Introduction

> ... though I bestowed some time in writing the book, yet
> it cost me not half so much labor as this very preface.
>
> *Miguel de Cervantes Saavedra*[1]

Background: In the early 1950s, Fermi, Pasta, and Ulam[2] (FPU), in the unpublished report by Fermi et al. (1955), analyzed numerically, in one of the first computer simulations performed, the behavior of oscillations in certain nonlinear lattices. Expecting equipartition of energy among the various modes, they were highly surprised to discover that the energy did not equidistribute, but rather they observed that the system seemed to return periodically to its initial state. Motivated by the surprising findings in FPU, several researchers, including Ford (1961), Ford and Waters (1964), Waters and Ford (1964), Atlee Jackson et al. (1968), Payton and Visscher (1967a,b; 1968), Payton et al. (1967), studied lattice models with different nonlinear interactions, observing close to periodic and solitary behavior. It was Toda who in 1967 isolated the exponential interaction, see Toda (1967a,b) and hence introduced a model that supported an exact periodic and soliton solution. The model, now called the Toda lattice, is a nonlinear differential-difference system continuous in time and discrete in space,

$$x_{tt}(n,t) = e^{(x(n-1,t)-x(n,t))} - e^{(x(n,t)-x(n+1,t))}, \quad (n,t) \in \mathbb{Z} \times \mathbb{R}, \tag{0.1}$$

[1] *Don Quixote*, (1605) preface.

[2] "We [Fermi and Ulam]... decided to attempt to formulate a problem simple to state but such that a solution ... could not be done with pencil and paper. ... Our problem turned out to be felicitously chosen. The results were entirely different from what even Fermi, with his great knowledge of wave motions, had expected. ... Fermi considered this to be, as he said, "a minor discovery." ... He intended to talk about this [at the Gibbs lecture; a lecture never given as Fermi became ill before the meeting]... ", see Ulam (1991, p. 226f).

where $x(n,t)$ denotes the displacement of the nth particle from its equilibrium position at time t. While nonlinear lattices are interesting objects of study and certainly of fundamental importance in their own right, it should be mentioned that already in the paper Toda (1967b), it is also shown that the Korteweg–de Vries (KdV) equation emerges in a certain scaling, or continuum limit, from the Toda lattice, creating a link with the theory of the KdV equation. Indeed, the theory for the Toda lattice is closely intertwined with the corresponding theory for the KdV equation on several levels. Most notably, the Toda lattice shares many of the properties of the KdV equation and other completely integrable equations. This applies, in particular, to the Hamiltonian and algebro-geometric formalism treated in detail in the present monograph. While the developments for the KdV equation preceded those for the Toda lattice, in the context of algebro-geometric solutions the actual developments for the latter rapidly followed the former as described below.

Before turning to a description of the main contributors and their accomplishments in connection with the Hamiltonian and algebro-geometric formalism for the Toda lattice, we briefly recall a few milestones in the development leading up to soliton and algebro-geometric solutions of the KdV equation (for an in-depth presentation of that theory, we refer to the introduction of Volume I). In 1965, Kruskal and Zabusky (cf. Zabusky and Kruskal (1965)), while analyzing the numerical results of FPU on heat conductivity in solids, discovered that pulselike solitary wave solutions of the KdV equation, for which the name "solitons" was coined, interacted elastically. This was followed by the 1967 discovery of Gardner, Greene, Kruskal, and Miura (cf. Gardner et al. (1967; 1974)) that the inverse scattering method allowed one to solve initial value problems for the KdV equation with sufficiently fast decaying initial data. Soon after, Lax (1968) found the explanation of the isospectral nature of KdV solutions using the concept of Lax pairs and introduced a whole hierarchy of KdV equations. Subsequently, in the early 1970s, Zakharov and Shabat (1972; 1973; 1974), and Ablowitz et al. (1973a,b; 1974) extended the inverse scattering method to a wide class of nonlinear partial differential equations of relevance in various scientific contexts, ranging from nonlinear optics to condensed matter physics and elementary particle physics. In particular, soliton solutions found numerous applications in classical and quantum field theory, in connection with optical communication devices, etc.

Another decisive step forward in the development of completely integrable soliton equations was taken around 1974. Prior to that period, inverse spectral methods in the context of nonlinear evolution equations had been restricted to spatially decaying solutions to enable the applicability of inverse scattering techniques. From 1975 on, following some pioneering work of Novikov (1974), the arsenal of inverse spectral methods was extended considerably in scope to include periodic and certain classes of quasi-periodic and almost periodic KdV finite-band solutions. This new approach to constructing solutions of integrable nonlinear evolution equations, based on solutions of the inverse periodic spectral problem and on algebro-geometric methods and theta function representations, developed by pioneers such as Dubrovin, Its, Kac, Krichever, Marchenko, Matveev, McKean, Novikov, and van Moerbeke, to name just a few, was followed by very rapid development in the field and within a few years of intense activity worldwide, the landscape of integrable systems was changed forever. By the early 1980s the theory was extended to a large class of nonlinear (including certain multi-dimensional) evolution equations beyond the KdV equation, and the explicit theta function representation of quasi-periodic solutions of integrable equations (including soliton solutions as special limiting cases) had introduced new algebro-geometric techniques into this area of nonlinear partial differential equations. Subsequently, this led to an interesting cross-fertilization between the areas of integrable nonlinear partial differential equations and algebraic geometry, culminating, for instance, in a solution of Schottky's problem (Shiota (1986; 1990), see also Krichever (2006) and the references cited therein).

The present monograph is devoted to hierarchies of completely integrable differential-difference equations and their algebro-geometric solutions, treating, in particular, the Toda, Kac–van Moerbeke, and Ablowitz–Ladik hierarchies. For brevity we just recall the early historical development in connection with the Toda lattice and refer to the Notes for more recent literature on this topic and for the corresponding history of the Kac–van Moerbeke and Ablowitz–Ladik hierarchies. After Toda's introduction of the exponential lattice in 1967, it was Flaschka who in 1974 proved its integrability by establishing a Lax pair for it with Lax operator a tri-diagonal Jacobi operator on $\mathbb{Z}$ (a discrete Sturm–Liouville-type operator, cf. Flaschka (1974a)). He used the variable transformation

$$\begin{aligned} a(n,t) &= \frac{1}{2}\exp\big(\tfrac{1}{2}(x(n,t)-x(n+1,t))\big), \\ b(n,t) &= -\frac{1}{2}x_t(n,t), \quad (n,t)\in\mathbb{Z}\times\mathbb{R}, \end{aligned} \tag{0.2}$$

which transforms (0.1) into a first-order system for a, b, the Toda lattice system, displayed in (0.3). Just within a few months, this was independently observed also by Manakov (1975). The corresponding integrability in the finite-dimensional periodic case had first been established by Hénon (1974) and shortly thereafter by Flaschka (1974b) (see also Flaschka (1975), Flaschka and McLaughlin (1976a), Kac and van Moerbeke (1975a), van Moerbeke (1976)).

Soon after, integrability of the finite nonperiodic Toda lattice was established by Moser (1975a). Returning to the Toda lattice (0.2) on $\mathbb{Z}$, infinitely-many constants of motion (conservation laws) were derived by Flaschka (1974a) and Manakov (1975) (see also McLaughlin (1975)), moreover, the Hamiltonian formalism, Poisson brackets, etc., were also established by Manakov (1975) (see also Flaschka and McLaughlin (1976b)). The theta function representation of b in the periodic case was nearly simultaneously derived by Dubrovin et al. (1976) and Date and Tanaka (1976a,b), following Its and Matveev (1975a,b) in their theta function derivation of the corresponding periodic finite-band KdV solution. An explicit theta function representation for a was derived a bit later by Krichever (1978) (see also Kričever (1982), Krichever (1982; 1983), and the appendix written by Krichever in Dubrovin (1981)). We also note that Dubrovin, Matveev, and Novikov as well as Date and Tanaka consider the special periodic case, but Krichever treats both the periodic and quasi-periodic cases.

