A lower bound for the representation dimension of kC_p^n

Steffen Oppermann

The representation dimension of a finite dimensional algebra has been introduced by Auslander in his Queen Mary College notes [1]. Rouquier has shown ([8]) that the representation dimension of the exterior algebra of an n-dimensional vector space is $n+1$, thus giving the first example of an algebra known to have representation dimension strictly larger than 3. To do so, he proved that the representation dimension is bounded below by the dimension of the stable module category plus two, and analyzed the latter with the help of Koszul duality. In characteristic 2, the exterior algebra is just kC_2^n, so Rouquier in particular has determined the representation dimension of these algebras. He pointed out that his result implies the case $p=2$ of the following conjecture of Benson: The Loewy length of a block of a group algebra in characteristic p is strictly larger than the p-rank of its defect group.

Here we generalize Rouquier's result to group algebras of elementary abelian groups. More precisely we will prove the following:

Theorem. *Let k be a field of characteristic p, C_p^n the elementary abelian group of order p^n. Then*

$$\dim kC_p^n\text{-}\underline{\mathrm{mod}} \geq n-1$$

One crucial idea of the proof of the $p=2$ case in [8] is to transfer the problem to commutative algebra with the help of Koszul duality. In contrast, we will work directly in the module category and show that certain morphisms vanish in the stable category.

By Rouquier's result mentioned above the following lower bound for the representation dimension follows:

Corollary 1. *Let k be a field of characteristic p. Then* $\operatorname{repdim} kC_p^n > n$.

Following arguments due to Rouquier, we can deduce a lower bound for the representation dimension of any block of a group algebra. The p-rank of a group G is the maximal n, such that G has an elementary abelian subgroup of order p^n.

Steffen Oppermann
Mathematisches Institut der Universität zu Köln, Weyertal 86-90, 50931 Köln, Germany
E-mail: sopperma@mi.uni-koeln.de

Corollary 2. *Let k be a field of characteristic p. Let G be a finite group, B a non-semisimple block of kG and D a defect group of B. Then*

$$\operatorname{repdim} B > p\text{-rank}(D).$$

Finally, by an argument of Auslander, the representation dimension of a self-injective algebra is bounded above by its Loewy length, that is the minimal n such that Rad^n vanishes. Using this, Benson's conjecture follows.

Corollary 3 (Benson's conjecture). *Let k be a field of characteristic p. Let G be a finite group, B a block of kG, $\operatorname{ll} B$ its Loewy length, and D a defect group of B. Then*

$$\operatorname{ll} B > p\text{-rank}(D).$$

In the first section we will recall the definitions and some basic properties of the representation dimension (due to Auslander) and the dimension of a triangulated category (due to Rouquier).

In the second section we explain the idea of the proof of the Theorem. We will assume to have a module M generating (as will be defined in Section 1) the stable module category in a finite number of steps. We will explain a method to show that another object N is not in the category generated by M in a fixed number of steps. To apply this method, we need to find a module N with a certain chain of endomorphisms having a non-zero composition in the stable module category, but such that every single endomorphism in the chain annihilates all morphisms from M.

For an infinite field, we will in Section 3 explicitly find a module N and endomorphisms meeting our requirements. More precisely, we will construct a family of modules in such a way that we can, depending on M, choose an adequate N (Proposition 11). We will show that this module N cannot be in the category generated by M in to few steps. This provides the lower bound for the dimension of the stable module category.

In the fourth section we will show that our result also holds for a finite field. We look at the algebraic closure, where we may use the result for infinite fields, and then see that everything actually happens in a finite extension of the given field.

Finally, in Section 5, we will see that our result also induces a lower bound for the representation dimension of any block of a group algebra. This implies Benson's conjecture.

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1. Definitions of the dimensions

Representation dimension:
Recall that the global dimension of an algebra, denoted by gld, is the maximum of the projective dimensions of the modules.

Definition. Let Λ be a finite dimensional algebra. Then the *representation dimension* of Λ is defined to be

$$\operatorname{repdim}\Lambda = \min\{\operatorname{gld}\operatorname{End}_\Lambda(M) \mid M \text{ generates and cogenerates } \Lambda\text{-mod}\}.$$

The representation dimension of a finite dimensional algebra has been introduced by Auslander [1] in order to measure how far an algebra is from being representation finite. He showed that an algebra is representation infinite if and only if its representation dimension is at least three. To date, no general method is known for calculating the representation dimension of a given algebra, and in fact the only examples known by now to have representation dimension greater than three are the exterior algebras ([8]) and the algebras in the family considered by Krause and Kussin ([6]). However, it has been shown by Iyama [5], that the representation dimension is always finite.

