{d\zeta}{\zeta}$ will be called the *principal* branch of the logarithm in Ω, although this term is most commonly used when the branch cut is the negative real axis.

Winding number. Consider now a path $\gamma\colon [a,b] \to \mathbb{C}\setminus\{z_0\}$, with $\gamma(a) = \alpha$, $\gamma(b) = \beta$. Assuming it lies in some simply connected subregion of $\mathbb{C}\setminus\{z_0\}$, we can write
$$\int_\gamma \frac{dz}{z-z_0} = \ln(\beta - z_0) - \ln(\alpha - z_0).$$
Taking the exponential of this, we find
$$\exp\int_\gamma \frac{dz}{z-z_0} = \frac{\beta - z_0}{\alpha - z_0}, \tag{4}$$
a result which is *independent* of the choice of branch for the logarithm.

In general, a path in $\mathbb{C}\setminus\{z_0\}$ can be divided into a finite number of subpaths for which the above reasoning holds, and so we can multiply together the results (4) for the individual subpaths and get the same formula (4) for the whole path. In particular, if γ is closed then $\exp\int_\gamma 1/(z-z_0)\,dz = 1$, so the integral is an integer multiple of $2\pi i$. In other words, we have proved

11 Proposition. *If γ is a closed path in $\mathbb{C}\setminus\{z\}_0$ then the number $\operatorname{ind}_\gamma(z_0)$ defined by*
$$\operatorname{ind}_\gamma(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_0}$$
is an integer.

$\operatorname{ind}_\gamma(z_0)$ is called the *index* of z_0 with respect to γ, or the *winding number* of γ around z_0. It quite literally measures the number of times γ winds around z_0.

Cauchy's integral formula

Let Ω be a region, and $z \in \Omega$. If the disk $B_\rho(z)$ is contained in Ω, then all circles centered at z and with radius $< r$ are homotopic via closed paths in $\Omega\setminus\{z\}$: More precisely, let the circle γ_r be given by
$$\gamma_r(t) = z + re^{it}, \qquad t \in [0, 2\pi]$$
and then that same formula provides the homotopy:[4]
$$H(t,r) = z + re^{it}, \qquad r \in [r_1, r_2],\ t \in [0, 2\pi]$$
defines a homotopy between γ_{r_1} and γ_{r_2}.

12 Theorem. (Cauchy's integral formula) *Assume that γ is a closed curve in Ω which is homotopic via closed paths in $\Omega\setminus\{z\}$ to one (and hence all) of the above small circles. If f is analytic in Ω, then*
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof: The integrand is an analytic function of ζ in $\Omega\setminus\{z\}$. Therefore the integral is unchanged if we replace γ by an arbitrarily small circle γ_r around z.

We also note that, by direct calculation
$$\int_\gamma \frac{1}{\zeta - z}\,d\zeta = 2\pi i.$$

[4] It was easier, when we proved general theorems on homotopies, to assume they were defined on $[0,1]\times[0,1]$. But nothing changes if we allow them to be defined on more general rectangles.

Multiplying this by $f(z)$ (which is independent of ζ, and therefore can be put inside the integral), we see that we only need to prove

$$\lim_{r\to 0}\int_{\gamma_r}\frac{f(z)-f(\zeta)}{\zeta - z}\,d\zeta = 0.$$

(The limit seems unnecessary, but is harmless, since the integral does not depend on r when r is small.)

But f is continuous, so $|f(z)-f(\zeta)|<\varepsilon$ for ζ on γ_r if r is small enough. Then the whole integrand has absolute value smaller than ε/r, and the integration path has length $2\pi r$, so the integral is smaller than $\varepsilon/r\cdot 2\pi r = 2\varepsilon\pi$. This completes the proof. ∎

From this we can already deduce the following. Recall that a function is called *entire* if it analytic in all of the complex plane.

13 Theorem. (Liouville) *A bounded, entire function is constant.*

Proof: Assume that $|f(z)|\le M$ for all $z\in\mathbb{C}$. Let $z\in\mathbb{C}$. If $r>|z|$ then

$$\begin{aligned} f(z)-f(0) &= \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\gamma_r}\frac{f(\zeta)}{\zeta}\,d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\Big(\frac{1}{\zeta - z}-\frac{1}{\zeta}\Big)\,d\zeta = \frac{1}{2\pi i}\int_{\gamma_r} f(\zeta)\frac{z}{(\zeta - z)\zeta}\,d\zeta \end{aligned}$$

so that

$$|f(z)-f(0)|\le \frac{2\pi r}{2\pi}M\frac{|z|}{(r-|z|)r}\to 0 \qquad \text{as } r\to 0.$$

This shows that $f(z)=f(0)$, and completes the proof. ∎

14 Theorem. (Cauchy's generalized integral formula) *Under the assumptions of theorem 12, f is infinitely differentiable, and*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta, \qquad n = 0,1,2,\ldots$$

Proof: For $n=0$, this is just the standard Cauchy formula.

The proof for the remaining n is just a matter of differentiating with respect to z under the integral sign. But in case you do not know when that is allowed, we can do it directly:

For $n=1$, notice that

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z)(\zeta - z_0)}$$

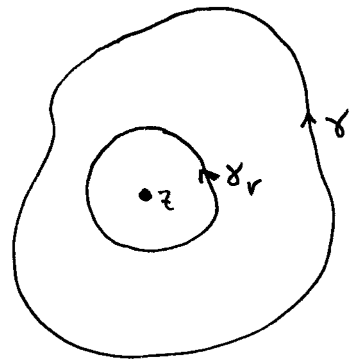
and substitute this value of $1/(\zeta - z)$ into the final term:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z-z_0)^2}{(\zeta - z)(\zeta - z_0)^2}$$

Substitute into Cauchy's formula and get

$$f(z) = f(z_0) + \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta}_{A}\cdot(z-z_0) + \underbrace{\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta\cdot(z-z_0)}_{e(z)}\cdot(z-z_0)$$

where the term marked A must be $f'(z_0)$, because the term marked $e(z)$ goes to zero as $z\to z_0$ (the integral is bounded, and the factor $z - z_0$ takes care of the rest).



