Linearization at equilibrium points

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This note is about the behaviour of a nonlinear autonomous system $\dot{x} = f(x)$ (where $x(t) \in \mathbb{R}^n$) near an equilibrium point x_0 (i.e., $f(x_0) = 0$).

The *Hartman–Grobman* theorem states that the system behaves "just like" its linearization near the equilibrium point. However, this theorem requires of the linearization that no eigenvalues have real part zero: Thus it is not the appropriate tool for deciding instability where some eigenvalue has a positive real part, since other eigenvalues may have a real part equal to zero in general. The Hartman–Grobman theorem will decide stability when all eigenvalues have negative real parts, but this is sort of a sledgehammer approach where simpler tools will do the job.

We will first develop and use these simpler tools, then return to the Hartman–Grobman theorem and its more appropriate uses later.

Stability and instability of equilibrium points

In this section, we use suitable Liapunov functions to prove the standard results on stability and instability of equilibria based on the eigenvalues of the linearization. We consider estimates for the linear part first.

After a change of variables, a linear system $\dot{x} = Ax$ can be written on the form $\dot{u} = Ju$, where J is a matrix on Jordan normal form: I assume that you know what this means, but remind you of the basic Jordan building block:

$$\lambda I + N = \begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \lambda & 1 \\ 0 & \dots & \dots & \dots & 0 & \lambda \end{pmatrix} \tag{1}$$

where N is the matrix with 1 just above the main diagonal, and zeroes elsewhere. For our purposes, it is useful to note that this is similar to a matrix $\lambda I + \varepsilon N$, where each 1 above the diagonal is replaced by an ε. To be more precise, $D^{-1}(\lambda I + N)D = \lambda I + \varepsilon N$ where D is the diagonal matrix with $1, \varepsilon, \varepsilon^2, \dots$ on the diagonal.

The reason this is interesting is the estimate

$$(\lambda - \varepsilon)|u|^2 \le u^{\mathrm{T}}(\lambda I + \varepsilon N)u \le (\lambda + \varepsilon)|u|^2 \tag{2}$$

for any vector u. It follows via a simple calculation from $|u^{\mathrm{T}}Nu| = |u_1u_2 + u_2u_3 + \cdots| \le |u|^2$, which in turn comes from the Cauchy–Schwarz inequality.

Now, I lied a bit above, for there are complex eigenvalues to be considered as well. To make a long story short, complex eigenvalues come in mutually conjugate pairs $\lambda = \sigma \pm i\omega$ where $\sigma, \omega \in \mathbb{R}$ and $\omega \ne 0$. These can give rise to Jordan blocks almost like (2), except each λ must be replaced by a 2×2matrix

$$\begin{pmatrix} \sigma & -\omega \\ \omega & \sigma \end{pmatrix}$$

and each 1 above the diagonal by a 2×2 identity matrix. But then we can perform the same rescaling trick as before, essentially replacing each of these identity matrices by ε times the identity, and we obtain an estimate just like (2), but with $\sigma = \operatorname{Re}\lambda$ replacing λ in the upper and lower bounds.

All this handwaving amounts to a proof of the following:

1 Lemma. *If A is a real, quadratic matrix, and each eigenvalue λ of A satisfies $\alpha \le \operatorname{Re}\lambda \le \beta$, then for any $\varepsilon > 0$, A is similar to a matrix $\tilde{A}$ satisfying*

$$(\alpha - \varepsilon)|u|^2 \le u^{\mathrm{T}}\tilde{A}u \le (\beta + \varepsilon)|u|^2$$

for each vector u.

We are now in a position to show the stability result:

2 Proposition. *Let x_0 be an equilibrium point of $\dot{x} = f(x)$, where f is a C^1 vector field. If $\operatorname{Re}\lambda < 0$ for each eigenvalue of the Jacobian matrix of f at x_0, the equilibrium is asymptotically stable.*

Proof: We may assume $x_0 = 0$ without loss of generality (after all, it's just a change of variables). So the system is of the form

$$\dot{x} = f(x) = Ax + o(|x|).$$

Now the Jacobian A of f at 0 has only a finite number of eigenvalues, all of which have negative real part – so there is some $\varepsilon > 0$ with $\operatorname{Re}\lambda \le -2\varepsilon$ for each eigenvalue λ. By Lemma 1, we can perform a further linear change of variables so that the system takes the form

$$\dot{u} = \tilde{A}u + o(|u|),$$

and where $u^{\mathrm{T}}\tilde{A}u \le -\varepsilon|u|^2$ for all u.