Since we know the module category of Λ better than the one of $\operatorname{End}_\Lambda(M)$, the following lemma may help us to understand the representation dimension.

Lemma 4 ([4, Lemma 2.1]). *Let Λ be a non-semisimple algebra, let $M \in \Lambda\text{-mod}$ be a generator and cogenerator, $n \in \mathbb{N}$. Then the following are equivalent:*

1. $\operatorname{gld}\operatorname{End}_\Lambda(M) = n$
2. *For every $N \in \Lambda\text{-mod}$ there is an exact sequence*

$$0 \longrightarrow M_{n-2} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow N \longrightarrow 0,$$

such that the induced sequence

$$0 \longrightarrow \operatorname{Hom}_\Lambda(M, M_{n-2}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_\Lambda(M, N) \longrightarrow 0$$

is also exact.

Such a sequence will be called a universal M-resolution of N.

This means, that the representation dimension of a non-semisimple algebra can also be defined to be the minimal n, such that there is a generator cogenerator M having the property that every module has a universal M-resolution of length at most $n-2$.

In order to get upper bounds for the representation dimension of some algebra, one can choose a generator cogenerator M and calculate the maximal length of a universal M-resolution of a module. The following lemma is an example of this technique.

Lemma 5 ([1, III.5, p.55], improved in [8, Proposition 3.9]). *Let Λ be self-injective. Then*

$$\operatorname{repdim}\Lambda \leq \operatorname{ll}\Lambda,$$

where $\operatorname{ll}\Lambda$ is the Loewy length of Λ.

Proof. Take $M = \bigoplus_i \Lambda/\operatorname{Rad}^i\Lambda$. Let us denote for a moment the kernel of the universal M-cover (that is the first morphism of the minimal universal M-resolution) of a module N by $\Omega_M N$. Now note that in every step of the resolution, the Loewy length of the module decreases by at least one, that means $\operatorname{ll}\Omega_M^{i+1}N \leq \operatorname{ll}\Omega_M^i N - 1$. If N is projective then it is in $\operatorname{add} M$, therefore we may assume that $\operatorname{ll} N < \operatorname{ll}\Lambda$. Putting this together we find $\operatorname{ll}\Omega_M^{\operatorname{ll}\Lambda-1}N = 0$, so $\Omega_M^{\operatorname{ll}\Lambda-1}N = 0$. Therefore $\operatorname{repdim}\Lambda \leq \operatorname{ll}\Lambda$ as claimed.

Alternatively, see [1, III.5, p.55], but note that if the Loewy length of the module is $\operatorname{ll}\Lambda$, then the module is projective, so this case can be excluded. $\square$

Dimension of a triangulated category: The notion of dimension of a triangulated category has been introduced by Rouquier in [7].

Let $\mathcal{T}$ be a triangulated category, $\mathcal{I} \subset \mathcal{O}b\,\mathcal{T}$. Then let $\langle\mathcal{I}\rangle$ be the full subcategory of $\mathcal{T}$ of all direct summands of finite direct sums of shifts of objects in $\mathcal{I}$. For two subclasses $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}b\,\mathcal{T}$ let $\mathcal{I}_1 * \mathcal{I}_2$ be the full subcategory of all extensions between them, that is the objects of $\mathcal{I}_1 * \mathcal{I}_2$ are exactly the M, such that there is a distinguished triangle $M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow M_1[1]$ in $\mathcal{T}$ with $M_i \in \mathcal{I}_i$. Now set

$$\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$$

and

$$\begin{aligned}
&\langle\mathcal{I}\rangle_0 = 0,\\
&\langle\mathcal{I}\rangle_1 = \langle\mathcal{I}\rangle\\
&\langle\mathcal{I}\rangle_{i+1} = \langle\mathcal{I}\rangle_i \diamond \langle\mathcal{I}\rangle\,.
\end{aligned}$$

Definition. The *dimension of a triangulated category* $\mathcal{T}$ is the minimal d such that there is an object $M \in \mathcal{T}$ with $\mathcal{T} = \langle M\rangle_{d+1}$.