The proof proceeds by induction. So assume the formula holds for a given n. Let γ_r be a small circle surrounding z and itself completely surrounded by γ. Then by the induction hypothesis,

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - w)^{n+1}}\,d\zeta, \qquad w \text{ on } \gamma_r.$$

We apply the case $n=1$ to the function $f^{(n)}$ and γ_r, then substitute the above result, interchange the order of integration, and rearrange a little:

$$\begin{aligned} f^{(n+1)}(z) &= \frac{1}{2\pi i}\int_{\gamma_r} \frac{f^{(n)}(w)}{(w-z)^2}\,dw \\ &= \frac{n!}{(2\pi i)^2}\int_{\gamma_r}\int_{\gamma} \frac{f(\zeta)}{(w-z)^2(\zeta-w)^{n+1}}\,d\zeta\,dw \\ &= \frac{n!}{(2\pi i)^2}\int_{\gamma}\int_{\gamma_r} \frac{f(\zeta)}{(w-z)^2(\zeta-w)^{n+1}}\,dw\,d\zeta \\ &= \frac{n!}{2\pi i}\int_{\gamma} f(\zeta)\Big(\frac{1}{2\pi i}\int_{\gamma_r}\frac{1}{(w-z)^2(\zeta-w)^{n+1}}\,dw\Big)\,d\zeta \end{aligned}$$

Now note that the function $w \mapsto (\zeta - w)^{-n+1}$ is analytic on and inside γ_r, so that the inner expression is just the generalized Cauchy formula for this function and $n = 1$. Thus the inner expression is the *derivative* of this function, at $w = z$, so

$$f^{(n+1)}(z) = \frac{n!}{2\pi i}\int_{\gamma} f(\zeta)\frac{n+1}{(\zeta-z)^{(n+2)}}\,d\zeta,$$

and the proof is complete. The only camel you have been asked to swallow concerns the possibility of interchanging the order of integration, and it seems like a fairly small camel. ∎

The global Cauchy integral formula

We sometimes need to work with the sum of integrals around several closed paths. A bit of notation is helpful. If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a finite collection of closed paths, we may call Γ a closed *multipath*.

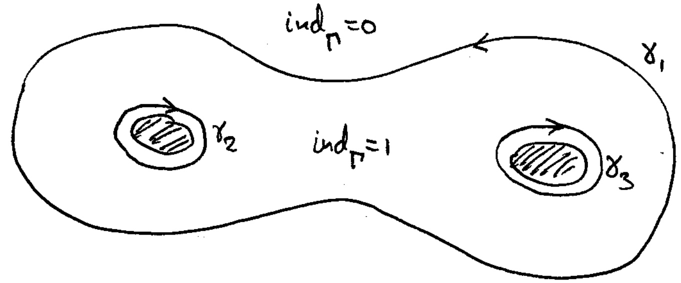
Write $\int_\Gamma f(z)\,dz = \sum_{k=1}^n \int_{\gamma_k} f(z)\,dz$ and $\operatorname{ind}_\Gamma(z) = \sum_{k=1}^n \operatorname{ind}_{\gamma_k}(z)$. Obviously, we will say a point is on Γ if it is on one of the paths γ_k.

15 Theorem. (Cauchy's formula, global (holonomy) version) *Assume that Ω is a region, and that Γ is a closed multipath in Ω so that $\operatorname{ind}_\Gamma(z) = 0$ for any $z \notin \Omega$. If f is analytic in Ω then for any $z \in \Omega$ that is not on Γ,*

$$\operatorname{ind}_\Gamma(z) f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Proof: Inserting the definition of the index on the left hand side, we see that what we must prove is

$$\int_\Gamma \frac{f(\zeta)-f(z)}{\zeta - z}\,d\zeta = 0.$$



The integrand here is more regular than it looks: We can define a continuous function g on $\Omega \times \Omega$ by

$$g(z,\zeta) = \begin{cases} \dfrac{f(\zeta)-f(z)}{\zeta - z}, & \zeta \neq z, \\ f'(z), & \zeta = z. \end{cases}$$

Now define the function h on $\mathbb{C}$ by

$$h(z) = \begin{cases} \displaystyle\int_\Gamma g(z,\zeta)\,d\zeta, & z \in \Omega, \\ \displaystyle\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta, & z \in \mathbb{C} \text{ not on } \Gamma \text{ and } \operatorname{ind}_\Gamma(z) = 0. \end{cases}$$

Note that for some z both cases in the above definition apply, but for those z the two definitions agree because of the assumption $\operatorname{ind}_\Gamma(z) = 0$ and the definition of the index.

The assumption that the index is zero for points outside Ω implies that at least one of the cases apply for any z, so h is indeed defined in the entire plane. Moreover each of the two parts of the definition defines an analytic function in an open set, so h is entire.

Finally, when $|z|$ is large then $\operatorname{ind}_\Gamma(z) = 0$ so the second part of the definition applies, and a simple estimate shows that $h(z) \to 0$ as $z \to \infty$. In particular h is bounded, so it is constant by Liouville's theorem. And then this constant must be zero by what we just showed, so $h(z) = 0$ for all z. This completes the proof. ∎