Consider the function $V(u) = \frac{1}{2}|u|^2$. Then

$$\dot{V} = u^{\mathrm{T}}\dot{u} = u^{\mathrm{T}}Au + o(|u|^2) \le -\varepsilon|u|^2 + o(|u|^2) < 0$$

when $|u|$ is small enough, so V is a strong Liapunov function, and 0 is indeed asymptotically stable. ∎

3 Lemma. *Consider an equilibrium point* 0 *for a dynamical system* $\dot{u} = g(u)$.

Let U *be a* C^1 *function so that* $U(0) = 0$*, every neighbourhood of* 0 *contains some* u *with* $U(u) > 0$*, and assume there is some neighbourhood of* 0 *so that whenever* u *belongs to this neighbourhood and* $U(u) > 0$*, then* $\dot{U}(u) > 0$ *as well. Then* 0 *is an unstable equilibrium point.*

Proof: Let $\varepsilon > 0$ be so that whenever $|u| \le \varepsilon$ and $U(u) > 0$, then $\dot{U}(u) > 0$.

Consider any $\delta > 0$. We shall prove that there exists an orbit starting within the δ-neighbourhood of 0 which must escape the ε-neighbourhood of 0.

So pick any u_0 with $|u_0| < \delta$ and $U(u_0) > 0$. Write

$$K = \big\{u\colon |u| \le \varepsilon \text{ and } |U(u)| \ge |U(u_0)|\big\}.$$

K is closed and bounded, therefore compact. Since $\dot{U} > 0$ on K, $\dot{U}$ has a positive lower bound on K, say $\dot{U}(u) \ge \gamma > 0$ whenever $u \in K$.

Now let u be the solution with initial value u_0. So long as $u(t) \in K$ then $U(u)$ will grow with a rate at least γ, so if $u(t) \in K$ for all K then $U\big(u(t)\big)$ will grow without bound, which is impossible because U is bounded on the compact set K. Therefore u must leave K, and it can only do that by getting $|u| > \varepsilon$, i.e., by escaping the ε-neighbourhood as claimed. ∎

4 Proposition. *Let* x_0 *be an equilibrium point of* $\dot{x} = f(x)$*, where* f *is a* C^1 *vector field. If* $\operatorname{Re}\lambda > 0$ *for some eigenvalue of the Jacobian matrix of* f *at* x_0*, the equilibrium is unstable.*

Proof: As before, assume $x_0 = 0$ without loss of generality. So the system is of the form

$$\dot{x} = f(x) = Ax + o(|x|).$$

We may as well assume we have already changed the variables so that A has Jordan normal form. We can also assume that the Jordan blocks of A appear in decreasing order of $\operatorname{Re}\lambda$. Lump together all the blocks with the largest value of $\operatorname{Re}\lambda$, and write u for the corresponding components of x. Write v for the remaining components. The system now has the form

$$\dot{u} = Bu + o(\sqrt{|u|^2 + |v^2|}),$$
$$\dot{v} = Cv + o(\sqrt{|u|^2 + |v^2|}),$$

where each eigenvalue of B satisfies $\operatorname{Re}\lambda = \beta > 0$, while each eigenvalue of C satisfies $\operatorname{Re}\lambda \le \alpha < \beta$. We can certainly insist that $\alpha > 0$ as well. Let $0 < \varepsilon < \frac{1}{2}(\beta - \alpha)$.

We shall change variables yet again, separately for u and v this time, but we will reuse the old variable names for u, v, B and C. We shall use Lemma 1 so that, after the variable change, we find

$$v^{\mathrm{T}}Cv < (\alpha + \varepsilon)|v|^2, \quad (\beta - \varepsilon)|u|^2 < u^{\mathrm{T}}Bu.$$

Let $U(u, v) = \frac{1}{2}(|u|^2 - |v|^2)$. We claim that U satisfies the conditions of Lemma 3, which will finish the proof.

