n-REPRESENTATION-FINITE ALGEBRAS AND n-APR TILTING

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ABSTRACT. We introduce the notion of n-representation-finiteness, generalizing representation-finite hereditary algebras. We establish the procedure of n-APR tilting, and show that it preserves n-representation-finiteness. We give some combinatorial description of this procedure, and use this to completely describe a class of n-representation-finite algebras called "type A".

CONTENTS

1. INTRODUCTION

One of the highlights in representation theory of algebras is given by representation-finite algebras, which provide a prototype of the use of functorial methods in representation theory. In 1971, Auslander gave a one-to-one correspondence between representation-finite algebras and Auslander algebras, which was a milestone in modern representation theory leading to later Auslander-Reiten theory. Many categorical properties of module categories can be understood as analogues of homological properties of Auslander algebras, and vice versa.

To study higher Auslander algebras, the notion of n-cluster tilting subcategories (=maximal $(n-1)$-orthogonal subcategories) was introduced in [Iya3], and a higher analogue of Auslander-Reiten theory was developed in a series of papers [Iya1, Iya2, IO], see also the survey paper

[Iya4]. Recent results (in particular [Iya1], but also this paper and [HI, HZ1, HZ2, HZ3, IO]) suggest, that n-cluster tilting modules behave very nicely if the algebra has global dimension n. In this paper, we call such algebras n-*representation-finite* and study them from the viewpoint of APR (=Auslander-Platzeck-Reiten) tilting theory (see [APR]).

For the case $n = 1$, 1-representation-finite algebras are representation-finite hereditary algebras. In the representation theory of path algebras, the notion of Bernstein-Gelfand-Ponomarev reflection functors play an important role. Nowadays they are formulated in terms of APR tilting modules from a functorial viewpoint (see [APR]). A main property is that the class of representation-finite hereditary algebras is closed under taking endomorphism algebras of APR tilting modules. By iterating the APR tilting process, we get a family of path algebras with the same underlying graph with different orientations

We follow this idea to construct from one given n-representation-finite algebra a family of n-representation-finite algebras. We introduce the general notion of n-APR tilting modules, which are explicitly constructed tilting modules associated with simple projective modules. The difference from the case $n = 1$ is that we need a certain vanishing condition of extension groups, but this is always satisfied if Λ is n-representation-finite.

In Section 3 we introduce n-APR tilting. We first introduce n-APR tilting modules. We give descriptions of the n-APR tilted algebra in terms of one-point (co)extensions (see Subsection 3.2, in particular Theorem 3.8), and for $n = 2$ also in terms of quivers with relations (see Subsection 3.3, in particular Theorem 3.11). Finally we introduce n-APR tilting in derived categories.

In Section 4 we apply n-APR tilts to n-representation-finite algebras. The first main result is that n-APR tilting preserves n-representation-finiteness (Theorems 4.2 and 4.7). In Subsections 4.3 and 4.4 we introduce the notions of slices and admissible sets in order to gain a better understanding of what algebras are iterated n-APR tilts of a given n-representation-finite algebra. More precisely we show that the iterated n-APR tilts are precisely the quotients of an explicitly constructed algebra by admissible sets (Theorem 4.23).

As an application of our general n-APR tilting theory, in Section 5, we give a family of n-representation-finite algebras by an explicit quivers with relations, which are iterated n-APR tilts of higher Auslander algebras given in [Iya1]. We call them n-representation-finite algebras of *type A*, since, for the case $n = 1$, they are path algebras of type A_s with arbitrary orientation. As shown in Section 4 in general, they form a family indexed by admissible sets. In contrast to the general setup, for type A we have a very simple combinatorial description of admissible sets (we call sets satisfying this description cuts until we can show that they coincide with admissible sets – see Definition 5.3 and Remark 5.13). Then the n-APR tiling process can be written purely combinatorially in terms of 'mutation' of admissible sets, and we can give a purely combinatorial proof of the fact that all admissible sets are transitive under successive mutation.

Summing up with results in [IO], we obtain self-injective weakly $(n + 1)$-representation-finite algebras as $(n + 1)$-preprojective algebras of n-representation-finite algebras of type A. This is a generalization of a result of Geiss, Leclerc, and Schröer [GLS1], saying that preprojective algebras of type A are weakly 2-representation-finite.

