Diagonals without tears

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This little note explores a minor innovation in the pedagogy of mathematical analysis. Maybe it's a rediscovery; I have not scoured the literature for previous work in this area.

Diagonal arguments play a minor but important role in many proofs of mathematical analysis: One starts with a sequence, extracts a subsequence with some desirable convergence property, then one obtains a subsequence of *that* sequence, and so forth. Finally, in what seems to the beginning analysis student like something of a sleight of hand, one "takes the diagonal" and ends up with a sequence sharing the nice properties of all the subsequences used in the construction.

One problem with the diagonal argument is that it quickly turns into something of a notational nightmare if you want a rigorous exposition, keeping careful track of things, as you should indeed do – particularly when teaching beginning students.

The only novelty presented here is a natural redefinition of the very concept of a sequence, and consequently that of a subsequence: Instead of insisting that sequences are indexed with the set of *all* natural numbers, we shall use *subsets* of the natural numbers instead. From this simple change of viewpoint, the rest follows naturally.

1 Definition. A *sequence* is a function defined on an infinite set I of natural numbers. The set I will be called the *index set* of the sequence. We use the customary index notation for sequences, writing $\langle x_n \rangle_{n\in I}$ for a sequence defined on the index set I.

So we can define sequences on the set of all natural numbers, or the even numbers, the powers of 2, or the prime numbers. The usual definition of convergence applies: A sequence $\langle x_n \rangle_{n\in I}$ of real numbers is said to *converge* to a limit a if, for each $\varepsilon > 0$, $|x_n - a| < \varepsilon$ for almost all $n \in I$. Here, *almost all* means all except a finite number of $n \in I$. We may then write $x_n \to a$ as $I \ni n \to \infty$ reading the latter formula as "$n \to \infty$ through I". However, this notation is cumbersome, so we will write $x_n \to a$ as $n \in I$ instead; or simply $\lim_{n\in I} x_n = a$.

We can now do subsequences without breaking a sweat, and more importantly, without indexing our indices:

2 Definition. A *subsequence* of a sequence $\langle x_n \rangle_{n\in I}$ is a sequence of the form $\langle x_n \rangle_{n\in J}$ where $J \subseteq I$. In other words, we obtain a subsequence by restricting the original sequence to a smaller index set.

A *tail* of an index set I is a set of the form $\{n \in I \mid n \geq m\}$. A tail of a sequence is a subsequence indexed by a tail of the original sequence.

The following lemma is utterly trivial, but utterly essential as well.

3 Lemma *Any subsequence of a convergent sequence is itself convergent, with the same limit. Further, if a sequence has a convergent tail, then the sequence itself is convergent.*

The following lemma is what makes diagonal arguments work:

4 Lemma (Diagonal lemma) *Assume that a given sequence $\langle x_n \rangle_{n\in I}$ has nested subsequences $\langle x_n \rangle_{n\in I_m}$, i.e., $I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. Then there exists an index set $J \subseteq I$ such that, for every m, some tail of J is contained in I_m.*

In other words, $\langle x_n \rangle_{n\in J}$ is *almost* a subsequence of $\langle x_n \rangle_{n\in I_m}$, with only finitely many exceptions, which clearly do not matter for convergence purposes.

We call $\langle x_n \rangle_{n\in J}$ a *diagonal sequence* associated with the given sequence of nested subsequences.

Proof: For each $m \in \mathbb{N}$, let i_m be the m'th smallest member of I_m, that is, the smallest member remaining after removing the smallest $m-1$ members. let $J = \{i_m \mid m \in \mathbb{N}\}$. It is easily seen that $i_{m+1} > i_m$, since $I_{m+1} \subseteq I_m$. Thus the tail $\{n \in J \mid n \geq i_m\}$ of J is contained in I_m. ∎

This proof justifies the use of the adjective *diagonal*: Create an infinite rectangular table in which the mth row contains the set I_m in increasing order. Then J is the set made up of the diagonal in this table.

Examples. Having revised the definitions of two familiar concepts, we now apply the revised definitions to improve the exposition of two proofs using the diagonal argument.

5 Theorem (Bolzano–Weierstrass) *Every bounded sequence of real numbers has a convergent subsequence.*

Proof: Let $\langle x_n \rangle_{n\in I}$ be a bounded sequence, and let $s = \overline{\lim}_{n\to\infty} x_n$. We shall find a subsequence converging to s. Define

$$I_m = \left\{ n \in I \colon x_n > s - \frac{1}{m} \right\}.$$

It is an easy exercise to show that $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$, and each I_m is infinite. Let $\langle x_n \rangle_{n\in J}$ be an associated diagonal subsequence. Recalling that any sort of limit depends only on the tails of a sequence, we find

$$\underline{\lim}_{n\in J} x_n \geq \underline{\lim}_{n\in I_m} x_n \geq s - \frac{1}{m}$$

for all m, and so

$$\underline{\lim}_{n\in J} x_n \geq s.$$

Also

$$\overline{\lim}_{n\in J} x_n \leq \overline{\lim}_{n\in I} x_n = s,$$

and the two inequalities above complete the proof. ∎

Our second example is the proof of a countable version of Tychonov's theorem, that the product of compact spaces (using $[0,1]$ as the canonical example) is compact.

6 Proposition *Let $\langle x_n \rangle_{n\in I}$ be a sequence of functions $x_n \colon \mathbb{N} \to [0,1]$. Then there exists a subsequence $\langle x_n \rangle_{n\in J}$ so that $\langle x_n(m) \rangle_{n\in J}$ converges in $[0,1]$ for all $m \in \mathbb{N}$.*

Proof: By the compactness of $[0,1]$, there exists an index set $I_1 \subseteq I$ so that $x_n(1)$ converges for $n \in I_1$. Next, there is some $I_2 \subseteq I_1$ so that $x_n(2)$ converges for $n \in I_2$, and so forth. In summary, we have a descending family of index sets

$$I \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

so that $x_n(m) \to y(m)$, say, for $n \in I_m$.

Let $\langle x_n \rangle_{n\in J}$ be a diagonal sequence associated with the subsequences $\langle x_n \rangle_{n\in I_m}$.

Now fix some $m \in \mathbb{N}$. Since some tail of $\langle x_n(m) \rangle_{n\in J}$ is a subsequence of $\langle x_n(m) \rangle_{n\in I_m}$, it is convergent by Lemma 3. ∎

Version history (ignoring the correction of minor misprints)

2007-03-05 First version.

2011-08-25 Extensive revision.

2013-09-07 Minor adjustments.

2021-01-27 Simplified version, much closer to the classical formulation.

2024-03-19 Simplified further.

2024-04-11 Explain the terminology.