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**Holden, Helge** (N-NTH-M); **Øksendal, Bernt** (N-OSLO); **Ubøe, Jan**; **Zhang, Tusheng**

★ **Stochastic partial differential equations. (English summary)**

A modeling, white noise functional approach.

Probability and its Applications.

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This book is based on research on stochastic partial differential equations (SPDEs) to describe the flow of fluid in a medium where some of the parameters are noisy. Actually this study started, to a large extent, around 1990, when a research project on fluid flow in stochastic reservoirs was initiated by a group including some of the authors. The book covers, in part, standard topics in white noise analysis as well as a number of more advanced probabilistic frameworks and also contains many deep discussions on stochastic partial differential equations in terms of the authors' original white noise functional approach. The authors have made significant contributions to each of the areas. As a whole, the book is well organized and very carefully written and the details of the proofs are basically spelled out. As the authors mention in the preface, no attempt has been made in this book to give a comprehensive account of the general theory of SPDEs. Rather, the main purpose consists of (1) providing a new machinery of analysis and (2) solving some of the basic SPDEs arising in various applications by applying the tools given in (1). We now wish to highlight some of the novel features of the book: approaches or methods that appear to be new or at least not well known and topics not usually included in books of comparable character in both areas of white noise analysis and SPDE.

First of all we observe a typical example and consider the existence of solutions in accordance with a discussion by J. B. Walsh [in *École d'été de probabilités de Saint-Flour, XIV–1984*, 265–439, Lecture Notes in Math., 1180, Springer, Berlin, 1986; [MR0876085 \(88a:60114\)](#)]. (1)  $\Delta u(x) = -W(x)$ ,  $x \in D$ , (2)  $u(x) = 0$ ,  $x \in \partial D$ , where  $D$  is a bounded domain in  $\mathbf{R}^d$  and  $W(x) = W(x_1, \dots, x_n, \omega)$  is a white noise. The above model for the temperature  $u(x)$  at a point  $x$  in  $D$  is obtained when  $u$  at the boundary  $\partial D$  is kept equal to 0 and there is a random heat source  $W$  in  $D$ . It was shown by Walsh [op. cit.] that for sufficiently large  $n \in \mathbf{N}$  there exist a Sobolev space  $H^{-n}(\mathbf{R}^d)$  and an  $H^{-n}(\mathbf{R}^d)$ -valued stochastic process  $u = u(\omega): \Omega \rightarrow H^{-n}(\mathbf{R}^d)$  such that (1)–(2) hold in the sense of distributions for almost all  $\omega$ . Therefore, this suggests that one can expect to have solutions represented as ordinary multiparameter processes only if the dimension given is sufficiently low. In other words, the Walsh construction, despite the elegant formulation, leads to the difficulty of defining the multiplication of distributions when one considers SPDEs in which the noise term appears multiplicatively.

Now let us consider the following stochastic pressure problem as a simple illustration: (3)  $\operatorname{div}(K(x) \cdot \nabla p(x)) = -f(x)$ ,  $x \in D$ , (4)  $p(x) = 0$ ,  $x \in \partial D$ . Here  $D$  is a given bounded domain in  $\mathbf{R}^d$ , and  $f(x)$  and  $K(x)$  are given functions. With the physical situation of fluid flow in a random medium,  $f(x)$  is the source rate of the fluid, and  $K(x) \geq 0$  is the permeability of the rock at the point  $x$ . Since  $K(x)$  is fluctuating in an irregular, unpredictable manner, e.g., in a typical porous

rock, it is natural to represent  $K(x)$  by a stochastic quantity. The physical consideration requires the permeability  $K(x)$  to be a positive noise, and furthermore the noise appears in a multiplicative way. The above-mentioned observation leads the authors to the recognition that something new that can treat the above at all needs to be established.

Beginning in Chapter 1 with this example, the authors present their original but rather personal view on the development of the SPDE theory via the white noise functional approach. Because the authors have made regular substantial contributions to the subject for more than six years, this point of view is a privileged one. Their methodology is largely analytical, using the Wick product, Hermite transform and techniques from partial differential equations (PDEs). Roughly speaking, a leading idea to overcome this difficulty is simply due to the following facts. (a) The Wick product  $\diamond$  possesses a striking feature in its relation to Skorokhod integration, namely,  $\int Y(t)\delta B(t) = \int Y(t)\diamond W(t)dt$  (Theorem 2.5.9 in §2.5). Here the left-hand side denotes the Skorokhod integral of the stochastic process  $Y(t)$ ,  $B(t)$  is a one-dimensional Brownian motion, and  $W(t)$  is the corresponding white noise. (b) The Hermite transform  $\mathcal{H}$  has a remarkable advantage. That is, if  $F$  and  $G$  belong to the Kondrat'ev space  $(S)_{-1}^N$ , then  $\mathcal{H}(F\diamond G) = \mathcal{H}(F) \cdot \mathcal{H}(G)$  holds as far as the right-hand side exists (Proposition 2.6.6 in §2.6). (c) Combining (a) with (b), the authors observe with an application of the Hermite transform for the stochastic integral that it turns out to be an ordinary integral of a multiplication of  $\mathcal{H}$ -transformed  $Y$  and  $W$  with respect to the Lebesgue measure.