2. Background and notation

Throughout this paper we assume Λ to be a finite dimensional algebra over some field k. We denote by $\mathrm{mod}\,\Lambda$ the category of finite dimensional Λ-modules (all modules are left modules).

2.1. n-representation-finiteness.

Definition 2.1 (see [Iya1])**.** A module $M \in \mathrm{mod}\,\Lambda$ is called n-*cluster tilting object* if
$$\mathrm{add}\,M = \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(M, X) = 0 \,\forall i \in \{1, \ldots, n-1\}\}$$
$$= \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(X, M) = 0 \,\forall i \in \{1, \ldots, n-1\}\}.$$

Clearly such an M is a generator-cogenerator, and *n-rigid* in the sense that $\mathrm{Ext}^i_\Lambda(M, M) = 0$ $\forall i \in \{1, \ldots, n-1\}$.

Note that a 1-cluster tilting object is just an additive generator of the module category.

Definition 2.2. Let Λ be a finite dimensional algebra. We say Λ is *weakly n-representation-finite* if there exists an n-cluster tilting object in $\mathrm{mod}\,\Lambda$. If moreover $\mathrm{gl.dim}\,\Lambda \leq n$ we say that Λ is *n-representation-finite*.

The main aim of this paper is to understand better n-representation-finite algebras, and to construct larger families of examples.

For $n \geq 1$ we define the following functors:

$$\mathrm{Tr}_n := \mathrm{Tr}\,\Omega^{n-1} \colon \underline{\mathrm{mod}}\,\Lambda \longleftrightarrow \underline{\mathrm{mod}}\,\Lambda^{\mathrm{op}},$$

$$\tau_n := D\,\mathrm{Tr}_n \colon \underline{\mathrm{mod}}\,\Lambda \longrightarrow \overline{\mathrm{mod}}\,\Lambda,$$

$$\tau_n^- := \mathrm{Tr}_n\,D \colon \overline{\mathrm{mod}}\,\Lambda \longrightarrow \underline{\mathrm{mod}}\,\Lambda.$$

(See [ARS] for definitions and properties of the functors Tr, D, and τ_1.)

Proposition 2.3 ([Iya3]). *Let M be an n-cluster tilting object in $\mathrm{mod}\,\Lambda$.*

- *We have an equivalence $\tau_n \colon \underline{\mathrm{add}}\,M \to \overline{\mathrm{add}}\,M$ with a quasi-inverse $\tau_n^- \colon \overline{\mathrm{add}}\,M \to \underline{\mathrm{add}}\,M$.*
- *We have functorial isomorphisms $\underline{\mathrm{Hom}}_\Lambda(\tau_n^- Y, X) \cong D\,\mathrm{Ext}^n_\Lambda(X, Y) \cong \overline{\mathrm{Hom}}_\Lambda(Y, \tau_n X)$ for any $X, Y \in \mathrm{add}\,M$.*
- *If $\mathrm{gl.dim}\,\Lambda \leq n$ then $\mathrm{add}\,M = \mathrm{add}\{\tau_n^{-i}\Lambda \mid i \in \mathbb{N}\} = \mathrm{add}\{\tau_n^i D\Lambda \mid i \in \mathbb{N}\}$.*

We have the following criterion for n-representation-finiteness:

Proposition 2.4 ([Iya3, Theorem 5.1(3)]). *Let Λ be a finite dimensional algebra and $n \geq 1$. Let M be an n-rigid generator-cogenerator. The following conditions are equivalent.*

1. *M is an n-cluster tilting object in $\mathrm{mod}\,\Lambda$.*
2. *$\mathrm{gl.dim}\,\mathrm{End}_\Lambda(M) \leq n+1$.*
3. *For any indecomposable object $X \in \mathrm{add}\,M$, there exists an exact sequence*

$$0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \overset{f}{\longrightarrow} X$$

with $M_i \in \mathrm{add}\,M$ and a right almost split map f in $\mathrm{add}\,M$.