The following lemma is an immediate consequence of this definition:

Lemma 6 ([7, Lemma 3.4]). *Let $\mathcal{S} \xrightarrow{\mathtt{F}} \mathcal{T}$ be a triangle functor between two triangulated categories. Assume any object in $\mathcal{T}$ is a direct summand of an object in the image of $\mathtt{F}$. Then*

$$\dim \mathcal{T} \leq \dim \mathcal{S}.$$

Here we will only be looking at the case $\mathcal{T} = \Lambda\text{-}\underline{\mathrm{mod}}$ of the stable module category of a self-injective algebra Λ. Since short exact sequences in Λ-mod become triangles in $\Lambda\text{-}\underline{\mathrm{mod}}$, any N having an M resolution of length n is contained in $\langle M \rangle_{n+1}$. Therefore we have the following lemma:

Lemma 7 ([8, part of Proposition 3.6]). *Let Λ be a non-semisimple self-injective algebra. Then*

$$\operatorname{repdim} \Lambda \geq \dim \Lambda\text{-}\underline{\mathrm{mod}} + 2.$$

In particular this shows that the Theorem implies Corollary 1.

2. Outline of the proof the Theorem:

Let k be a field of characteristic p, V an n-dimensional k-vector space. Let

$$\Lambda = S(V)/(v^p \mid v \in V)$$

be the symmetric algebra modulo all p-th powers. Then $\Lambda \cong kC_p^n$.

We will use the following lemma to get a lower bound for the dimension of $\Lambda\text{-}\underline{\mathrm{mod}}$:

Lemma 8 ([6, Lemma 2.3] and [7, Lemma 4.11]). *Let $\mathcal{T}$ be a triangulated category and let*

$$H_1 \xrightarrow{f_1} H_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} H_{n-1} \xrightarrow{f_{n-1}} H_n$$

be a sequence of morphisms between cohomological functors $\mathcal{T}^{\mathrm{op}} \longrightarrow \mathtt{Ab}$. For every i, let $\mathcal{I}_i$ be a subcategory of $\mathcal{T}$ closed under shifts and on which f_i vanishes. Then $f_1 \cdots f_{n-1}$ vanishes on $\mathcal{I}_1 \diamond \cdots \diamond \mathcal{I}_{n-1}$.

Proof. This can easily be shown by induction on n. See [6] or [7]. □

To make use of it, we let $M \in \mathcal{O}b(\Lambda\text{-}\underline{\mathrm{mod}})$ such that M realizes the minimal d in the definition of $\dim \Lambda\text{-}\underline{\mathrm{mod}}$, so $\langle M \rangle_{\dim(\Lambda\text{-}\underline{\mathrm{mod}})+1} = \Lambda\text{-}\underline{\mathrm{mod}}$. We will find a module N, depending on M, and morphisms $f_i : N \longrightarrow N$ such that $\mathrm{Hom}(-, N) \xrightarrow{f_{i*}} \mathrm{Hom}(-, N)$ is 0 on $\langle M \rangle$ but $f_1 \cdots f_{n-1} \neq 0$. Thus, by the above lemma, $\langle M \rangle_{n-1}$ cannot be the entire stable category. Therefore $\dim(\Lambda\text{-}\underline{\mathrm{mod}}) + 1 > n - 1$.

3. Proof of the Theorem in the case k is infinite

In this section we assume the field k to be infinite. Whenever we are talking about open or closed sets we are referring to the classical Zariski topology.

Lemma 9. *Let $\mathcal{Y} \subset \text{mod-}\Lambda$ be a finite set of right Λ-modules. Then there is an open, nonempty subset $\mathcal{U} \subset V$, such that for any $u \in \mathcal{U}$, any $y \in Y \in \mathcal{Y}$, and any $1 \leq s < p$ we have*

$$yu = 0 \Rightarrow y \cdot \operatorname{Rad}^s \Lambda \subset Yu^s.$$

Proof. Fix $1 \leq s < p$ and $Y \in \mathcal{Y}$. Any $v \in V$ induces a linear map

$$\rho_v^s : Y \longrightarrow Y : y \longmapsto yv^s.$$

Now we can find a set with the desired property for our fixed s and Y:

$$\mathcal{U}_{s,Y} = \{v \in V \mid \operatorname{rk} \rho_v^s \text{ maximal}\}.$$