Here is the general strategy. With this motivation, in Chapter 4 the authors apply the techniques developed in Chapter 2 to SPDEs of specific type. They interpret possible products that occur in the question as Wick products, even for the cases where the solution will be a stochastic distribution and one cannot in general take the product of two distributions. Subsequently, they take the Hermite transform of the resulting equation and obtain an equation that they try to solve, where the random variables have been replaced by complex-valued functions of infinitely many complex variables. Finally, they use the inverse Hermite transform to obtain a solution of the regularized, original equation. On this account, one may make use of these brilliant techniques to derive the new explicit representations of solutions to SPDEs if the solution formula for  $\mathcal{H}$ -transformed equations is available. In most cases one obtains the final solution in closed form, expressed as an expectation of an auxiliary Brownian motion. Also explained are other methods for solving equations where the solution cannot be obtained in closed form (Benth, 1996 and Våge, 1995).

The topic of Chapter 2 is the general framework, in particular, oriented to the white noise functional approach to SPDEs. This chapter develops the apparatus to derive the stochastic representation formula of solutions to Wick-type SPDEs, which is precisely discussed later in the succeeding chapter. The ingredients are white noise, Wiener-Itô chaos expansions, Kondrat'ev spaces of stochastic test functions and stochastic distributions, the Wick product, the Hermite transform, and functional processes. Chapter 2 includes a nice heuristic explanation and presentations of singular [resp., smoothed] white noise processes  $W(x)$  [resp.,  $W_\varphi(x)$ ], which play important roles as “elementary particles” of a noisy source. Section 2.6 includes a proof of a characterization theorem for Kondrat'ev spaces  $(S)_{-1}^N$ , and Section 2.8 is devoted to a discussion on the topology of Kondrat'ev spaces, which gives a useful description of the topology in terms of the

transforms. This argument is essential and crucial, too, because in the standard theory solutions to SDEs or SPDEs are regarded as stochastic distribution processes (in §2.8) lying in the Kondrat'ev distribution space, and also because the authors often work with the Hermite transforms  $\mathcal{H}$  of elements of  $(S)_{-1}^N$ . Section 2.7 is a short excursion, where the relation between the  $S$ -transform in the Hida calculus and the Hermite transform is stated. Section 2.10 treats the translation operator  $T_\omega$  relative to the Wick product and includes Gjessing's lemma (Theorem 2.10.7), which is used effectively to solve quasilinear SPDEs in §4.8. The chapter concludes with the positivity in the case of distributions and also in the case of functional processes. Because "permeability" is always a nonnegative quantity, the positive white noise process naturally appears in the stochastic pressure equation (3)–(4).

The framework developed in Chapter 2 can also be employed to obtain new results as well as new proofs of old results for SDEs. Chapter 3 illustrates this by discussing some important examples. Results on stochastic ordinary differential equations in the chapter allude to the validity and effectiveness of a general method (functional process approach) for SPDEs involving a smoothed white noise source that is more completely described in Chapter 4. In Section 3.2 the authors consider a stochastic version of the Malthus model for population growth in terms of general  $(S)_{-1}$  solutions. Section 3.3 includes existence and uniqueness theorems for general linear SDEs (Våge, 1995) and general linear multi-dimensional Wick SDEs. The authors discuss the classical and Wick stochastic Volterra equations and give several interesting examples, including the model of oscillations in a stochastic medium. Section 3.5 is another short excursion on Wick products versus ordinary products, where an experiment comparing the two products is described. The chapter concludes with Wick approximation of quasilinear SDEs, where Gjessing's result (1994) (cf. Theorem 3.6.1) for quasilinear, anticipating (= Skorokhod type) stochastic differential equations is introduced.