The only property of U that is nontrivial to prove is the one on the sign of $\dot{U}$. Now we find

$$\begin{aligned}\dot{U} &= u^{\mathrm{T}}\dot{u} - v^{\mathrm{T}}\dot{v}\\ &= u^{\mathrm{T}}Bu - v^{\mathrm{T}}Cv + o(|u|^2 + |v^2|)\\ &> (\beta - \varepsilon)|u|^2 - (\alpha + \varepsilon)|v|^2 + o(|u|^2 + |v^2|),\end{aligned}$$

and when $U > 0$ we have $|u| > |v|$, so we find

$$\dot{U} > (\beta - \alpha - 2\varepsilon)|u|^2 + o(|u|^2) > 0$$

when $|u|$ is small enough. ∎

The Hartman–Grobman theorem

Consider the autonomous system $\dot{x} = f(x)$ with an equilibrium point x_0. We shall assume that f is a C^1 function. The linearization of this system is $\dot{u} = Au$, where A is the Jacobian matrix of f at x_0. The general solution of the linearized system is $u = e^{tA}u_0$.

The proof of the following theorem is beyond the scope of this text. A relatively easy proof can be found in [4]. However, the proof is done in a Banach space setting, which might make it less accessible. The theorem was originally proved independently by Grobman [1] and Hartman [2].

5 Theorem. (Hartman–Grobman) *Under the above assumptions, and with the extra condition that every eigenvalue of A has nonzero real part, there is a homeomorphism H from a neighbourhood S of 0 to a neighbourhood R of x_0, so that $x(t) = H(e^{tA}u_0)$ is a solution to $\dot{x} = f(x)$ whenever $e^{tA}u_0 \in S$.*

A *homeomorphism* is just a continuous map with a continuous inverse. Note that it follows from the uniqueness theorem for solutions of ODEs that all solutions in R have the form given above, for any point $x \in R$ can be written $H(u_0)$ for some $u_0 \in S$, and then $H(e^{tA}u_0)$ is a solution passing through x (at $t = 0$).

One weakness of the Hartman–Grobman theorem is the assumption on the eigenvalues, which cannot be avoided: When some eigenvalues have real part zero, the detailed behaviour of the system near the equilibrium cannot be derived from the linearization.

Another weakness of the theorem is that the conclusion is too weak for many applications: A homeomorphism can map a node to a focus!

For example, consider the function

$$H(u, v) = (u\cos s - v\sin s, u\sin s + v\cos s), \qquad s = -\tfrac{1}{2}\ln(u^2 + v^2).$$

Assuming that (u, v) satisfy the equations $\dot{u} = -u$ and $\dot{v} = -v$ (corresponding to a stable node), we find $\dot{s} = 1$, and $(x, y) = H(u, v)$ solves the system

$$\dot{x} = -x - y, \quad \dot{y} = x - y,$$

which corresponds to a stable focus.

Of course, the function H above is not differentiable at 0. (It is continuous there, if we add $H(0,0) = (0,0)$ to the definition.) If we require that H and its inverse be differentiable, such behaviour as seen in the example above becomes impossible.

Unfortunately, we cannot guarantee differentiability of H in general. But the following result [3] helps:

6 Theorem. (Hartman) *Under the assumptions of the Hartman–Grobman theorem, if additionally f is a C^2 function and the real parts of all the eigenvalues of A have the same sign, then the homeomorphism H can in fact be chosen to be a C^1 diffeomorphism.*

And by that we mean that H is C^1, and so is its inverse.

As a consequence of this, so long as the vector field f is C^2, any equilibrium whose linearization is a node or focus is itself of the same type.

No matter how differentiable f may be, we cannot conclude any higher degree of differentiability for H. And when eigenvalues exist in both the left and right halfplanes, a homeomorphism is all we can hope for. All this makes the Hartman–Grobman theorem quite a bit less useful than it looks at first sight.

References

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[3] Philip Hartman, On local homeomorphisms of Euclidean spaces. *Bol. Soc. Math. Mexicana* **5** (1960) 220–241.

[4] Charles C. Pugh, On a theorem of P. Hartman. *Amer. J. Math.* **91** (1969) 363–367.