By choosing a basis for V and Y, the maps ρ_v^s induce a polynomial map

$$k^{\dim V} \cong V \longrightarrow \operatorname{End}_k Y \cong k^{\dim Y \times \dim Y}.$$

We then compose this map with taking subdeterminants of size r, where r is the maximal rank in the definition of $\mathcal{U}_{s,Y}$ above. This results in polynomial maps $k^{\dim V} \longrightarrow k$, such that $\mathcal{U}_{s,Y}$ is just the set where not all of these polynomials are zero. Thus $\mathcal{U}_{s,Y}$ is open and obviously it is non-empty.

Now fix $u \in \mathcal{U}_{s,Y}$. Let $\{a_i \mid 1 \leq i \leq A\}$ be a basis of $\operatorname{Ker} \rho_u^1$, complement it to a basis $\{a_i, b_j \mid 1 \leq i \leq A, 1 \leq j \leq B\}$ of $\operatorname{Ker} \rho_u^s$, and finally to a basis $\{a_i, b_j, c_l \mid 1 \leq i \leq A, 1 \leq j \leq B, 1 \leq l \leq C\}$ of Y.

Let $v \in V$. The rank of ρ_u^s is maximal, so in particular $\operatorname{rk} \rho_{u+\varepsilon v}^s \leq \operatorname{rk} \rho_u^s$ for any ε. Fix $1 \leq i \leq A$. Therefore, for all ε, the tuple

$$(a_i \rho_{u+\varepsilon v}^s, c_1 \rho_{u+\varepsilon v}^s, \ldots, c_C \rho_{u+\varepsilon v}^s)$$

is linearly dependent. Since $a_i u = 0$, for $\varepsilon \neq 0$ the tuple

$$(a_i \rho_v^s, c_1 \rho_{u+\varepsilon v}^s, \ldots, c_C \rho_{u+\varepsilon v}^s)$$

also is linearly dependent. But the set of all ε such that it is linearly dependent is closed (the subdeterminants of $\left(a_i \rho_v^s, c_l \rho_{u+\varepsilon v}^s\right)$ are polynomials in ε), hence it has to be all of k. Therefore, especially $(a_i \rho_v^s, c_l \rho_u^s \mid 1 \leq l \leq C)$ is linearly dependent, so $a_i \rho_v^s \in \langle c_l \rho_u^s \mid 1 \leq l \leq C\rangle$. But the v^s generate $\operatorname{Rad}^s \Lambda$ as a k-vector space, since s is strictly smaller than p.

Finally set $\mathcal{U} = \bigcap_{Y \in \mathcal{Y}} \bigcap_{s=1}^{p-1} \mathcal{U}_{s,Y}$. $\square$

Lemma 10. *Let $\mathcal{X} \subset \Lambda$-mod be a finite set of Λ-modules. Then there is an open, nonempty subset $\mathcal{U} \subset V$, such that for any $u \in \mathcal{U}$, any $\varphi \in \operatorname{Hom}_\Lambda(X, \Lambda)$ with $X \in \mathcal{X}$, and any $1 \leq s < p$ we have*

$$\varphi u = 0 \Rightarrow \varphi \cdot \operatorname{Rad}^s \Lambda \subset \operatorname{Hom}_\Lambda(X, \Lambda) u^s.$$

Proof. Set $\mathcal{Y} = \{\operatorname{Hom}_\Lambda(X, \Lambda) \mid X \in \mathcal{X}\}$ in Lemma 9. □

Proposition 11. *Let $\mathcal{X} \subset \Lambda$-mod be a finite set of Λ-modules. Let $u \in \mathcal{U}$ of Lemma 10, $N = \Lambda u^{p-1}$. Then, for any $X \in \mathcal{X}$ and any $f \in \operatorname{Rad}^{p-1} \Lambda$, any composition $X \longrightarrow N \xrightarrow{\cdot f} N$ factors through Λ-proj.*

