Sectors and Bendixson's index formula

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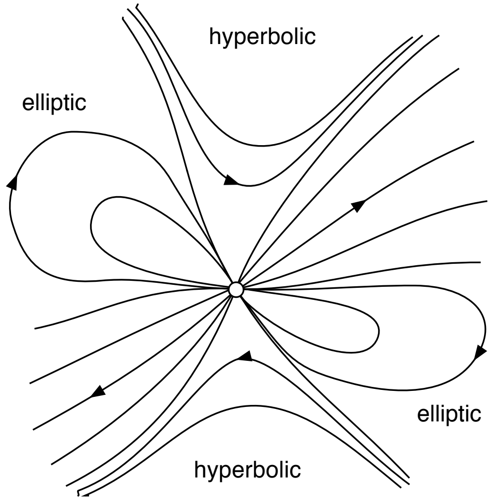
The picture below shows a phase portrait in the vicinity of an equilibrium point.

In this picture, the neighbourhood contains two *elliptic sectors*, recognizable by orbits starting and ending at the equilibrium itself (and these orbits forming ever smaller loops converging on the equilibrium point).

There are also two *hyperbolic sectors*, filled with orbits roughly resembling hyperbolas. The sectors are separated in one case, by *separatrices*, being orbits with one end at the equilibrium separating the sectors.

Finally, there are two *parabolic sectors*, filled with orbits having just one end at the equilibrium, and surrounded by other orbits of the same kind.



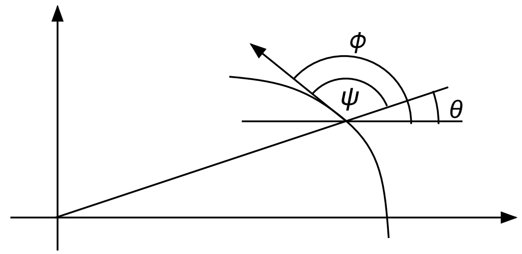
Bendixson's index formula states that the index of the equilibrium point is

$$1+\frac{e-h}{2}$$

where e is the number of elliptic sectors and h is the number of hyperbolic sectors.

To see why this is true, recall the definition of the index: At any non-equilibrium point, let ϕ be the angle made by the underlying vector field with the x axis. As we traverse a small closed curve (small enough not to surround any other equilibrium) going once around the equilibrium point in the positive direction, keep track of ϕ as a continuously varying quantity. The index is the increase in ϕ, divided by 2π. Written symbolically, it is $\Delta\phi/(2\pi)$.

Now let (r,θ) be polar coordinates centered at the equilibrium point (move the equilibrium to the origin if you prefer). Write $\phi=\theta+\psi$, where ψ must then be the angle between the line from the equilibrium to the current point and the vector field at that point (see the figure).



Since θ by definition increases by 2π as we go around a curve of the prescribed sort, symmbolically $\Delta\theta=2\pi$, it follows that the index must be $1+\Delta\psi/(2\pi)$.

Now it only remains to notice that ψ increases by π across an elliptic sector, ψ decreases by π across a hyperbolic sector, and it does not change at all across a parabolic sector.

Add these changes to ψ over all sectors, with the result $\Delta\psi=(e-h)\pi$, and Bendixson's formula is the result.

The above argument is exact if the separatrices are straight lines. In practice they need not be, but if we move in sufficiently close to the equilibrium, they can be approximated by straight lines. Since the total increase in ψ must be an integer times 2π, any small errors in estimating the angles will not matter.

Notice finally that e and h must both be odd numbers, or else they must both be even. To see this more directly, note that any separatrix or parabolic sector has a definite direction: Either towards the equilibrium point or away from it. And opposite sides of an elliptic or hyperbolic sectors have opposite directions. Thus there must be in total an even number of these sectors.