Scope: We aim for an elementary, yet self-contained, and precise presentation of hierarchies of integrable soliton differential-difference equations and their algebro-geometric solutions. Our point of view is predominantly influenced by analytical methods. We hope this will make the presentation accessible and attractive to analysts working outside the traditional areas associated with soliton equations. Central to our approach is a simultaneous construction of all algebro-geometric solutions and their theta function representation of a given hierarchy. In this volume we focus on some of the key hierarchies in $(1+1)$-dimensions associated with differential-difference integrable models such as the Toda lattice hierarchy (Tl), the Kac–van Moerbeke hierarchy (KM), and the Ablowitz–Ladik hierarchy (AL). The key equations, defining the corresponding hierarchies, read[1]

$$
\begin{aligned}
&\text{Tl:} && \begin{pmatrix} a_t - a(b^+ - b) \\ b_t - 2\big(a^2 - (a^-)^2\big) \end{pmatrix} = 0, \\
&\text{KM:} && \rho_t - \rho\big((\rho^+)^2 - (\rho^-)^2\big) = 0, \\
&\text{AL:} && \begin{pmatrix} -i\alpha_t - (1-\alpha\beta)(\alpha^+ + \alpha^-) + 2\alpha \\ -i\beta_t + (1-\alpha\beta)(\beta^+ + \beta^-) - 2\beta \end{pmatrix} = 0.
\end{aligned} \tag{0.3}
$$

Our principal goal in this monograph is the construction of algebro-geometric solutions of the hierarchies associated with the equations listed in (0.3). Interest in the class of algebro-geometric solutions can be motivated in a variety of ways: It represents a natural extension of the classes of soliton solutions

[1] Here, and in the following, $\phi^{\pm}$ denotes the shift of a lattice function ϕ, that is, $\phi^{\pm}(n) = \phi(n \pm 1)$, $n \in \mathbb{Z}$.

and similar to these, its elements can still be regarded as explicit solutions of the nonlinear integrable evolution equation in question (even though their complexity considerably increases compared to soliton solutions due to the underlying analysis on compact Riemann surfaces). Moreover, algebro-geometric solutions can be used to approximate more general solutions (such as almost periodic ones) although this is not a topic pursued in this monograph. Here we primarily focus on the construction of explicit solutions in terms of certain algebro-geometric data on a compact hyperelliptic Riemann surface and their representation in terms of theta functions. Solitons arise as the special case of solutions corresponding to an underlying singular hyperelliptic curve obtained by confluence of pairs of branch points. The theta function associated with the underlying singular curve then degenerates into appropriate determinants with exponential entries.

We use basic techniques from the theory of differential-difference equations, some spectral analysis, and elements of algebraic geometry (most notably, the basic theory of compact Riemann surfaces). In particular, we do not employ more advanced tools such as loop groups, Grassmanians, Lie algebraic considerations, formal pseudo-differential expressions, etc. Thus, this volume strays off the mainstream, but we hope it appeals to spectral theorists and their kin and convinces them of the beauty of the subject. In particular, we hope a reader interested in quickly reaching the fundamentals of the algebro-geometric approach of constructing solutions of hierarchies of completely integrable evolution equations will not be disappointed.

Completely integrable systems, and especially nonlinear evolution equations of soliton-type, are an integral part of modern mathematical and theoretical physics, with far-reaching implications from pure mathematics to the applied sciences. It is our intention to contribute to the dissemination of some of the beautiful techniques applied in this area.

Contents: In the present volume we provide an effective approach to the construction of algebro-geometric solutions of certain completely integrable nonlinear differential-difference evolution equations by developing a technique which simultaneously applies to all equations of the hierarchy in question.

Starting with a specific integrable differential-difference equation, one can build an infinite sequence of higher-order differential-difference equations, the so-called hierarchy of the original soliton equation, by developing an explicit recursive formalism that reduces the construction of the entire hierarchy to elementary manipulations with polynomials and defines the associated Lax pairs or zero-curvature equations. Using this recursive polynomial formalism, we simultaneously construct algebro-geometric solutions for the entire hierarchy of soliton equations at hand. On a more technical level, our point of departure for the construction of algebro-geometric solutions is not directly based on Baker–Akhiezer functions and axiomatizations of algebro-geometric data, but rather on a canonical meromorphic function ϕ on the underlying hyperelliptic Riemann surface $\mathcal{K}_p$ of genus $p \in \mathbb{N}_0$. More precisely, this fundamental meromorphic function ϕ carries the spectral information of the underlying Lax operator (such as the Jacobi operator in context of the Toda lattice) and in many instances represents a direct generalization of the Weyl–Titchmarsh m-function, a fundamental device in the spectral theory of difference operators. Riccati-type difference equations satisfied by ϕ separately in the discrete space and continuous time variables then govern the time evolutions of all quantities of interest (such as that of the associated Baker–Akhiezer function). The basic meromorphic function ϕ on $\mathcal{K}_p$ is then linked with solutions of equations of the underlying hierarchy via trace formulas and Dubrovin-type equations for (projections of) the pole divisor of ϕ. Subsequently, the Riemann theta function representation of ϕ is then obtained more or less simultaneously with those of the Baker–Akhiezer function and the algebro-geometric solutions of the (stationary or time-dependent) equations of the hierarchy of evolution equations. This concisely summarizes our approach to all the (1+1)-dimensional discrete integrable models discussed in this volume.

In the following we will detail this verbal description of our approach to algebro-geometric solutions of integrable hierarchies with the help of the Toda hierarchy.

The Toda lattice, in Flaschka's variables, reads

$$\begin{aligned} a_t - a\big(b^+ - b\big) &= 0, \\ b_t - 2\big((a^+)^2 - (a^-)^2\big) &= 0, \end{aligned} \tag{0.4}$$

where $a = \{a(n,t)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, $b = \{b(n,t)\}_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$, $t \in \mathbb{R}$. The system (0.4) is equivalent to the Lax equation

$$L_t(t) - [P_2(t), L(t)] = 0.$$

Here L and P_2 are the difference expressions of the form

$$L = aS^+ + a^-S^- + b, \quad P_2 = aS^+ - a^-S^-,$$

and $S^\pm$ denote the shift operators

$$(S^\pm f)(n) = f(n \pm 1), \quad n \in \mathbb{Z}, \ f = \{f(m)\}_{m\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}},$$

with $\mathbb{C}^{\mathbb{Z}}$ abbreviating the set of complex-valued sequences indexed by $\mathbb{Z}$.

In this introduction we will indicate how to construct all real-valued algebro-geometric quasi-periodic finite-band solutions of a hierarchy of nonlinear evolution equations of which the first equation is the Toda lattice, abbreviated Tl. The approach is similar to the one advocated for the Korteweg–de Vries (KdV) and Zakharov–Shabat (ZS), or equivalently, Ablowitz–Kaup–Newell–Segur (AKNS), equations and their hierarchies in Chapters 1 and 3 of Volume I.

This means that we construct a hierarchy of difference operators P_{2p+2} such that the Lax relation

$$L_{t_p} - [P_{2p+2}, L] = 0,$$

defines a hierarchy of differential-difference equations where the time variation is continuous and space is considered discrete. We let each equation in this hierarchy run according to its own time variable t_p. The operators P_{2p+2} are defined recursively. In the stationary case, where we study

$$[P_{2p+2}, L] = 0,$$

there is a hyperelliptic curve $\mathcal{K}_p$ of genus p which is associated with the equation in a natural way. This relation is established by introducing the analog of Burchnall–Chaundy polynomials, familiar from the KdV and ZS-AKNS theory. The basic relations for both the time-dependent and stationary Toda hierarchy as well as the construction of the Burchnall–Chaundy polynomials are contained in Section 1.2.

In Section 1.3 we discuss the stationary case in detail. We introduce the Baker–Akhiezer function ψ which is the common eigenfunction of the commuting difference operators L and P_{2p+2}. The main result of this section is the proof of theta function representations of $\phi = \psi^+/\psi$ and ψ, as well as the solutions a and b of the stationary Toda hierarchy.

In Section 1.4 we analyze the algebro-geometric initial value problem for the Toda hierarchy. By that we mean the following: Given a nonspecial Dirichlet divisor of degree p at one fixed lattice point, we explicitly construct an algebro-geometric solution, which equals the given data at the lattice point, of the qth stationary Toda lattice, $q \in \mathbb{N}$.

Section 1.5 parallels that of Section 1.3, but it discusses the time-dependent case. The goal of the section is to construct the solution of the rth equation in the Toda hierarchy with a given stationary solution of the pth equation in the Toda hierarchy as initial data. We construct the solution in terms of theta functions.

Section 1.6 treats the algebro-geometric time-dependent initial value problem for the Toda hierarchy. Given a stationary solution of an arbitrary equation in the Toda hierarchy and its associated nonsingular hyperelliptic curve as initial data, we construct explicitly the solution of any other time-dependent equation in the Toda hierarchy with the given stationary solution as initial data.

Finally, in Section 1.7 we construct an infinite sequence of local conservation laws for each of the equations in the Toda hierarchy. Moreover, we derive two Hamiltonian structures for the Toda hierarchy.

We now return to a more detailed survey of the results in this monograph for the Toda hierarchy. The Toda hierarchy is the simplest of the hierarchies of nonlinear differential-difference evolution equations studied in this volume, but the same strategy, with modifications to be discussed in the individual chapters, applies to the integrable systems treated in this monograph and is in fact typical for all $(1+1)$-dimensional integrable differential-difference hierarchies of soliton equations.