Proof. Fix $X \in \mathcal{X}$. Since N is a submodule of Λ, we may identify

$$\begin{aligned} \operatorname{Hom}_\Lambda(X, N) &= \{\varphi \in \operatorname{Hom}_\Lambda(X, \Lambda) \mid \varphi(X) \subset N\} \\ &= \{\varphi \in \operatorname{Hom}_\Lambda(X, \Lambda) \mid \varphi(X) \subset \Lambda u^{p-1}\} \\ &= \{\varphi \in \operatorname{Hom}_\Lambda(X, \Lambda) \mid \varphi(X) u = 0\} \end{aligned}$$

Let $\varphi \in \operatorname{Hom}_\Lambda(X, N)$, that is $\varphi \in \operatorname{Hom}_\Lambda(X, \Lambda)$ with $\varphi u = 0$. Then, by Lemma 10, $\varphi f \in \operatorname{Hom}_\Lambda(X, \Lambda) u^{p-1}$.

The projective cover of N is induced by the endomorphism u^{p-1} of Λ. Therefore, the maps $X \longrightarrow N$ factoring through a projective module are exactly the elements of $\operatorname{Hom}_\Lambda(X, \Lambda) u^{p-1}$. □

Proposition 12. *Let $v \in V \setminus \{0\}$, $N = \Lambda v^{p-1}$. Then the composition*

$$N \twoheadrightarrow \operatorname{hd} N \cong k \cong \operatorname{Soc} N \rightarrowtail N$$

does not factor through Λ-proj.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis of V, $v_1 = v$. Then the composition above is (up to a scalar) multiplication with $\prod_{i \neq 1} v_i^{p-1}$. Let f be any map $\Lambda \longrightarrow N$. It is defined by the image of 1_Λ in $N \subset \Lambda$, which we will also call f. Now assume that the following diagram commutes.

$$\begin{tikzcd} N \arrow[r, "\prod_{i\neq 1} v_i^{p-1}"] \arrow[d, hook] & N \\ \Lambda \arrow[ur, "f"] & \end{tikzcd}$$

The image of v^{p-1} has to be the same on both ways, that is $v^{p-1} f = \prod_i v_i^{p-1}$. Therefore $f - \prod_{i \neq 1} v_i^{p-1}$ is a multiple of v. Since $f \in N$, f is also a multiple of v. So $\prod_{i \neq 1} v_i^{p-1}$ would be a multiple of v, but that is not true in Λ.

Therefore the morphism cannot factor through Λ-proj. □

Proof of the Theorem for infinite fields. Let M be a Λ-module realizing the minimal d in the definition of the dimension of the stable module category. Let $\mathcal{X} = \{M, \mho M\}$, where $\mho$ is the cosyzygy functor, which is the shift in the stable category. Then choose $u \in \mathcal{U}$ as in Lemma 10, and complement it to a basis $\{u_1, \dots u_n\}$ of V with $u_1 = u$. Let $N = \Lambda u^{p-1}$. Then we have the following sequence of cohomological functors $\Lambda\text{-}\underline{\text{mod}} \longrightarrow k\text{-mod}$.

$$\underline{\mathrm{Hom}}_\Lambda(-, N) \xrightarrow{\cdot u_2^{p-1}} \underline{\mathrm{Hom}}_\Lambda(-, N) \cdots \xrightarrow{\cdot u_n^{p-1}} \underline{\mathrm{Hom}}_\Lambda(-, N)$$

Its composition is nonzero by Proposition 12. So by Lemma 8 we only need to show that $\underline{\mathrm{Hom}}_\Lambda(\mho^i M, N) \xrightarrow{\cdot u_j^{p-1}} \underline{\mathrm{Hom}}_\Lambda(\mho^i M, N)$ is zero for any $i \in \mathbb{Z}$ and any $2 \le j \le n$.

