Linear systems of ODEs with variable coefficients

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Let A be a matrix valued function defined on some interval, with each $A(t)$ bein an $n \times n$ matrix. A is supposed to be a Lipschitz continuous function of its argument.

This note is about the linear system

$$\dot{x} = Ax + b(t) \tag{1}$$

where $x(t)$ is a (column) n-vector for each t, and b is a vector valued function of t, assumed throughout to be continuous.

Consider the following ODE for a matrix valued function Φ, where each $\Phi(t)$ is also supposed to be an $n \times n$ matrix:

$$\dot{\Phi} = A\Phi \tag{2}$$

1 Proposition. *Let Φ be a matrix valued function satisfying* (2). *If $\Phi(t_0)$ is invertible for some t_0 then $\Phi(t)$ is in fact invertible for every t, and the inverse $\Psi(t) = \Phi(t)^{-1}$ satisfies the differential equation*

$$\dot{\Psi} = -\Psi A. \tag{3}$$

Proof: The differential equation for Ψ is easy to derive: Just differentiate the relation $\Psi\Phi = I$ to get

$$0 = \frac{d}{dt}(\Psi\Phi) = \dot{\Psi}\Phi + \Psi\dot{\Phi} = \dot{\Psi}\Phi + \Psi A\Phi,$$

which when multiplied on the right by Ψ (and using $\Phi\Psi = I$) yields (3).

The above proof requires of course not only that Φ is invertible for all t, but also that the inverse is differentiable.

We can make the argument more rigourous by turning inside out, *defining* Ψ to be the solution of (3) satisfying the initial condition $\Psi(t_0) = \Phi(t_0)^{-1}$. Then we differentiate:

$$\frac{d}{dt}(\Psi\Phi) = \dot{\Psi}\Phi + \Psi\dot{\Phi} = -\Psi A\Phi + \Psi A\Phi = 0,$$

so that $\Psi\Phi = I$ for all t, since it so at $t = t_0$. ∎

2 Definition. A matrix valued solution of (2), which is invertible for all t, is called a *fundamental matrix* for (1).

Clearly, there are many fundamental matrices, for if Φ is one such and B is any constant invertible matrix, then ΦB is also a fundamental matrix.

However, a fundamental matrix is uniquely determined by its value at any given t_0, and if Φ_1 and Φ_2 are two fundamental matrices, we can set $B = \Phi_1^{-1}(t_0)\Phi_2(t_0)$, so that $\Phi_1 B = \Phi_2$ – at $t = t_0$, and hence for all t.

We now show how the fundamental matrix solves the general initial-value problem for (1).

In fact, let x be any solution of (1). Let Φ be a fundamental matrix, and write $x = \Phi y$. Then $\dot{x} = \dot{\Phi}y + \Phi\dot{y} = A\Phi y + \Phi\dot{y}$, so that (1) becomes

$$A\Phi y + \Phi\dot{y} = A\Phi y + b.$$

Two terms cancel of course, and after multiplying both sides by Φ^{-1} on the left what remains is

$$\dot{y} = \Phi^{-1}b,$$

which is trivial to solve. Given the initial condition $x(t_0) = x_0$, that translates into $y(t_0) = \Phi(t_0)^{-1}x_0$, so the solution for y is

$$y(t) = \Phi(t_0)^{-1}x_0 + \int_{t_0}^{t} \Phi(s)^{-1}b(s)\,ds.$$

Multiplying by $\Phi(t)$ on the left we finally have the solution

$$x(t) = \Phi(t)\Phi(t_0)^{-1}x_0 + \Phi(t)\int_{t_0}^{t} \Phi(s)^{-1}b(s)\,ds.$$