A discussion of the Toda case then proceeds as follows.[1] In order to define the Lax pairs and zero-curvature pairs for the Toda hierarchy, one assumes a, b to be bounded sequences in the stationary context and smooth functions in the time variable in the time-dependent case. Next, one introduces the recursion relation for some polynomial functions f_ℓ, g_ℓ of a, b and certain of its shifts by

$$\begin{aligned} &f_0 = 1, \quad g_0 = -c_1, \\ &2f_{\ell+1} + g_\ell + g_\ell^- - 2bf_\ell = 0, \quad \ell \in \mathbb{N}_0, \\ &g_{\ell+1} - g_{\ell+1}^- + 2\big(a^2 f_\ell^+ - (a^-)^2 f_\ell^-\big) - b\big(g_\ell - g_\ell^-\big) = 0, \quad \ell \in \mathbb{N}_0. \end{aligned} \tag{0.5}$$

Here c_1 is a given constant. From the recursively defined sequences $\{f_\ell, g_\ell\}_{\ell\in\mathbb{N}_0}$ (whose elements turn out to be difference polynomials with respect to a, b, defined up to certain summation constants) one defines the *Lax pair* of the Toda hierarchy by

$$L = aS^+ + a^-S^- + b, \tag{0.6}$$

$$P_{2p+2} = -L^{p+1} + \sum_{\ell=0}^{p} (g_{p-\ell} + 2af_{p-\ell}S^+)L^\ell + f_{p+1}. \tag{0.7}$$

The commutator of P_{2p+2} and L then reads[2]

$$[P_{2p+2}, L] = -a\big(g_p^+ + g_p + f_{p+1}^+ + f_{p+1} - 2b^+ f_p^+\big)S^+$$

[1] All details of the following construction are to be found in Chapter 1.

[2] The quantities P_{2p+2} and $\{f_\ell, g_\ell\}_{\ell=0,\dots,p}$ are constructed such that all higher-order difference operators in the commutator (0.8) vanish. Observe that the factors multiplying $S^\pm$ are just shifts of one another.

$$+ 2\big(- b(g_p + f_{p+1}) + a^2 f_p^+ - (a^-)^2 f_p^- + b^2 f_p\big) \tag{0.8}$$
$$- a^- \big(g_p + g_p^- + f_{p+1} + f_{p+1}^- - 2bf_p\big)S^-,$$

using the recursion (0.5). Introducing a deformation (time) parameter[1] $t_p \in \mathbb{R}$, $p \in \mathbb{N}_0$, into a, b, the *Toda hierarchy* of nonlinear evolution equations is then defined by imposing the *Lax commutator relation*

$$\frac{d}{dt_p} L - [P_{2p+2}, L] = 0, \tag{0.9}$$

for each $p \in \mathbb{N}_0$. By (0.8), the latter are equivalent to the collection of evolution equations[2]

$$\mathrm{Tl}_p(a,b) = \begin{pmatrix} a_{t_p} - a(f_{p+1}^+(a,b) - f_{p+1}(a,b)) \\ b_{t_p} + g_{p+1}(a,b) - g_{p+1}^-(a,b) \end{pmatrix} = 0, \quad p \in \mathbb{N}_0. \tag{0.10}$$

Explicitly,

$$\begin{aligned}
\mathrm{Tl}_0(a,b) &= \begin{pmatrix} a_{t_0} - a(b^+ - b) \\ b_{t_0} - 2(a^2 - (a^-)^2) \end{pmatrix} = 0,\\
\mathrm{Tl}_1(a,b) &= \begin{pmatrix} a_{t_1} - a((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2) \\ b_{t_1} + 2(a^-)^2(b + b^-) - 2a^2(b^+ + b) \end{pmatrix}\\
&\quad + c_1 \begin{pmatrix} -a(b^+ - b) \\ -2(a^2 - (a^-)^2) \end{pmatrix} = 0,\\
\mathrm{Tl}_2(a,b) &= \begin{pmatrix} a_{t_2} - a((b^+)^3 - b^3 + 2(a^+)^2 b^+ - 2(a^-)^2 b \\ +a^2(b^+ - b) + (a^+)^2 b^{++} + (a^-)^2 b^-) \\ b_{t_2} - 2a^2(b^2 + bb^+ + (b^+)^2 + a^2 + (a^+)^2) \\ +2(a^-)^2(b^2 + bb^- + (b^-)^2 + (a^-)^2 + (a^{--})^2) \end{pmatrix}\\
&\quad + c_1 \begin{pmatrix} -a((a^+)^2 - (a^-)^2 + (b^+)^2 - b^2) \\ 2(a^-)^2(b + b^-) - 2a^2(b^+ + b) \end{pmatrix}\\
&\quad + c_2 \begin{pmatrix} -a(b^+ - b) \\ -2(a^2 - (a^-)^2) \end{pmatrix} = 0, \text{ etc.},
\end{aligned}$$

represent the first few equations of the time-dependent Toda hierarchy. For $p = 0$ one obtains *the* Toda lattice (0.4). Introducing the polynomials ($z \in \mathbb{C}$),

$$F_p(z) = \sum_{\ell=0}^{p} f_{p-\ell} z^\ell, \tag{0.11}$$

[1] Here we follow Hirota's notation and introduce a separate time variable t_p for the pth level in the Toda hierarchy.

[2] In a slight abuse of notation we will occasionally stress the functional dependence of f_ℓ, g_ℓ on a, b, writing $f_\ell(a,b), g_\ell(a,b)$.

$$G_{p+1}(z) = -z^{p+1} + \sum_{\ell=0}^{p} g_{p-\ell} z^\ell + f_{p+1}, \tag{0.12}$$

one can alternatively introduce the Toda hierarchy as follows. One defines a pair of 2×2 matrices $(U(z), V_{p+1}(z))$ depending polynomially on z by

$$U(z) = \begin{pmatrix} 0 & 1 \\ -a^-/a & (z-b)/a \end{pmatrix}, \tag{0.13}$$

$$V_{p+1}(z) = \begin{pmatrix} G^-_{p+1}(z) & 2a^- F^-_p(z) \\ -2a^- F_p(z) & 2(z-b)F_p + G_{p+1}(z) \end{pmatrix}, \quad p \in \mathbb{N}_0, \tag{0.14}$$

and then postulates the discrete *zero-curvature equation*

$$0 = U_{t_p} + UV_{p+1} - V^+_{p+1}U. \tag{0.15}$$

One verifies that both the Lax approach (0.10), as well as the zero-curvature approach (0.15), reduce to the basic equations,

$$\begin{aligned} a_{t_p} &= -a\big(2(z-b^+)F^+_p + G^+_{p+1} + G_{p+1}\big), \\ b_{t_p} &= 2\big((z-b)^2 F_p + (z-b)G_{p+1} + a^2 F^+_p - (a^-)^2 F^-_p\big). \end{aligned} \tag{0.16}$$

Each one of (0.10), (0.15), and (0.16) defines the Toda hierarchy by varying $p \in \mathbb{N}_0$.

The strategy we will be using is then the following: First we assume the existence of a solution a, b, and derive several of its properties. In particular, we deduce explicit Riemann's theta function formulas for the solution a, b, the so-called Its–Matveev formulas (cf. (0.41) in the stationary case and (0.53) in the time-dependent case). As a second step we will provide an explicit algorithm to construct the solution given appropriate initial data.

The Lax and zero-curvature equations (0.9) and (0.15) imply a most remarkable isospectral deformation of L as will be discussed later in this introduction. At this point, however, we interrupt our time-dependent Toda considerations for a while and take a closer look at the special stationary Toda equations defined by

$$a_{t_p} = b_{t_p} = 0, \quad p \in \mathbb{N}_0. \tag{0.17}$$

By (0.8)–(0.10) and (0.15), (0.16), the condition (0.17) is then equivalent to each one of the following collection of equations, with p ranging in $\mathbb{N}_0$, defining the *stationary Toda hierarchy* in several ways,

$$[P_{2p+2}, L] = 0, \tag{0.18}$$

$$f^+_{p+1} - f_{p+1} = 0, \quad g_{p+1} - g^-_{p+1} = 0, \tag{0.19}$$

$$UV_{p+1} - V^+_{p+1}U = 0, \tag{0.20}$$

$$\begin{aligned} 2(z-b^+)F_p^+ + G_{p+1}^+ + G_{p+1} &= 0, \\ (z-b)^2F_p + (z-b)G_{p+1} + a^2F_p^+ - (a^-)^2F_p^- &= 0. \end{aligned} \tag{0.21}$$

To set the stationary Toda hierarchy apart from the general time-dependent one, we will denote it by

$$\text{s-Tl}_p(a,b) = \begin{pmatrix} f_{p+1}^+(a,b) - f_{p+1}(a,b) \\ g_{p+1}(a,b) - g_{p+1}^-(a,b) \end{pmatrix} = 0, \quad p \in \mathbb{N}_0.$$