The following diagrams have short exact rows and commute for any $f \in \Lambda$.

$$\begin{tikzcd}
N \arrow[r, tail] \arrow[d, "\cdot f"'] & \Lambda \arrow[r, two heads] \arrow[d, "\cdot f"'] & \Lambda/(u^{p-1}) \arrow[d, "\cdot f"'] \\
N \arrow[r, tail] & \Lambda \arrow[r, two heads] & \Lambda/(u^{p-1})
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda/(u^{p-1}) \arrow[r, tail, "\cdot u"] \arrow[d, "\cdot f"'] & \Lambda \arrow[r, two heads, "\cdot u^{p-1}"] \arrow[d, "\cdot f"'] & N \arrow[d, "\cdot f"'] \\
\Lambda/(u^{p-1}) \arrow[r, tail, "\cdot u"] & \Lambda \arrow[r, two heads, "\cdot u^{p-1}"] & N
\end{tikzcd}$$

Therefore $\mho N \cong \Lambda/(u^{p-1})$, $\mho(\Lambda/(u^{p-1})) \cong N$ and $\mho(\xrightarrow{\cdot f}) \cong \xrightarrow{\cdot f}$. So there is a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Hom}}_\Lambda(\mho^i M, N) \arrow[r, "\cdot u_j^{p-1}"] \arrow[d, "\cong"'] & \underline{\mathrm{Hom}}_\Lambda(\mho^i M, N) \arrow[d, "\cong"'] \\
\underline{\mathrm{Hom}}_\Lambda(\mho^{i+2n} M, \mho^{2n} N) \arrow[r, "\mho^{2n}(\cdot u_j^{p-1})"] \arrow[d, "\cong"'] & \underline{\mathrm{Hom}}_\Lambda(\mho^{i+2n} M, \mho^{2n} N) \arrow[d, "\cong"'] \\
\underline{\mathrm{Hom}}_\Lambda(\mho^{i+2n} M, N) \arrow[r, "\cdot u_j^{p-1}"] & \underline{\mathrm{Hom}}_\Lambda(\mho^{i+2n} M, N)
\end{tikzcd}$$

By choosing n appropriately we can get $i + 2n \in \{0, 1\}$, so $\mho^{i+2n} M \in \mathcal{X}$ and the claim follows from Proposition 11. $\square$

4. The case of a finite field

Now let k be finite, $\overline{k}$ an algebraic closure. Denote by $\overline{\Lambda} = \overline{k} \otimes_k \Lambda$ the induced algebra. For any Λ-module X let $\overline{X} = \overline{k} \otimes_k X \in \overline{\Lambda}$-mod, and for $\mathcal{X} \subset \Lambda\text{-mod}$

let $\overline{\mathcal{X}} = \{\overline{X} \mid X \in \mathcal{X}\}$. Whenever we are talking about an extension field $\widehat{k}$ of k let $\widehat{\Lambda}$, $\widehat{X}$ and $\widehat{\mathcal{X}}$ be the obvious variations of the above.

Lemma 13. *Let $\mathcal{X} \subset \Lambda$-mod be finite. Then there is a finite extension $\widehat{k}$ of k and $u \in \widehat{V}$ such that any $\varphi \in \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda})$ with $X \in \mathcal{X}$, and any $1 \leq s < p$ we have*

$$\varphi u = 0 \Rightarrow \varphi \cdot \operatorname{Rad}^s \widehat{\Lambda} \subset \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda}) u^{p-1}.$$

Proof. By Lemma 10 there is $u \in \overline{V}$ such that for any $\varphi \in \operatorname{Hom}_{\overline{\Lambda}}(\overline{X}, \overline{\Lambda})$ and any $1 \leq s < p$ we have $\varphi u = 0 \Rightarrow \varphi \cdot \operatorname{Rad}^s \overline{\Lambda} \subset \operatorname{Hom}_{\overline{\Lambda}}(\overline{X}, \overline{\Lambda}) u^s$. Choose $\widehat{k}$ finite over k such that $u \in \widehat{V}$. Since $\overline{k} \otimes_{\widehat{k}} \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda}) = \operatorname{Hom}_{\overline{\Lambda}}(\overline{X}, \overline{\Lambda})$ ([3, 29.5]), we may identify $\operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda})$ with the subset of morphisms in $\operatorname{Hom}_{\overline{\Lambda}}(\overline{X}, \overline{\Lambda})$ mapping $\widehat{X}$ to $\widehat{\Lambda}$. Therefore, for any $\varphi \in \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda})$ with $\varphi u = 0$, we have $\varphi \cdot \operatorname{Rad}^s \widehat{\Lambda} \subset \operatorname{Hom}_{\overline{\Lambda}}(\overline{X}, \overline{\Lambda}) u^s \cap \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda}) = \operatorname{Hom}_{\widehat{\Lambda}}(\widehat{X}, \widehat{\Lambda}) u^s$. The right equality holds because $\widehat{\Lambda}$ is a direct summand of $\overline{\Lambda}$ as $\widehat{\Lambda}$-module. $\square$