2.2. **Derived categories and n-cluster tilting.** Let Λ be a finite dimensional algebra of finite global dimension. We denote by

$$\mathcal{D}_\Lambda := \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)$$

the bounded derived category of $\mathrm{mod}\,\Lambda$. We denote by

$$\nu := D\Lambda \otimes^{\mathbf{L}}_\Lambda - \cong D\mathbf{R}\,\mathrm{Hom}_\Lambda(-, \Lambda) \colon \mathcal{D}_\Lambda \longrightarrow \mathcal{D}_\Lambda$$

the Nakayama-functor in $\mathcal{D}_\Lambda$. Clearly ν restricts to the usual Nakayama functor

$$\nu \colon \mathrm{add}\,\Lambda \longrightarrow \mathrm{add}\,D\Lambda.$$

We denote by ν_n the n-th desuspension of ν, that is $\nu_n = \nu[-n]$.

Note that if $\mathrm{gl.dim}\,\Lambda \leq n$ then $\tau_n^\pm = H^0(\nu_n^\pm -)$.

We set

$$\mathcal{U} = \mathcal{U}_\Lambda^n := \mathrm{add}\{\nu_n^i \Lambda \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}_\Lambda.$$

Theorem 2.5 ([Iya1, Theorem 1.22]). *Let Λ be an algebra of $\mathrm{gl.dim}\,\Lambda \leq n$, such that $\tau_n^{-i}\Lambda = 0$ for sufficiently large i. Then the category $\mathcal{U}$ is an n-cluster tilting subcategory of $\mathcal{D}_\Lambda$.*

In particular, if Λ is n-representation-finite, then $\mathcal{U}$ is n-cluster tilting.

We have the following criterion for n-representation-finiteness in terms of the derived category:

Theorem 2.6 ([IO, Theorem 3.1]). *Let Λ be an algebra with $\mathrm{gl.dim}\,\Lambda \leq n$. Then the following are equivalent.*

1. *Λ is n-representation-finite,*
2. *$D\Lambda \in \mathcal{U}$,*
3. *$\nu\mathcal{U} = \mathcal{U}$.*

2.3. n-**Amiot-cluster categories and** $(n+1)$-**preprojective algebras.**

Definition 2.7 (see [Ami1, Ami2])**.** We denote by $\mathcal{D}_\Lambda/\nu_n$ the *orbit category*, that is $\mathcal{Ob}\,\mathcal{D}_\Lambda/\nu_n = \mathcal{Ob}\,\mathcal{D}_\Lambda$, and

$$\operatorname{Hom}_{\mathcal{D}_\Lambda/\nu_n}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{D}_\Lambda}(X,\nu_n^i Y).$$

We denote by $\mathcal{C}_\Lambda^n$ the *n-Amiot-cluster category*, that is the triangulated hull (see [Ami1, Ami2] – we do not give a definition because for the purposes in this paper it does not matter if we think of the orbit category or the n-Amiot-cluster category). We denote by $\pi\colon \mathcal{D}_\Lambda \to \mathcal{C}_\Lambda^n$ the functor induced by projection onto the orbit category.

Lemma 2.8 (Amiot [Ami1, Ami2])**.** *Let Λ be an algebra with* $\operatorname{gl.dim}\Lambda \leq n$. *The n-Amiot-cluster category $\mathcal{C}_\Lambda^n$ is* Hom-*finite if and only if $\tau_n^{-i}\Lambda = 0$ for sufficiently large i.*
In particular it is Hom-*finite for any n-representation-finite algebra.*

Theorem 2.9 (Amiot [Ami1, Ami2])**.** *Let Λ be an algebra with* $\operatorname{gl.dim}\Lambda \leq n$ *such that $\mathcal{C}_\Lambda^n$ is* Hom-*finite. Then $\pi\Lambda$ is an n-cluster tilting object in $\mathcal{C}_\Lambda^n$.*

Observation 2.10. Note that $\operatorname{add}\pi\Lambda$ is the image of $\mathcal{U}$ under the functor of the derived category to the n-Amiot-cluster category as indicated in the following diagram.