Explicitly, the first few equations of the stationary Toda hierarchy then read as follows

$$\begin{aligned} \text{s-Tl}_0(a,b) &= \begin{pmatrix} b^+ - b \\ 2((a^-)^2 - a^2) \end{pmatrix} = 0, \\ \text{s-Tl}_1(a,b) &= \begin{pmatrix} (a^+)^2 - (a^-)^2 + (b^+)^2 - b^2 \\ 2(a^-)^2(b+b^-) - 2a^2(b^+ + b) \end{pmatrix} \\ &\quad + c_1 \begin{pmatrix} b^+ - b \\ 2((a^-)^2 - a^2) \end{pmatrix} = 0, \\ \text{s-Tl}_2(a,b) &= \begin{pmatrix} (b^+)^3 - b^3 + 2(a^+)^2b^+ - 2(a^-)^2b \\ +a^2(b^+ - b) + (a^+)^2b^{++} + (a^-)^2b^- \\ 2(a^-)^2(b^2 + bb^- + (b^-)^2 + (a^-)^2 + (a^{--})^2) \\ -2a^2(b^2 + bb^+ + (b^+)^2 + a^2 + (a^+)^2) \end{pmatrix} \\ &\quad + c_1 \begin{pmatrix} (a^+)^2 - (a^-)^2 + (b^+)^2 - b^2 \\ 2(a^-)^2(b+b^-) - 2a^2(b^+ + b) \end{pmatrix} \\ &\quad + c_2 \begin{pmatrix} b^+ - b \\ 2((a^-)^2 - a^2) \end{pmatrix} = 0, \text{ etc.} \end{aligned}$$

The class of *algebro-geometric* Toda potentials, by definition, equals the set of solutions a, b of the stationary Toda hierarchy. In the following analysis we fix the value of a, b in (0.18)–(0.21), and hence we now turn to the investigation of algebro-geometric solutions a, b of the pth equation within the stationary Toda hierarchy. Equation (0.18) is of special interest since by the discrete analog of a 1923 result of Burchnall and Chaundy, proven by Naiman in 1962, commuting difference expressions (due to a common eigenfunction, to be discussed below, cf. (0.34), (0.35)) give rise to an algebraic relationship between the two difference expressions. Similarly, (0.20) permits the important conclusion that

$$\det(yI_2 - V_{p+1}(z,n)) = \det(yI_2 - V_{p+1}(z,n+1)), \tag{0.22}$$

(with I_2 the identity matrix in $\mathbb{C}^2$) and hence

$$\det(yI_2 - V_{p+1}(z,n)) = y^2 + \det(V_{p+1}(z,n))$$

$$
\begin{aligned}
&= y^2 - G_{p+1}^-(z,n)^2 + 4a^-(n)^2 F_p^-(z,n)F_p(z,n) \\
&= y^2 - R_{2p+2}(z),
\end{aligned}
\tag{0.23}
$$

for some n-independent monic polynomial R_{2p+2}, which we write as

$$
R_{2p+2}(z) = \prod_{m=0}^{2p+1}(z - E_m) \text{ for some } \{E_m\}_{m=0,\dots,2n} \subset \mathbb{C}.
$$

In particular, the combination

$$
G_{p+1}(z,n)^2 - 4a(n)^2 F_p(z,n)F_p^+(z,n) = R_{2p+2}(z) \tag{0.24}
$$

is n-independent. Moreover, one can rewrite (0.21) to yield

$$
\begin{aligned}
&(z-b)^4 F_p^2 - 2a^2(z-b)^2 F_p F_p^+ - 2(a^-)^2(z-b)^2 F_p F_p^- + a^4(F_p^+)^2 \\
&\qquad + (a^-)^4(F_p^-)^2 - 2a^2(a^-)^2 F_p^+ F_p^- = (z-b)^2 R_{2p+2}(z), \\
&(z-b)(z-b^+)G_{p+1}^2 - a^2(G_{p+1}^- + G_{p+1})(G_{p+1} + G_{p+1}^+) \\
&\qquad\qquad = (z-b)(z-b^+)R_{2p+2},
\end{aligned}
\tag{0.25}
$$

with precisely the same integration constant $R_{2p+2}(z)$ as in (0.23). In fact, by (0.11) and (0.12), equations (0.24) and (0.25) are simply identical. Incidentally, the algebraic relationship between L and P_{2p+2} alluded to in connection with the vanishing of their commutator in (0.18) can be made precise as follows: Restricting P_{2p+1} to the (algebraic) kernel $\ker(L-z)$ of $L-z$, one computes, using (0.7) and (0.25),

$$
\begin{aligned}
\big(P_{2p+2}\big|_{\ker(L-z)}\big)^2 &= \big((2aF_pS^+ + G_{p+1})\big|_{\ker(L-z)}\big)^2 \\
&= \big(2aF_p(G_{p+1}^+ + G_{p+1} + 2(z-b^+)F_p^+)S^+ \\
&\quad + G_{p+1}^2 - 4a^2F_pF_p^+\big)\big|_{\ker(L-z)} \\
&= (G_{p+1}^2 - 4a^2F_pF_p^+)\big|_{\ker(L-z)} = R_{2p+2}(L)\big|_{\ker(L-z)}.
\end{aligned}
$$

Thus, P_{2p+2}^2 and $R_{2p+2}(L)$ coincide on the finite-dimensional nullspace of $L-z$. Since $z \in \mathbb{C}$ is arbitrary, one infers that

$$
P_{2p+2}^2 - R_{2p+2}(L) = 0 \tag{0.26}
$$

holds once again with the same polynomial R_{2p+2}. The characteristic equation of V_{p+1} (cf. (0.23)) and (0.26) naturally leads one to the introduction of the *hyperelliptic curve* $\mathcal{K}_p$ of genus $p \in \mathbb{N}_0$ defined by

$$
\mathcal{K}_p\colon \mathcal{F}_p(z,y) = y^2 - R_{2p+2}(z) = 0, \quad R_{2p+2}(z) = \prod_{m=0}^{2p+1}(z - E_m). \tag{0.27}
$$

One compactifies the curve by adding two distinct points $P_{\infty_-}, P_{\infty_+}$ (still denoting the curve by $\mathcal{K}_p$ for simplicity) and notes that points $P \neq P_{\infty_\pm}$ on the curve are denoted by $P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_-}, P_{\infty_+}\}$, where $y(\,\cdot\,)$ is the meromorphic function on $\mathcal{K}_p$ satisfying[1] $y^2 - R_{2p+2}(z) = 0$. For simplicity, we will assume in the following that the (affine part of the) curve $\mathcal{K}_p$ is nonsingular, that is, the zeros E_m of R_{2p+2} are all simple. Remaining within the stationary framework a bit longer, one can now introduce the fundamental meromorphic function ϕ on $\mathcal{K}_p$ alluded to earlier, as follows,

$$\phi(P, n) = \frac{y - G_{p+1}(z, n)}{2a(n)F_p(z, n)} \tag{0.28}$$

$$= \frac{-2a(n)F_p^+(z, n)}{y + G_{p+1}(z, n)}, \quad P = (z, y) \in \mathcal{K}_p. \tag{0.29}$$

(We mention in passing that via (C.9) and (C.17), the two branches $\phi_\pm$ of ϕ are directly connected with the diagonal Green's function of the Lax operator L.) Equality of the two expressions (0.28) and (0.29) is an immediate consequence of the identity (0.24) and the fact $y^2 = R_{2p+p}(z)$. A comparison with (0.20) then readily reveals that ϕ satisfies the Riccati-type equation

$$a\phi(P) + a^-\phi^-(P)^{-1} = z - b. \tag{0.30}$$

The next step is crucial. It concerns the zeros and poles of ϕ and hence involves the zeros of $F_p(\,\cdot\,, n)$. Isolating the latter by introducing the factorization

$$F_p(z, n) = \prod_{j=1}^{p} (z - \mu_j(n)),$$

one can use the zeros of F_p and F_p^+ to define the following points $\hat{\mu}_j(n)$ and $\hat{\mu}_j^+(n)$ on $\mathcal{K}_p$,

$$\hat{\mu}_j(n) = (\mu_j(n), -G_{p+1}(\mu_j(n), n)), \quad j = 1, \dots, p, \tag{0.31}$$

$$\hat{\mu}_j^+(n) = (\mu_j^+(n), G_{p+1}(\mu_j^+(n), n)), \quad j = 1, \dots, p, \tag{0.32}$$

where μ_j^+, $j = 1, \dots, p$, denote the zeros of F_p^+. The motivation for this choice stems from $y^2 = R_{2p+2}(z)$ by (0.23), the identity (0.24) (which combines to $G_{p+1}^2 - 4a^2 F_p F_p^+ = y^2$), and a comparison of (0.28) and (0.29). Given (0.28)–(0.32) one obtains for the divisor $(\phi(\,\cdot\,, n))$ of the meromorphic function ϕ,

$$(\phi(\,\cdot\,, n)) = \mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}^+(n)} - \mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n)}. \tag{0.33}$$

Here we abbreviated $\underline{\hat{\mu}} = \{\hat{\mu}_1, \dots, \hat{\mu}_p\}, \underline{\hat{\mu}}^+ = \{\hat{\mu}_1^+, \dots, \hat{\mu}_p^+\} \in \mathrm{Sym}^p(\mathcal{K}_p)$, with

[1] For more details we refer to Appendix B and Chapter 1.