Chapter 4 is the main part of the book. The chief motivation for setting up the machinery (in Chapter 2) is to enable the authors to solve some of the basic SPDEs that appear frequently in applications. The chapter outlines such important examples. The tools developed in Chapter 2 are used to prove the general theorems, which are then applied to study various kinds of SPDEs, including the principal theme, namely, the stochastic pressure equation (in §4.6). Section 4.1 provides the basic general theorem (Theorem 4.1.1) which gives sufficient conditions for the above-mentioned "strategy" to work. It says: "Suppose  $u(t, x, z)$  is a solution of the equation (5)  $\mathcal{H}A(t, x, \partial_t, \nabla_x, u, z) = 0$  for  $(t, x)$  in some bounded open set  $G \subset \mathbf{R} \times \mathbf{R}^d$ , and for all  $z \in \mathbf{K}_q(R) \subset \mathbf{C}^N$  for some  $q, R$ . Moreover, suppose that  $u$  and all its partial derivatives involved in (5) are uniformly bounded for  $(t, x, z) \in G \times \mathbf{K}_q(R)$ , continuous in  $(t, x) \in G$  for each  $z$ , and analytic with respect to  $z \in \mathbf{K}_q(R)$  for all  $(t, x) \in G$ . Then there exists  $U \in (S)_{-1}$  such that  $u(t, x, z) = (\mathcal{H}U(t, x))(z)$  for all  $t, x, z$  and  $U$  solves the Wick-type equation (6)  $A^\diamond(t, x, \partial_t, \nabla_x, U, \omega) = 0$  in  $(S)_{-1}$ . The proof of Theorem 4.1.1 in this general context uses results on the convergence of stochastic distribution processes (Lemma 2.8.4, §2.8). Section 4.2 treats a stochastic Poisson equation similar to (1)–(2). The problem with smoothed white noise  $W_\psi(x)$  is also considered in line with the functional process approach. The reason for doing this could be simply technical. Indeed, by smoothing the white noise we get less-singular equations to work with and therefore, we expect, less-singular solutions. But the reason could also come from the model. In fact, in some

cases the smoothed process  $W_\psi(x)$  simply gives a more realistic model for the noise considered. In Section 4.3 the stochastic transport equation is considered and a probabilistic representation of the solution is also derived. This section includes the heat equation with a stochastic potential as its simple example. In Section 4.4 the stochastic Schrödinger equation is considered and  $L^1(\mu)$ -properties of the solution are discussed in connection with the smoothing procedure of white noise with a test function  $\varphi \in \mathcal{S}(\mathbf{R})$ . Section 4.5 is devoted to an introduction of the authors' recent results on the study of the viscous Burgers equation with a stochastic source. The section includes a Wick version of the Cole-Hopf solution method and the relation between the stochastic Burgers equation and the stochastic heat equation. The result in the previous section is used to express the solution of the stochastic Burgers equation. See a paper by H. Holden et al. [in *Stochastic partial differential equations (Edinburgh, 1994)*, 141–161, Cambridge Univ. Press, Cambridge, 1995; [MR1352740 \(97c:60159\)](#)]. In Section 4.6 the authors return to the stochastic pressure problem (3)–(4) that they discussed in the introduction (Chapter 1). First, the equation with smoothed positive noise case is considered, that is, the permeability is given by the Wick exponential:  $K_\varphi(x) := \exp^\diamond W_\varphi(x)$ . Theorem 4.6.1 asserts that, under suitable analytical conditions on the fluid source rate  $f$ , the smoothed stochastic pressure equation with  $K_\varphi$  has a unique  $(S)_{-1}$ -valued solution  $p \equiv p_\varphi \in C^2(D) \cap C(\overline{D})$  given by

$$(7) \quad p_\varphi(x) = \frac{1}{2} \exp^\diamond \left[ -\frac{1}{2} W_\varphi(x) \right] \diamond \hat{\mathbf{E}}^x \left[ \int_0^{\tau_D} f(b_t) \diamond \exp^\diamond \left\{ -\frac{1}{2} W_\varphi(b_t) - \frac{1}{4} \int_0^t \left[ \frac{1}{2} (\nabla W_\varphi(y))^{\diamond 2} + \Delta W_\varphi(y) \right]_{y=b_s} ds \right\} dt \right],$$

where  $(b_t, P_x)$  is a standard Brownian motion in  $\mathbf{R}^d$  (independent of  $B_x$ ),  $\hat{\mathbf{E}}^x$  denotes expectation with respect to  $P_x$ , and  $\tau_D$  is the first exit time of  $b_t$  from  $D$ . Next the singular case is considered. The authors emphasize the analogy with derivation of the probabilistic representation of solutions for the case of the singular positive white noise. A full description can be found in a paper by H. Holden et al. [Potential Anal. **4** (1995), no. 6, 655–674; [MR1361382 \(97d:60105\)](#)]. Section 4.7 includes an updated version of the above Theorem 4.1.1. The point is as follows. When discussing the SPDE of the more general form where the explicit solution formula is not necessarily available, the emphasis moves naturally to a study of the abstract existence and uniqueness of solutions (to SPDEs). In that case it must be based upon a new result of analytic extension of the functions given. If we look at the usual case for a while, when taking the Hermite transform the authors got an equation in  $U(x, z)$  that could be solved for real values  $\lambda_k$  of the parameters  $z_k$ . Then, from the solution formula for  $u(x, \lambda)$ , it was apparent that it had an analytic extension to  $u(x, z)$  for complex  $z$ . Finally, to prove that  $u(x, z)$  is the Hermite transform of an element in  $(S)_{-1}$ , they proved boundedness for  $z$  in some  $\mathbf{K}_q(\delta)$ . However, in other equations with more general form in question the extensions from the real case  $z_k = \lambda_k \in \mathbf{R}$  to the complex-analytic case  $z_k \in \mathbf{C}$  may not be as obvious, and it is natural to ask if it is sufficient with good enough estimates in the real case alone to obtain the same conclusion. This line of investigation has led the authors to an updated version (Theorem 4.7.3). As an example the section treats the heat equation in a stochastic anisotropic medium as well, and Gjerde's theorem (1995) (Theorem 4.7.4) is introduced with its companion Gjerde's criterion, whereby one can obtain existence and uniqueness more generally