Proposition 14. *Let $\mathcal{X} \subset \Lambda$-mod be finite, u and $\widehat{k}$ as in Lemma 13, and set $N = \widehat{\Lambda} u^{p-1}$. Then, for any $X \in \mathcal{X}$ and any $f \in \operatorname{Rad}^{p-1} \widehat{\Lambda}$, any composition $\widehat{X} \longrightarrow N \xrightarrow{\cdot f} N$ of $\widehat{\Lambda}$-morphisms factors through $\widehat{\Lambda}$-proj.*

Proof. This is just the proof of Proposition 11, replacing the reference to Lemma 10 by a reference to Lemma 13. $\square$

Proposition 15. *Let $\mathcal{X} \subset \Lambda$-mod finite, u and $\widehat{k}$ as in Lemma 13, and set $N = \widehat{\Lambda} u^{p-1}$. Then, for any $X \in \mathcal{X}$ and any $f \in \operatorname{Rad}^{p-1} \widehat{\Lambda}$, any composition $X \longrightarrow N \xrightarrow{\cdot f} N$ of Λ-morphisms factors through Λ-proj.*

Proof. Any Λ-morphism $\varphi : X \longrightarrow N$ lifts to a $\widehat{\Lambda}$-morphism $\widehat{\varphi} : \widehat{X} \longrightarrow N$ as indicated in the following diagram.

$$\begin{tikzcd}
 & N \arrow[r, "u_i^{p-1}"] & N \\
 & \widehat{X} \arrow[u, "\widehat{\varphi}"] \arrow[r, dashed] & \widehat{\Lambda} \arrow[u] \\
X \arrow[uur, "\varphi"] \arrow[ur] & &
\end{tikzcd}$$

The dashed arrow now exists by Proposition 14 making the square commutative, so the composition factors through $\widehat{\Lambda}$. Clearly this is a projective Λ-module. $\square$

We note that Proposition 12 does not depend on the field at all. To see that the morphism is non-zero in Λ-$\underline{\mathrm{mod}}$, not just in $\widehat{\Lambda}$-$\underline{\mathrm{mod}}$, we need to recall the following lemma.

Lemma 16. *Let $\Lambda \subset \widehat{\Lambda}$ be finite dimensional algebras. Assume $\widehat{\Lambda}$ is projective as Λ-module and a direct summand of $\widehat{\Lambda} \otimes_\Lambda \widehat{\Lambda}$ as $(\widehat{\Lambda}, \widehat{\Lambda})$-bimodule. Then restriction induces an injective map $\underline{\mathrm{Hom}}_{\widehat{\Lambda}}(X,Y) \longrightarrow \underline{\mathrm{Hom}}_\Lambda(X,Y)$.*

Proof. Assume a $\widehat{\Lambda}$-morphism $\varphi : X \longrightarrow Y$ vanishes in $\underline{\mathrm{Hom}}_\Lambda(X,Y)$. Then it factors through a finite number of copies of Λ, as indicated in the following diagram.

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr] & & Y \\ & \Lambda^n \arrow[ur] & \end{tikzcd}$$

Let $\widehat{\Lambda} \underset{\pi}{\overset{\iota}{\rightleftarrows}} \widehat{\Lambda} \otimes_\Lambda \widehat{\Lambda}$ be the maps inducing the direct sum decomposition of $\widehat{\Lambda} \otimes_\Lambda \widehat{\Lambda}$. Tensoring the above diagram with $\widehat{\Lambda}$ we find the triangle in the following diagram. The rest of the diagram commutes since tensoring commutes with direct sums.

$$\begin{tikzcd} \widehat{\Lambda} \otimes_{\widehat{\Lambda}} X \arrow[r, equal] \arrow[d, "\iota \otimes 1"'] & X \arrow[rr, "\varphi"] & & Y \arrow[r, equal] & \widehat{\Lambda} \otimes_{\widehat{\Lambda}} Y \\ \widehat{\Lambda} \otimes_\Lambda \widehat{\Lambda} \otimes_{\widehat{\Lambda}} X \arrow[r, equal] & \widehat{\Lambda} \otimes_\Lambda X \arrow[rr, "1 \otimes \varphi"] \arrow[dr] & & \widehat{\Lambda} \otimes_\Lambda Y \arrow[r, equal] & \widehat{\Lambda} \otimes_\Lambda \widehat{\Lambda} \otimes_{\widehat{\Lambda}} Y \arrow[u, "\pi \otimes 1"'] \\ & & \widehat{\Lambda}^n \arrow[ur] & & \end{tikzcd}$$