$$
\begin{array}{ccc}
\mathcal{U} & \longrightarrow\!\!\!\!\rightarrow & \operatorname{add}\pi(\Lambda) \\
\big\downarrow & & \big\downarrow \\
\mathcal{D}_\Lambda & \xrightarrow{\ \ \pi\ \ } & \mathcal{C}_\Lambda^n
\end{array}
$$

Definition 2.11. Let Λ be an algebra with $\operatorname{gl.dim}\Lambda \leq n$. The $(n+1)$-*preprojective algebra* $\widehat{\Lambda}$ of Λ is the tensor algebra of the bimodule $\operatorname{Ext}_\Lambda^n(D\Lambda, \Lambda)$ over Λ

$$\widehat{\Lambda} := \mathrm{T}_\Lambda \operatorname{Ext}_\Lambda^n(D\Lambda, \Lambda).$$

(See [Kel1] or [Kel3] for a motivation for this name.)

Proposition 2.12. *The $(n+1)$-preprojective algebra $\widehat{\Lambda}$ is isomorphic to the endomorphism ring*

$$\operatorname{End}_{\mathcal{D}_\Lambda/\nu_n}(\Lambda) \cong \operatorname{End}_{\mathcal{C}_\Lambda^n}(\pi\Lambda).$$

Proof. The proof of [Ami2, Proposition 5.2.1] or [Ami1, Proposition 4.7] carries over. $\square$

3. n-APR TILTING

In this section we introduce n-APR tilting, and prove some general properties.

In Subsection 3.1 we introduce the notion of (weak) n-APR tilting modules and study their basic properties.

In Subsection 3.2 we will give a concrete description of the n-APR tilted algebra in terms of one-point (co)extensions. Namely, if Λ is a one-point coextension of $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ by a module M, then the n-APR tilt is the one-point extension of $\operatorname{End}_\Lambda(Q)^{\mathrm{op}}$ by $\operatorname{Tr}_{n-1} M$. This result will allow us to give an explicit description of the quivers and relations in case $n = 2$ in Subsection 3.3.

Finally, in Subsection 3.4, we introduce a version version of APR tilting in the language of derived categories.

3.1. n-**APR tilting modules.**

Definition 3.1. Let Λ be a basic finite dimensional algebra and $n \geq 1$. Let P be a simple projective Λ-module satisfying $\operatorname{Ext}_\Lambda^i(D\Lambda, P) = 0$ for any $0 \leq i < n$. We decompose $\Lambda = P \oplus Q$. We call

$$T := (\tau_n^- P) \oplus Q$$

the *weak n-APR tilting module* associated with P. If moreover $\operatorname{id} P = n$, then we call T an *n-APR tilting module*, and we call $\operatorname{End}_\Lambda(T)^{\mathrm{op}}$ an *n-APR tilt* of Λ.

Dually we define *(weak) n-APR cotilting modules.*

The more general notion of n-BB tilting modules has been introduced in [HX].

The following result shows that weak n-APR tilting modules are in fact tilting Λ-modules.

Theorem 3.2. *Let Λ be a basic finite dimensional algebra, and let T be a weak n-APR tilting Λ-module (as in Definition 3.1). Then T is a tilting Λ-module with $\mathrm{pd}_\Lambda T = n$.*

We also have the following useful properties.

Proposition 3.3. *Let $T = (\tau_n^- P) \oplus Q$ be a weak n-APR tilting Λ-module.*
  (1) $\mathrm{Ext}_\Lambda^i(T, \Lambda) = 0$ *for any* $0 < i < n$.
  (2) *If moreover T is n-APR tilting, then $\mathrm{Hom}_\Lambda(\tau_n^- P, \Lambda) = 0$.*

For the proof of Theorem 3.2 and Proposition 3.3, we use the following observation on tilting mutation due to Riedtmann-Schofield [RS].