without explicit solution formulas. The chapter also includes the case of Poissonian noise and places the concepts of Kondrat'ev spaces of Poissonian distributions, Poissonian Wick product and Poissonian Hermite transform in the context of the Wick-type SPDE. The book closes with a celebrated theme, "SPDEs driven by Poissonian noise" (§4.9). There is a close mathematical connection between SPDEs driven by Gaussian and Poissonian noises, at least for Wick-type equations. More precisely, the unitary operator  $\mathcal{U}$  enables the authors to transform any Wick-type SPDE with Poissonian white noise into a Wick-type SPDE with Gaussian white noise and vice versa (cf. Theorem 4.9.15 (Benth and Gjerde, 1995)). Thus, one can obtain the solution of the Poissonian SPDE simply by applying this map to that of the corresponding Gaussian SPDE. Because the purpose has shifted from instruction to overview, Chapter 4 is less self-contained. So, it seems necessary for readers to consult occasionally the original papers cited therein.

There are a large number of substantial exercises after each chapter; many are extensions of the theory or even fundamental results. Appendices include not only a brief review of Itô calculus, some fundamental properties of Hermite polynomials and Wick products, but also a proof of the Bochner-Minlos theorem that provides Gaussian and Poissonian white noise probability measures indispensable to the basic framework of the theory. Overall, the book provides a good introduction to the subject of the white noise functional approach to SPDEs. The prospective reader must possess solid knowledge of elementary stochastic calculus, and may sometimes need to consult other sources for certain topics in white noise analysis, stochastic analysis, and PDEs. The book may be used as a text for an analytically oriented graduate course in probability theory (especially stochastic equation theory), or as a supplement to other more purely probabilistic expositions. This is a rich and demanding book with a definite point of view. It will be of great value for students of probability theory or SPDEs with an interest in the subject, and also for professional probabilists. My personal choice for the book would be to have it serve as the basis for a reading course for a graduate student who would like to specialize in white noise analysis and its applications to SPDEs.

For more on the subject (i.e., the white noise functional approach to SPDEs), consult the monographs and papers of the Norwegian school which are listed in the references. The paper by H. Holden et al. [*Probab. Theory Related Fields* **95** (1993), no. 3, 391–419; [MR1213198 \(94c:60107\)](#)] is a good place to start. For the reader who wants to acquire more details about the background, e.g., white noise analysis, the book by T. Hida et al. [*White noise*, Kluwer Acad. Publ., Dordrecht, 1993; [MR1244577 \(95f:60046\)](#)] is recommended.

It is also true that the basic idea behind functional processes is inspired by Colombeau's theory of distributions, which makes it possible to define the product of certain distributions [see J.-F. Colombeau, *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), no. 2, 251–268; [MR1028141 \(91c:46053\)](#); see also F. Russo, in *Stochastic analysis and applications in physics (Funchal, 1993)*, 329–349, Kluwer Acad. Publ., Dordrecht, 1994; [MR1337971 \(96k:60148\)](#); M. B. Oberguggenberger, in *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1993)*, 215–229, Birkhäuser, Basel, 1995; [MR1360278 \(97k:60111\)](#)].

As far as modeling is concerned, the functional process  $u(\varphi, x): \mathcal{S}(\mathbf{R}^d) \times \mathbf{R}^d \rightarrow (S)_{-1}$  (defined in §2.9) is of independent interest, not simply as an admissible approximation from a technical point of view. Indeed, mathematically it is quite interesting to ask what happens if we take the

limit  $\varphi \rightarrow \delta_0$  where  $\delta_0$  is the Dirac measure at 0. This problem is highly interesting and stimulating as well, not only from the point of view of approximation methods but also in connection with numerical methods. Unfortunately, the book does not deal with this question in depth, but only gives some examples in Section 3.6.

Reviewed by *Isamu Dôku*

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