$\mathrm{Sym}^p(\mathcal{K}_p)$ the pth symmetric product of $\mathcal{K}_p$, and used our conventions[1] (A.39), (A.43), and (A.44) to denote positive divisors of degree p and $p+1$ on $\mathcal{K}_p$. Given $\phi(\,\cdot\,,n)$ one defines the *stationary Baker–Akhiezer function* $\psi(\,\cdot\,,n,n_0)$ on $\mathcal{K}_p \setminus \{P_{\infty_\pm}\}$ by

$$\psi(P,n,n_0) = \begin{cases} \prod_{n'=n_0}^{n-1} \phi(P,n'), & n \geq n_0+1, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} \phi(P,n')^{-1}, & n \leq n_0 - 1. \end{cases}$$

In particular, this implies

$$\phi = \psi^+/\psi,$$

and the following normalization[2] of ψ, $\psi(P,n_0,n_0) = 1,\ P \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}$. The Riccati-type equation (0.30) satisfied by ϕ then shows that the Baker–Akhiezer function ψ is the common formal eigenfunction of the commuting pair of Lax difference expressions L and P_{2p+2},

$$L\psi(P) = z\psi(P), \tag{0.34}$$
$$P_{2p+2}\psi(P) = y\psi(P), \quad P=(z,y) \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}, \tag{0.35}$$

and at the same time the Baker–Akhiezer vector Ψ defined by

$$\Psi(P) = \begin{pmatrix} \psi(P) \\ \psi^+(P) \end{pmatrix}, \quad P \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}, \tag{0.36}$$

satisfies the zero-curvature equations,

$$\Psi(P) = U(z)\Psi^-(P), \tag{0.37}$$
$$y\Psi^-(P) = V_{p+1}(z)\Psi^-(P), \quad P=(z,y) \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}. \tag{0.38}$$

Moreover, one easily verifies that away from the branch points $(E_m,0)$, $m = 0,\dots,2p+1$, of the two-sheeted Riemann surface $\mathcal{K}_p$, the two branches of ψ constitute a fundamental system of solutions of (0.34) and similarly, the two branches of ψ yield a fundamental system of solutions of (0.37). Since $\psi(\,\cdot\,,n,n_0)$ vanishes at $\hat\mu_j(n)$, $j=1,\dots,p$, and $\psi^+(\,\cdot\,,n,n_0)$ vanishes at $\hat\mu_j^+(n)$, $j=1,\dots,p$, we may call $\{\hat\mu_j(n)\}_{j=1,\dots,p}$ and $\{\hat\mu_j^+(n)\}_{j=1,\dots,p}$ the *Dirichlet* and *Neumann data* of L at the point $n \in \mathbb{Z}$, respectively.

Now the stationary formalism is almost complete; we only need to relate

[1] $\mathcal{D}_{\underline{Q}}(P) = m$ if P occurs m times in $\{Q_1,\dots,Q_p\}$ and zero otherwise, $\underline{Q} = \{Q_1,\dots,Q_p\} \in \mathrm{Sym}^p(\mathcal{K}_p)$. Similarly, $\mathcal{D}_{Q_0\underline{Q}} = \mathcal{D}_{Q_0} + \mathcal{D}_{\underline{Q}}$, $\mathcal{D}_{\underline{Q}} = \mathcal{D}_{Q_1} + \cdots + \mathcal{D}_{Q_p}$, $Q_0 \in \mathcal{K}_p$, and $\mathcal{D}_Q(P) = 1$ for $P=Q$ and zero otherwise.

[2] This normalization is less innocent than it might appear at first sight. It implies that $\mathcal{D}_{\underline{\hat\mu}(n)}$ and $\mathcal{D}_{\underline{\hat\mu}(n_0)}$ are the divisors of zeros and poles of $\psi(\,\cdot\,,n,n_0)$ on $\mathcal{K}_p \setminus \{P_{\infty_\pm}\}$.

the solution a, b of the pth stationary Toda equation and $\mathcal{K}_p$-associated data. This can be accomplished as follows.

First we relate a, b and the zeros μ_j of F_p. This is easily done by comparing the coefficients of the power z^{2p} in (0.25) and results in the *trace formulas,*[1]

$$\begin{aligned} a^2 &= \frac{1}{2}\sum_{j=1}^{p} y(\hat{\mu}_j)\prod_{\substack{k=1\\k\neq j}}^{p}(\mu_j-\mu_k)^{-1} + \frac{1}{4}\big(b^{(2)}-b^2\big), \\ b &= \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{j=1}^{p}\mu_j, \end{aligned} \tag{0.39}$$

where $b^{(2)} = \frac{1}{2}\sum_{m=0}^{2p+1} E_m^2 - \sum_{j=1}^{p}\mu_j^2$. However, the formula for a^2 is not useful for the algebro-geometric initial value problem as the quantities μ_j indeed may collide.[2] A more elaborate reconstruction algorithm, as described below, is required.

We will now indicate how to reconstruct a, b from $\mathcal{K}_p$ and given Dirichlet data at just one fixed point n_0. Due to the discrete spatial variation, this is considerably more involved than, say, for the KdV equation. Consider first the simplest case of self-adjoint Jacobi operators where a and b are real-valued and bounded sequences. In that case we are given Dirichlet divisors $\mathcal{D}_{\underline{\hat{\mu}}(n_0)} \in \operatorname{Sym}^p(\mathcal{K}_p)$ with corresponding Dirichlet eigenvalues in appropriate spectral gaps of L (more precisely, in appropriate spectral gaps of a bounded operator realization of L in $\ell^2(\mathbb{Z})$, but for simplicity this aspect will be ignored in the introduction). Next one develops an algorithm that provides finite nonspecial divisors $\mathcal{D}_{\underline{\hat{\mu}}(n)} \in \operatorname{Sym}^p(\mathcal{K}_p)$ in real position for all $n \in \mathbb{Z}$. In the self-adjoint case, the Dirichlet eigenvalues remain in distinct spectral gaps, and hence the expression for (0.39) for a^2 remains meaningful.

The self-adjoint situation is in sharp contrast to the general non-self-adjoint case in which the Dirichlet eigenvalues no longer are confined to distinct spectral gaps on the real axis. Moreover, Dirichlet eigenvalues are not necessarily separated and hence might coincide (i.e., collide), at particular lattice points. In addition, they may not remain finite and hit P_{∞_+} or P_{∞_-}. The algorithm has to take that into consideration; it is handled by further restricting the permissible set of initial data, which, however, remains a dense set of full measure

[1] Observe that only a^2 enters, and thus the sign of a is left undetermined.
[2] In the continuous case, e.g., for the Korteweg–de Vries equation, the situation is considerably simpler: The spatial variation of the μ_j, $j = 1, \dots, p$, is determined by the Dubrovin equations, a first-order system of ordinary differential equations. Assuming the μ_j, $j = 1, \dots, p$, are distinct at a given spatial point, there exists a small neighborhood around that point for which they remain distinct. This has no analog in the discrete case.

even in this more involved setting. A key element in the construction is the discrete dynamical system[1]

$$\underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n)}) = \underline{\alpha}_{Q_0}(\mathcal{D}_{\underline{\hat{\mu}}(n_0)}) - (n-n_0)\underline{A}_{P_{\infty_-}}(P_{\infty_+}),$$
$$\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0),\dots,\hat{\mu}_p(n_0)\} \in \operatorname{Sym}^p(\mathcal{K}_p),$$

where $Q_0 \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}$ is a given base point. Starting from a nonspecial finite initial divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$, we find that as n increases, $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ stays nonspecial as long as it remains finite. If it becomes infinite, then it is still nonspecial and contains P_{∞_+} at least once (but not P_{∞_-}). Further increasing n, all instances of P_{∞_+} will be rendered into P_{∞_-} step by step, until we have again a nonspecial divisor that has the same number of P_{∞_-} as the first infinite one had P_{∞_+}. Generically, one expects the subsequent divisor to be finite and nonspecial again. A central part of the algorithm is to prove that for a full set of initial data, the iterates stay away from $P_{\infty_\pm}$. Summarizing, we solve the following inverse problem: Given $\mathcal{K}_p$ and appropriate initial data

$$\underline{\hat{\mu}}(n_0) = \{\hat{\mu}_1(n_0),\dots,\hat{\mu}_p(n_0)\} \in \mathcal{M}_0,$$
$$\hat{\mu}_j(n_0) = \big(\mu_j(n_0), -G_{p+1}(\mu_j(n_0),n_0)\big), \quad j=1,\dots,p,$$

where $\mathcal{M}_0 \subset \operatorname{Sym}^p(\mathcal{K}_p)$ is the set of nonspecial Dirichlet divisors, we develop an algorithm that defines finite nonspecial divisors $\underline{\hat{\mu}}(n)$ for all $n\in\mathbb{Z}$.