Lemma 3.4 (Riedtmann-Schofield [RS]). *Let T be a Λ-module and $Y \xrightarrow{\ g\ } T' \xrightarrow{\ f\ } X$ an exact sequence with $T' \in \mathrm{add}\, T$. Then the following conditions are equivalent.*
  • $T \oplus X$ *is a tilting Λ-module and f is a right $(\mathrm{add}\, T)$-approximation.*
  • $T \oplus Y$ *is a tilting Λ-module and g is a left $(\mathrm{add}\, T)$-approximation.*

Proof of Theorem 3.2 and Proposition 3.3. Take a minimal injective resolution

$$\text{(1)} \qquad\qquad 0 \longrightarrow P \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_{n-1} \xrightarrow{\ g\ } I_n.$$

Applying D, we have an exact sequence

$$\text{(2)} \qquad\qquad DI_n \xrightarrow{\ Dg\ } DI_{n-1} \longrightarrow \cdots \longrightarrow DI_0 \longrightarrow DP \longrightarrow 0.$$

Applying the functor $(-)^* = \mathrm{Hom}_{\Lambda^{\mathrm{op}}}(-, \Lambda)$, to this projective resolution of DP we obtain a complex

$$0 \longrightarrow (DI_0)^* \longrightarrow \cdots \longrightarrow (DI_n)^* \longrightarrow 0.$$

By definition the homology in its rightmost term is $\tau_n^- P$, and since $\mathrm{Ext}_\Lambda^i(D\Lambda, P) = 0$ for $0 \le i < n$ all other homologies vanish. Since $(DI_0)^*$ is an indecomposable projective Λ-module with $\mathrm{top}(DI_0)^* = \mathrm{Soc}\, I_0 = P$, we have $(DI_0)^* = P$. Thus we have an exact sequence

$$\text{(3)} \qquad\qquad 0 \longrightarrow P \longrightarrow (DI_1)^* \longrightarrow \cdots \xrightarrow{(Dg)^*} (DI_n)^* \xrightarrow{\ f\ } \tau_n^- P \longrightarrow 0.$$

Thus we have $\mathrm{pd}_\Lambda T = n$. Since P is a simple projective Λ-module, we have $(DI_i)^* \in \mathrm{add}\, Q$ for $0 < i \le n$.

Applying the functor $(-)^*$ to the sequence (3), we have an exact sequence (2). Thus we have Proposition 3.3(1). If $\mathrm{id}\, P = n$, then g in (1) is surjective and Dg in (2) is injective. Since $(Dg)^{**} = Dg$ we have

$$\mathrm{Hom}_\Lambda(\tau_n^- P, \Lambda) = (\tau_n^- P)^* = (\mathrm{Cok}(Dg)^*)^* = \mathrm{Ker}(Dg)^{**} = \mathrm{Ker}\, Dg = 0.$$

Thus we have Proposition 3.3(2).

Note that we have a functorial isomorphism

$$\mathrm{Hom}_\Lambda((DI_i)^*, -) \cong (DI_i) \otimes_\Lambda -.$$

Apply the functors $- \otimes_\Lambda Q$ and $\mathrm{Hom}_\Lambda(-, Q)$ to Sequences (2) and (3) respectively, the above isomorphism gives rise to to a commutative diagram

$$
\begin{array}{ccccccccc}
(DI_n) \otimes_\Lambda Q & \longrightarrow & \cdots & \longrightarrow & (DI_1) \otimes_\Lambda Q & \longrightarrow & (DI_0) \otimes_\Lambda Q & \longrightarrow & 0 \\
\wr\| & & & & \wr\| & & \wr\| & & \\
\mathrm{Hom}_\Lambda((DI_n)^*, Q) & \longrightarrow & \cdots & \longrightarrow & \mathrm{Hom}_\Lambda((DI_1)^*, Q) & \longrightarrow & \mathrm{Hom}_\Lambda((DI_0)^*, Q) & \longrightarrow & 0
\end{array}
$$

of exact sequences. Thus (3) is a left $(\mathrm{add}\, Q)$-approximation sequence of P, and we have that T is a tilting Λ-module by using Lemma 3.4 repeatedly. $\qquad\square$

We recall the following result from tilting theory [Hap]: For a tilting Λ-module T with $\Gamma :=$ $\mathrm{End}_\Lambda(T)^{\mathrm{op}}$, we have functors

$$\mathbf{F} := \mathbf{R}\,\mathrm{Hom}_\Lambda(T,-)\colon \mathcal{D}_\Lambda \to \mathcal{D}_\Gamma,$$

$$\mathbf{F}_i := \mathrm{Ext}_\Lambda^i(T,-)\colon \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Gamma \quad (i \geq 0).$$