Therefore φ also vanishes in $\underline{\mathrm{Hom}}_{\widehat{\Lambda}}(X,Y)$. ☐

Corollary 17. *Let $\widehat{k}$ be a finite extension field of k, $v \in \widehat{V}$. Then the composition $\psi : \widehat{\Lambda} v^{p-1} \longrightarrow \widehat{k} \longrightarrow \widehat{\Lambda} v^{p-1}$ does not factor through a projective Λ-module.*

Proof. Since k is a finite field the extension is separable. Therefore, by [3, Corollary 69.8] $\widehat{k}$ is a direct summand of $\widehat{k} \otimes_k \widehat{k}$ as $(\widehat{k}, \widehat{k})$-bimodule. Tensoring with Λ we find that the assumptions of Lemma 16 are satisfied. ☐

Proof of the Theorem for finite fields. The argument is the same as the one in the proof of the Theorem for infinite fields at the end of Section 3, with references to Propositions 11 and 12 replaced by references to Proposition 15 and Corollary 17 respectively. However, when we choose M as before and set $\mathcal{X} = \{M, \mho M\}$, we need to check that $\widehat{\mathcal{X}}$ is indeed $\{\widehat{M}, \mho_{\widehat{\Lambda}} \widehat{M}\}$, or, in other words, that taking cosyzygies and tensoring with $\widehat{k}$ commutes. This is the case because tensoring with $\widehat{k}$ is exact and $\widehat{k} \otimes \Lambda$ is projective over itself. $\square$

5. Applications

The applications here are obtained by applying the ideas of [8] to the more general result.

Proposition 18 (implicit in [8, Theorem 4.9]). *Let $H \leq G$ be finite groups. Then*

$$\dim kG\text{-}\underline{\mathrm{mod}} \geq \dim kH\text{-}\underline{\mathrm{mod}}\,.$$

Proof. We have the exact functors

$$\begin{aligned} \mathtt{res} : kG\text{-mod} &\longrightarrow kH\text{-mod} \text{ and} \\ \mathtt{ind} : kH\text{-mod} &\longrightarrow kG\text{-mod}\,. \end{aligned}$$

Since both of them map projective modules to projective ones there are induced triangle functors $kG\text{-}\underline{\mathrm{mod}} \rightleftarrows kH\text{-}\underline{\mathrm{mod}}$. By Lemma 6, it suffices to show that every kH-module is a direct summand of a module in the image of $\mathtt{res}$. But kH is a direct summand of kG as kH-kH-bimodule, so $1_{kH\text{-mod}}$ is a direct summand of $\mathtt{res} \circ \mathtt{ind}$. $\square$

Corollary 19. *Let G be a finite group, k a field of characteristic p, such that p devides the order of G. Then*

$$\mathrm{ll}\, kG \geq \mathrm{repdim}\, kG \geq \dim kG\text{-}\underline{\mathrm{mod}} + 2 > p\text{-rank}(G).$$

Proof. The first inequality is Lemma 5, the second one is Lemma 7. The third inequality follows from the Theorem and Proposition 18. $\square$

Proposition 20 ([8, Proposition 4.7]). *Let G be a finite group and B a block of kG. Let D be a defect group of B. Then* $\dim B\text{-}\underline{\mathrm{mod}} = \dim kD\text{-}\underline{\mathrm{mod}}$.

The proof is similar to the one of Proposition 18. See [8].

Corollary 21. *Let G be a finite group, B a non-semisimple block of kG,* $\mathrm{char}\, k = p$. *Let D be a defect group of B. Then*

$$\mathrm{ll}(B) \geq \mathrm{repdim}\, B \geq \dim B\text{-}\underline{\mathrm{mod}} + 2 = \dim D\text{-}\underline{\mathrm{mod}} + 2 > p\text{-rank}(D).$$

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