Having constructed $\mu_j(n)$, $j=1,\dots,p$, $n\in\mathbb{Z}$, using an elaborate twelve-step procedure, one finds that the quantities a and b are given by

$$\begin{aligned} a(n)^2 &= \frac{1}{2}\sum_{k=1}^{q(n)} \frac{\big(d^{p_k(n)-1}y(P)/d\zeta^{p_k(n)-1}\big)\big|_{P=(\zeta,\eta)=\hat{\mu}_k(n)}}{(p_k(n)-1)!} \\ &\quad\times \prod_{k'=1,\,k'\neq k}^{q(n)} (\mu_k(n)-\mu_{k'}(n))^{-p_k(n)} + \frac{1}{4}\big(b^{(2)}(n)-b(n)^2\big), \\ b(n) &= \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{k=1}^{q(n)} p_k(n)\mu_k(n), \quad n\in\mathbb{Z}, \end{aligned} \tag{0.40}$$

where $p_k(n)$ are associated with degeneracies of the μ_j, $j=1,\dots,p$, and $\sum_{k=1}^{q(n)} p_k(n) = p$, see Theorem 1.32. We stress the resemblance between (0.40) and (0.39). Formulas (0.40) then yield a solution a, b of the pth stationary Toda equation.

An alternative reconstruction of a, b, nicely complementing the one just

[1] Here $\underline{\alpha}_{Q_0}$ and $\underline{A}_{Q_0}$ denote Abel maps, see (A.30) and (A.29), respectively.

discussed, can be given with the help of the *Riemann theta function*[1] associated with $\mathcal{K}_p$ and an appropriate homology basis of cycles on it. The known zeros and poles of ϕ (cf. (0.33)), and similarly, the set of zeros $\{P_{\infty_+}\} \cup \{\hat{\mu}_j(n)\}_{j=1,\dots,p}$ and poles $\{P_{\infty_-}\} \cup \{\hat{\mu}_j(n_0)\}_{j=1,\dots,p}$ of the Baker–Akhiezer function $\psi(\,\cdot\,, n, n_0)$, then permit one to find theta function representations for ϕ and ψ by referring to Riemann's vanishing theorem and the Riemann–Roch theorem. The corresponding theta function representation of the algebro-geometric solution a, b of the pth stationary Toda equation then can be obtained from that of ψ by an asymptotic expansion with respect to the spectral parameter near the point P_{∞_+}. The resulting final expression for a, b, the analog of the *Its–Matveev formula* in the KdV context, is of the type

$$\begin{aligned} a(n)^2 &= \tilde{a}^2 \frac{\theta(\underline{A} - \underline{B} + \underline{B}n)\theta(\underline{A} + \underline{B} + \underline{B}n)}{\theta(\underline{A} + \underline{B}n)^2}, \\ b(n) &= \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{j=1}^{p} \lambda_j \\ &\quad - \sum_{j=1}^{p} c_j(p) \frac{\partial}{\partial w_j} \ln\left(\frac{\theta(\underline{A} + \underline{B}n + \underline{w})}{\theta(\underline{A} - \underline{B} + \underline{B}n + \underline{w})} \right)\bigg|_{\underline{w}=0}. \end{aligned} \tag{0.41}$$

Here the constants $\tilde{a}, \lambda_j, c_j(p) \in \mathbb{C}$, $j = 1, \dots, p$, and the constant vector $\underline{B} \in \mathbb{C}^p$ are uniquely determined by $\mathcal{K}_p$ (and its homology basis), and the constant vector $\underline{A} \in \mathbb{C}^p$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(n_0) = (\hat{\mu}_1(n_0), \dots, \hat{\mu}_p(n_0)) \in \mathrm{Sym}^p(\mathcal{K}_p)$ at the initial point n_0 as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0)}$ is assumed to be nonspecial.[2] Moreover, the theta function representation (0.41) remains valid as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n)}$ stays nonspecial. We emphasize the remarkable fact that the argument of the theta functions in (0.41) is linear with respect to n.

This completes our somewhat lengthy excursion into the stationary Toda hierarchy. In the following we return to the time-dependent Toda hierarchy and describe the analogous steps involved to construct solutions $a = a(n, t_r)$, $b = b(n, t_r)$ of the rth Toda equation with initial values being algebro-geometric solutions of the pth stationary Toda equation. More precisely, we are seeking

[1] For details on the p-dimensional theta function $\theta(\underline{z})$, $\underline{z} \in \mathbb{C}^p$, we refer to Appendices A and B.

[2] If $\mathcal{D} = n_1\mathcal{D}_{Q_1} + \cdots + n_k\mathcal{D}_{Q_k} \in \mathrm{Sym}^p(\mathcal{K}_p)$ for some $n_\ell \in \mathbb{N}$, $\ell = 1, \dots, k$, with $n_1 + \cdots + n_k = p$, then $\mathcal{D}$ is called nonspecial if there is no nonconstant meromorphic function on $\mathcal{K}_p$ which is holomorphic on $\mathcal{K}_p \setminus \{Q_1, \dots, Q_k\}$ with poles at most of order n_ℓ at Q_ℓ, $\ell = 1, \dots, k$.

a solution a, b of

$$\begin{aligned}
\widetilde{\mathrm{Tl}}_r(a,b) &= \begin{pmatrix} a_{t_r} - a\big(\tilde f^+_{p+1}(a,b) - \tilde f_{p+1}(a,b)\big) \\ b_{t_r} + \tilde g_{p+1}(a,b) - \tilde g^-_{p+1}(a,b) \end{pmatrix} = 0, \\
(a,b)\big|_{t_r = t_{0,r}} &= \big(a^{(0)}, b^{(0)}\big),
\end{aligned} \tag{0.42}$$

$$\text{s-Tl}_p\,\big(a^{(0)}, b^{(0)}\big) = \begin{pmatrix} f^+_{p+1}\big(a^{(0)}, b^{(0)}\big) - f_{p+1}\big(a^{(0)}, b^{(0)}\big) \\ g_{p+1}\big(a^{(0)}, b^{(0)}\big) - g^-_{p+1}\big(a^{(0)}, b^{(0)}\big) \end{pmatrix} = 0 \tag{0.43}$$

for some $t_{0,r} \in \mathbb{R}$, $p, r \in \mathbb{N}_0$ and a prescribed curve $\mathcal{K}_p$ associated with the stationary solution $a^{(0)}, b^{(0)}$ in (0.43).

We pause for a moment to reflect on the pair of equations (0.42), (0.43): As it turns out, it represents a dynamical system on the set of algebro-geometric solutions isospectral to the initial value $a^{(0)}, b^{(0)}$. By isospectral we here allude to the fact that for any fixed t_r, the solution $a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)$ of (0.42), (0.43) is a stationary solution of (0.43),

$$\begin{aligned}
&\text{s-Tl}_p\,\big(a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)\big) \\
&\quad = \begin{pmatrix} f^+_{p+1}(a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)) - f_{p+1}(a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)) \\ g_{p+1}(a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)) - g^-_{p+1}(a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)) \end{pmatrix} = 0
\end{aligned}$$

associated with the fixed underlying algebraic curve $\mathcal{K}_p$ (the latter being independent of t_r). Put differently, $a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)$ is an isospectral deformation of $a^{(0)}, b^{(0)}$ with t_r the corresponding deformation parameter. In particular, $a(\,\cdot\,, t_r), b(\,\cdot\,, t_r)$ traces out a curve in the set of algebro-geometric solutions isospectral to $a^{(0)}, b^{(0)}$.

Since the summation constants in the functionals f_ℓ of a, b in the stationary and time-dependent contexts are independent of each other, we indicate this by adding a tilde on all the time-dependent quantities. Hence we shall employ the notation $\widetilde P_{2r+2}$, $\widetilde V_{r+1}$, $\widetilde F_r$, etc., in order to distinguish them from P_{2p+2}, V_{p+1}, F_p, etc. Thus $\widetilde P_{2r+2}$, $\widetilde V_{r+1}$, $\widetilde F_r$, $\widetilde G_{r+1}$, $\tilde f_s$, $\tilde g_s$, $\tilde c_s$ are constructed in the same way as P_{2p+2}, V_{p+1}, F_p, G_p, f_ℓ, g_ℓ, c_ℓ using the recursion (0.5) with the only difference being that the set of summation constants $\tilde c_r$ in $\tilde f_s$ is independent of the set c_k used in computing f_ℓ.