Put

$$\mathcal{F}_i := \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}_\Lambda^j(T,X) = 0 \;\text{ for any }\; j \neq i\},$$

$$\mathcal{X}_i := \{Y \in \mathrm{mod}\,\Gamma \mid \mathrm{Tor}_j^\Gamma(Y,T) = 0 \;\text{ for any }\; j \neq i\}.$$

Lemma 3.5 (Happel [Hap]).   • $\mathbf{F} = \mathbf{R}\,\mathrm{Hom}_\Lambda(T,-)\colon \mathcal{D}_\Lambda \to \mathcal{D}_\Gamma$ *is an equivalence.*
- *For any $i \geq 0$, we have an equivalence $\mathbf{F}_i := \mathrm{Ext}_\Lambda^i(T,-)\colon \mathcal{F}_i \to \mathcal{X}_i$ which is isomorphic to the restriction of $[i] \circ \mathbf{F}$.*
- *For any $X \in \mathcal{F}_0$, there exists an exact sequence*

$$0 \longrightarrow T_m \longrightarrow \cdots \longrightarrow T_0 \longrightarrow X \longrightarrow 0$$

with $T_i \in \mathrm{add}\,T$ and $m \leq \mathrm{gl.dim}\,\Lambda$.

We now prove the following result, saying that the class of algebras of global dimension at most n is closed under n-APR tilting.

Proposition 3.6. *If* $\mathrm{gl.dim}\,\Lambda = n$, *and T is an n-APR tilting Λ-module, then* $\mathrm{gl.dim}\,\Gamma = n$ *holds for* $\Gamma := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$.

Proof. We only have to show that $\mathrm{pd}_\Gamma(\mathrm{top}\,\mathbf{F}_0 X) \leq n$ for any indecomposable $X \in \mathrm{add}\,T$.

(i) First we consider the case $X \in \mathrm{add}\,Q$. Since $\mathrm{gl.dim}\,\Lambda = n$, we can take a minimal projective resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\;f\;} X \longrightarrow \mathrm{top}\,X \longrightarrow 0.$$

Since $\mathrm{Hom}_\Lambda(\tau_n^- P, \Lambda) = 0$ by Proposition 3.3(2), we have that any morphism $T \to X$ which is not a split epimorphism factors through f.

Applying $\mathrm{Hom}_\Lambda(T,-)$, we have an exact sequence

$$0 \longrightarrow \mathbf{F}_0 P_n \longrightarrow \cdots \longrightarrow \mathbf{F}_0 P_1 \xrightarrow{\;\mathbf{F}_0 f\;} \mathbf{F}_0 X$$

since we have $\mathrm{Ext}_\Lambda^i(T,\Lambda) = 0$ for any $0 < i < n$ by Proposition 3.3(1). Moreover the above observation implies $\mathrm{Cok}\,\mathbf{F}_0 f = \mathrm{top}\,\mathbf{F}_0 X$. Thus we have $\mathrm{pd}_\Gamma(\mathrm{top}\,\mathbf{F}_0 X) \leq n$.

(ii) Next we consider the case $X = \tau_n^- P$. We will show that $\mathrm{pd}_\Gamma(\mathrm{top}\,\mathbf{F}_0 \tau_n^- P)$ is precisely n. Applying $\mathbf{F}_0$ to the sequence (3) in the proof of Theorem 3.2, we have an exact sequence

$$0 = \mathbf{F}_0 P \longrightarrow \mathbf{F}_0(DI_1)^* \longrightarrow \cdots \longrightarrow \mathbf{F}_0(DI_n)^* \xrightarrow{\;\mathbf{F}_0 f\;} \mathbf{F}_0 \tau_n^- P$$

since we have $\mathrm{Ext}_\Lambda^i(T,\Lambda) = 0$ for any $0 < i < n$ by Proposition 3.3(1).