Our strategy will be the same as in the stationary case: Assuming existence of a solution a, b, we will deduce many of its properties which in the end will yield an explicit expression for the solution. In fact, we will go a step further, postulating the equations

$$\begin{aligned}
a_{t_r} &= -a\big(2(z - b^+)\widetilde F^+_r + \widetilde G^+_{r+1} + \widetilde G_{r+1}\big), \\
b_{t_r} &= 2\big((z-b)^2 \widetilde F_r + (z-b)\widetilde G_{r+1} + a^2 \widetilde F^+_r - (a^-)^2 \widetilde F^-_r\big),
\end{aligned} \tag{0.44}$$

$$
\begin{aligned}
&0 = 2(z-b^+)F_p^+ + G_{p+1}^+ + G_{p+1},\\
&0 = (z-b)^2F_p + (z-b)G_{p+1} + a^2F_p^+ - (a^-)^2F_p^-,
\end{aligned} \tag{0.45}
$$

where $a^{(0)} = a^{(0)}(n)$, $b^{(0)} = b^{(0)}(n)$ in (0.43) has been replaced by $a = a(n,t_r)$, $b = b(n,t_r)$ in (0.45). Here

$$
\begin{aligned}
F_p(z) &= \sum_{\ell=0}^{p} f_{p-\ell}z^\ell = \prod_{j=1}^{p}(z-\mu_j), \quad \widetilde{F}_r(z) = \sum_{s=0}^{r} \tilde{f}_{r-s}z^s,\\
G_{p+1}(z) &= -z^{p+1} + \sum_{\ell=0}^{p} g_{p-\ell}z^\ell + f_{p+1},\\
\widetilde{G}_{r+1}(z) &= -z^{r+1} + \sum_{s=0}^{r} \tilde{g}_{r-s}z^s + \tilde{f}_{r+1},
\end{aligned}
$$

for fixed $p, r \in \mathbb{N}_0$. Introducing G_{p+1}, U, V_{p+1} and $\widetilde{G}_{r+1}$, $\widetilde{V}_{r+1}$ (replacing F_p by $\widetilde{F}_r$) as in (0.12)–(0.14), the basic equations (0.44), (0.45) are equivalent to the *Lax equations*

$$
\begin{aligned}
\frac{d}{dt_r}L - \big[\widetilde{P}_{2r+2}, L\big] &= 0,\\
\big[P_{2p+2}, L\big] &= 0,
\end{aligned}
$$

and to the *zero-curvature equations*

$$
U_{t_r} + U\widetilde{V}_{r+1} - \widetilde{V}_{r+1}^+U = 0, \tag{0.46}
$$

$$
UV_{p+1} - V_{p+1}^+U = 0. \tag{0.47}
$$

Moreover, one computes in analogy to (0.22) and (0.23) that

$$
\begin{aligned}
\det(yI_2 - V_{p+1}(z,n+1,t_r)) - \det(yI_2 - V_{p+1}(z,n,t_r)) &= 0,\\
\partial_{t_r}\det(yI_2 - V_{p+1}(z,n,t_r)) &= 0,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\det(yI_2 - V_{p+1}(z,n,t_r)) = y^2 + \det(V_{p+1}(z,n,t_r))\\
&= y^2 - G_{p+1}^-(z,n)^2 + 4a^-(n)^2F_p^-(z,n)F_p(z,n) = y^2 - R_{2p+2}(z),
\end{aligned} \tag{0.48}
$$

is independent of $(n,t_r) \in \mathbb{Z}\times\mathbb{R}$. Thus,

$$
G_{p+1}^2 - 4a^2F_pF_p^+ = R_{2p+2},
$$

$$\begin{aligned}
&(z-b)^4F_p^2-2a^2(z-b)^2F_pF_p^+-2(a^-)^2(z-b)^2F_pF_p^-+a^4(F_p^+)^2\\
&\qquad\qquad+(a^-)^4(F_p^-)^2-2a^2(a^-)^2F_p^+F_p^-=(z-b)^2R_{2p+2}(z),\\
&(z-b)(z-b^+)G_{p+1}^2-a^2(G_{p+1}^-+G_{p+1})(G_{p+1}+G_{p+1}^+)\\
&\qquad\qquad\qquad\qquad=(z-b)(z-b^+)R_{2p+2},
\end{aligned}$$

hold as in the stationary context. The independence of (0.48) of t_r can be interpreted as follows: The rth Toda flow represents an isospectral deformation of the curve $\mathcal{K}_p$ defined in (0.27), in particular,[1] the branch points of $\mathcal{K}_p$ remain invariant under these flows,

$$\partial_{t_r}E_m=0,\quad m=0,\dots,2p+1. \tag{0.49}$$

As in the stationary case, one can now introduce the basic meromorphic function ϕ on $\mathcal{K}_p$ by

$$\begin{aligned}
\phi(P,n,t_r)&=\frac{y-G_{p+1}(z,n,t_r)}{2a(n,t_r)F_p(z,n,t_r)}\\
&=\frac{-2a(n,t_r)F_p(z,n+1,t_r)}{y+G_{p+1}(z,n,t_r)},\quad P(z,y)\in\mathcal{K}_p,
\end{aligned}$$

and a comparison with (0.46) and (0.47) then shows that ϕ satisfies the Riccati-type equations

$$a\phi(P)+a^-(\phi^-(P))^{-1}=z-b, \tag{0.50}$$

$$\begin{aligned}
\phi_{t_r}(P)&=-2a\big(\widetilde{F}_r(z)\phi(P)^2+\widetilde{F}_r^+(z)\big)+2(z-b^+)\widetilde{F}_r^+(z)\phi(P)\\
&\quad+\big(\widetilde{G}_{r+1}^+(z)-\widetilde{G}_{r+1}(z)\big)\phi(P).
\end{aligned} \tag{0.51}$$

Next, factorizing F_p as before,

$$F_p(z)=\prod_{j=1}^p(z-\mu_j),$$

one introduces points $\hat{\mu}_j(n,t_r)$, $\hat{\mu}_j^+(n,t_r)$ on $\mathcal{K}_p$ by

$$\begin{aligned}
\hat{\mu}_j(n,t_r)&=(\mu_j(n,t_r),-G_{p+1}(\mu_j(n,t_r),n,t_r)),\quad j=1,\dots,p,\\
\hat{\mu}_j^+(n,t_r)&=(\mu_j^+(n,t_r),G_{p+1}(\mu_j^+(n,t_r),n,t_r)),\quad j=1,\dots,p,
\end{aligned}$$

and obtains for the divisor $(\phi(\,\cdot\,,n,t_r))$ of the meromorphic function ϕ,

$$(\phi(\,\cdot\,,n,t_r))=\mathcal{D}_{P_{\infty_+}\underline{\hat{\mu}}^+(n,t_r)}-\mathcal{D}_{P_{\infty_-}\underline{\hat{\mu}}(n,t_r)},$$

[1] Property (0.49) is weaker than the usually stated isospectral deformation of the Lax operator $L(t_r)$. However, the latter is a more delicate functional analytic problem since a,b need not be bounded and by the possibility of non-self-adjointness of $L(t_r)$. See, however, Theorem 1.62.

as in the stationary context. Given $\phi(\,\cdot\,, n, t_r)$ one then defines the *time-dependent Baker–Akhiezer vector* $\psi(\,\cdot\,, n, n_0, t_r, t_{0,r})$ on $\mathcal{K}_p \setminus \{P_{\infty_\pm}\}$ by

$$\begin{aligned}
&\psi(P, n, n_0, t_r, t_{0,r}) \\
&\quad = \exp\left(\int_{t_{0,r}}^{t_r} ds \big(2a(n_0, s)\widetilde{F}_r(z, n_0, s)\phi(P, n_0, s) + \widetilde{G}_{r+1}(z, n_0, s)\big)\right) \\
&\qquad \times \begin{cases} \prod_{n'=n_0}^{n-1} \phi(P, n', t_r), & n \geq n_0 + 1, \\ 1, & n = n_0, \\ \prod_{n'=n}^{n_0-1} \phi(P, n', t_r)^{-1}, & n \leq n_0 - 1, \end{cases}
\end{aligned}$$

with

$$\phi(P, n, t_r) = \psi^+(P, n, n_0, t_r, t_{0,r})/\psi(P, n, n_0, t_r, t_{0,r}).$$

The Riccati-type equations (0.50), (0.51) satisfied by ϕ then show that

$$-V_{p+1,t_r} + \big[\widetilde{V}_{r+1}, V_{p+1}\big] = 0$$

in addition to (0.46), (0.47). Moreover, they yield again that the Baker–Akhiezer function ψ is the common formal eigenfunction of the commuting pair of Lax differential expressions $L(t_r)$ and $P_{2p+2}(t_r)$,

$$\begin{aligned}
L\psi(P) &= z\psi(P), \\
P_{2p+2}\psi(P) &= y\psi(P), \quad P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}, \\
\psi_{t_r}(P) &= \widetilde{P}_{2r+2}\psi(P) \\
&= 2a\widetilde{F}_r(z)\psi^+(P) + \widetilde{G}_{r+1}(z)\psi(P),
\end{aligned}$$

and at the same time the Baker–Akhiezer vector Ψ (cf. (0.36)) satisfies the zero-curvature equations,

$$\begin{aligned}
\Psi(P) &= U(z)\Psi^-(P), \\
y\Psi^-(P) &= V_{p+1}(z)\Psi^-(P), \quad P = (z, y) \in \mathcal{K}_p \setminus \{P_{\infty_\pm}\}, \\
\Psi_{t_r}(P) &= \widetilde{V}_{r+1}^+(z)\Psi(P).
\end{aligned}$$

The remaining time-dependent constructions closely follow our stationary outline. The time variation of the μ_j, $j = 1, \ldots, p$, is given by the *Dubrovin* *equations*[1]

$$\mu_{j,t_r} = -2\widetilde{F}_r(\mu_j)y(\hat{\mu}_j)\prod_{\substack{\ell=1\\ \ell\neq j}}^{p}(\mu_j-\mu_\ell)^{-1}, \quad j=1,\dots,p. \tag{0.52}$$

However, as in the stationary case, the formula (0.52) as well as (0.39) are not useful in the general complex-valued case where the μ_j, $j=1,\dots,p$ may be degenerate and may not remain bounded. Thus, a more elaborate procedure is required.

Let us first consider the case of real-valued and bounded sequences a, b, that is, the situation when the Lax operator L is self-adjoint. Given the curve $\mathcal{K}_p$ and an initial nonspecial Dirichlet divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})} \in \mathrm{Sym}^p(\mathcal{K}_p)$ at a point $(n_0,t_{0,r})$, one follows the stationary algorithm to construct a solution s-$\mathrm{Tl}_p(a^{(0)},b^{(0)})=0$. From each lattice point $n\in\mathbb{Z}$ one can use the time-dependent Dubrovin equations (0.52) to construct locally the solution $\mu_j(n,t_r)$ for t_r near $t_{0,r}$. Using the formulas (0.39) we find solutions of $\mathrm{Tl}_r(a,b)=0$ with $(a,b)|_{t_r=t_{0,r}}=(a^{(0)},b^{(0)})$. However, this construction requires that the eigenvalues $\mu_j(n,t_r)$, $j=1,\dots,p$, remain distinct, which generally is only true in the self-adjoint case with real-valued initial data. In contrast, in the general complex-valued case where eigenvalues can be expected to collide, a considerably more refined approach is required. The Dubrovin equations (0.52) are replaced by a first-order autonomous system of $2p$ differential equations in the variables f_j, $j=1,\dots,p$, g_j, $j=1,\dots,p-1$, and g_p+f_{p+1} which can be solved locally in a neighborhood $(t_{0,r}-T_0,t_{0,r}+T_0)$ of $t_{0,r}$. Next, one uses the general stationary algorithm to extend this solution from $\{n_0\}\times(t_{0,r}-T_0,t_{0,r}+T_0)$ to $\mathbb{Z}\times(t_{0,r}-T_0,t_{0,r}+T_0)$. For a carefully selected set $\mathcal{M}_1$ of full measure of initial divisors, the solution can even be extended to a global solution on $\mathbb{Z}\times\mathbb{R}$. Summarizing, we solve the following inverse problem: Given $\mathcal{K}_p$ and appropriate initial data

$$\begin{aligned}
&\underline{\hat{\mu}}(n_0,t_{0,r}) = \{\hat{\mu}_1(n_0),\dots,\hat{\mu}_p(n_0,t_{0,r})\}\in\mathcal{M}_1,\\
&\hat{\mu}_j(n_0,t_{0,r}) = \big(\mu_j(n_0,t_{0,r}),-G_{p+1}(\mu_j(n_0,t_{0,r}),n_0,t_{0,r})\big), \quad j=1,\dots,p,
\end{aligned}$$

where $\mathcal{M}_1\subset\mathrm{Sym}^p(\mathcal{K}_p)$ is an appropriate set of nonspecial Dirichlet divisors, we develop an algorithm that defines finite nonspecial divisors $\underline{\hat{\mu}}(n,t_r)$ for all $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$.

Having constructed $\mu_j(n,t_r)$, $j=1,\dots,p$, $(n,t_r)\in\mathbb{Z}\times\mathbb{R}$, one then shows that the analog of (0.40) remains valid and then leads to a solution a,b of (0.42), (0.43).

[1] To obtain a closed system of differential equations, one has to express $\widetilde{F}_r(\mu_j)$ solely in terms of $\mu_1,\dots,\mu_p$ and $E_0,\dots,E_{2p+1}$, see Lemma D.4.

The corresponding representations of a, b, ϕ, and ψ in terms of the *Riemann theta function* associated with $\mathcal{K}_p$ are then obtained in close analogy to the stationary case. In particular, in the case of a, b, one obtains the *Its–Matveev formula*

$$\begin{aligned}
a(n,t_r)^2 &= \tilde{a}^2 \frac{\theta(\underline{A}-\underline{B}+\underline{B}n+\underline{C}_r t_r)\theta(\underline{A}+\underline{B}+\underline{B}n+\underline{C}_r t_r)}{\theta(\underline{A}+\underline{B}n+\underline{C}_r t_r)^2}, \\
b(n,t_r) &= \frac{1}{2}\sum_{m=0}^{2p+1} E_m - \sum_{j=1}^{p} \lambda_j \\
&\quad - \sum_{j=1}^{p} c_j(p) \frac{\partial}{\partial w_j} \ln\left(\frac{\theta(\underline{A}+\underline{B}n+\underline{C}_r t_r+\underline{w})}{\theta(\underline{A}-\underline{B}+\underline{B}n+\underline{C}_r t_r+\underline{w})} \right)\bigg|_{\underline{w}=0}.
\end{aligned} \tag{0.53}$$

Here the constants $\tilde{a}, \lambda_j, c_j(p) \in \mathbb{C}$, $j = 1, \dots, p$, and the constant vectors $\underline{B}, \underline{C}_r \in \mathbb{C}^p$ are uniquely determined by $\mathcal{K}_p$ (and its homology basis) and r, and the constant vector $\underline{A} \in \mathbb{C}^p$ is in one-to-one correspondence with the Dirichlet data $\underline{\hat{\mu}}(n_0, t_{0,r}) = (\hat{\mu}_1(n_0, t_{0,r}), \dots, \hat{\mu}_p(n_0, t_{0,r})) \in \mathrm{Sym}^p(\mathcal{K}_p)$ at the initial point $(n_0, t_{0,r})$ as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n_0,t_{0,r})}$ is assumed to be nonspecial. Moreover, the theta function representation (0.53) remains valid as long as the divisor $\mathcal{D}_{\underline{\hat{\mu}}(n,t_r)}$ stays nonspecial. Again, one notes the remarkable fact that the argument of the theta functions in (0.53) is linear with respect to both n and t_r.

The reader will have noticed that we used terms such as *completely integrable*, *soliton equations*, *isospectral deformations*, etc., without offering a precise definition for them. Arguably, an integrable system in connection with nonlinear evolution equations should possess several properties, including, for instance,

- infinitely many conservation laws
- isospectral deformations of a Lax operator
- action-angle variables, Hamiltonian formalism
- algebraic (spectral) curves
- infinitely many symmetries and transformation groups
- "explicit" solutions.

While many of these properties apply to particular systems of interest, there is simply no generally accepted definition to date of what constitutes an integrable system.[1] Thus, different schools have necessarily introduced different shades of integrability (Liouville integrability, analytic integrability, algebraically complete integrability, etc.); in this monograph we found it useful

[1] See, also, Lakshmanan and Rajasekar (2003, Chs. 10, 14, App. I) and several contributions to Zakharov (1991) for an extensive discussion of various aspects of integrability.

to focus on the existence of underlying algebraic curves and explicit representations of solutions in terms of corresponding Riemann theta functions and limiting situations thereof.

Finally, a brief discussion of the content of each chapter is in order (additional details are collected in the list of contents at the beginning of each chapter). Chapter 1 is devoted to the Toda hierarchy and its algebro-geometric solutions. In Chapter 2 we turn to the Kac–van Moerbeke equation. Rather than studying this equation independently, we exploit its intrinsic connection with the Toda lattice. Indeed, there exists a Miura-like transformation between the two integrable systems, allowing for a transfer of solutions between them. Next, in Chapter 3, we consider the Ablowitz–Ladik (AL) hierarchy (a complexified discrete nonlinear Schrödinger hierarchy) of differential-difference evolution equations and its algebro-geometric solutions.

Presentation: Each chapter, together with appropriate appendices compiled in the second part of this volume, is intended to be essentially self-contained and hence can be read independently from the remaining chapters. This attempt to organize chapters independently of one another comes at a price, of course: Similar arguments in the construction of algebro-geometric solutions for different hierarchies are repeated in different chapters. We believe this makes the results more easily accessible.

While we kept the style of presentation and the notation employed as close as possible to that used in Volume I, we emphasize that this volume is entirely self-contained and hence can be read independently of Volume I.

References are deferred to detailed notes for each section at the end of every chapter. In addition to a comprehensive bibliographical documentation of the material dealt with in the main text, these notes also contain numerous additional comments and results (and occasionally hints to the literature of topics not covered in this monograph).

Succinctly written appendices, some of which summarize subjects of interest on their own, such as compact (and, in particular, hyperelliptic) Riemann surfaces, guarantee a fairly self-contained presentation, accessible at the advanced graduate level.

An extensive bibliography is included at the end of this volume. Its size reflects the enormous interest this subject generated over the past four decades. It underscores the wide variety of techniques employed to study completely integrable systems. Even though we undertook every effort to provide an exhaustive list of references, the result in the end must necessarily be considered incomplete. We regret any omissions that have occurred. Publications with three or more authors are abbreviated “First author et al. (year)” in the text. If more than one publication yields the same abbreviation, latin letters a,b,c, etc., are added after the year. In the bibliography, publications are alphabetically ordered using all authors' names and year of publication.