Since Q, $(DI_n)^*$, and $\tau_n^- P$ are in $\mathcal{F}_0$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Hom}_\Gamma(\mathbf{F}_0 Q, \mathbf{F}_0(DI_n)^*) & \xrightarrow{\;\mathbf{F}_0 f\;} & \mathrm{Hom}_\Gamma(\mathbf{F}_0 Q, \mathbf{F}_0 \tau_n^- P) \\
\wr\Vert & & \wr\Vert \\
\mathrm{Hom}_\Lambda(Q, (DI_n)^*) & \xrightarrow{\;\;f\;\;} & \mathrm{Hom}_\Lambda(Q, \tau_n^- P) \longrightarrow 0
\end{array}
$$

where the lower sequence is exact since Q is a projective Λ-module. Since

$$
\begin{aligned}
\mathrm{End}_\Gamma(\mathbf{F}_0 \tau_n^- P) &\cong \mathrm{End}_\Lambda(\tau_n^- P) \\
&= \underline{\mathrm{End}}_\Lambda(\tau_n^- P) && \text{(Proposition 3.3(2))} \\
&\cong \overline{\mathrm{End}}_\Lambda(\Omega^{-(n-1)} P) && \text{(AR-translation)} \\
&\cong \overline{\mathrm{End}}_\Lambda(P) && \text{(since } \mathrm{Ext}_\Lambda^i(D\Lambda, P) = 0 \;\forall i \in \{1, \ldots, n-1\}, \\
& && \qquad \text{see for instance [AB])},
\end{aligned}
$$

any non-zero endomorphism of $\mathbf{F}_0\tau_n^- P$ is an automorphism. Thus $\mathbf{F}_0 f$ is a right almost split map in $\operatorname{add}\Gamma$, and we have $\operatorname{Cok}\mathbf{F}_0 f = \operatorname{top}\mathbf{F}_0\tau_n^- P$ and $\operatorname{pd}_\Gamma(\operatorname{top}\mathbf{F}_0\tau_n^- P) = n$. $\qquad\square$

Later we shall use the following observation.

Lemma 3.7. *Under the circumstances in Theorem 3.2, we have the following.*

   (1) *$P \in \mathcal{F}_n$.*
   (2) *$\mathbf{F}_n P$ is a simple Γ-module. If $\operatorname{id} P = n$, then $\mathbf{F}_n P$ is an injective Γ-module.*

Proof. (1) Follows immediately from Proposition 3.3 and the fact that P is simple.
   (2) By AR-duality we have

$$\mathbf{F}_n P = \operatorname{Ext}_\Lambda^n(T, P) \cong \operatorname{Ext}_\Lambda^1(T, \Omega^{-(n-1)}P) \cong D\underline{\operatorname{Hom}}_\Lambda(\tau_n^- P, T).$$

First we show that $\mathbf{F}_n P$ is a simple Γ-module. Since $\mathbf{F}_n P \cong D\underline{\operatorname{Hom}}_\Lambda(\tau_n^- P, T) = D\underline{\operatorname{End}}_\Lambda(\tau_n^- P)$, any composition factor of the Γ-module $\mathbf{F}_n P$ is isomorphic. Thus we only have to show that $\operatorname{End}_\Gamma(\mathbf{F}_n P)$ is a division ring. By Lemma 3.5, we have $\operatorname{End}_\Gamma(\mathbf{F}_n P) \cong \operatorname{End}_\Lambda(P)$. Thus the assertion follows.

Next we show the second assertion. Since we have $\operatorname{Hom}_\Lambda(\tau_n^- P, \Lambda) = 0$ by Proposition 3.3, we have $\mathbf{F}_n P \cong D\underline{\operatorname{Hom}}_\Lambda(\tau_n^- P, T) = D\operatorname{Hom}_\Lambda(\tau_n^- P, T)$. Thus $\mathbf{F}_n P$ is an injective Γ-module. $\qquad\square$

3.2. n-APR tilting as one-point extension. Let Λ be a finite dimensional algebra, $M \in \operatorname{mod}\Lambda^{\operatorname{op}}$ and $N \in \operatorname{mod}\Lambda$. Slightly more general than "classical" one-point (co)extensions, we consider the algebras $\begin{pmatrix} K & M \\ & \Lambda \end{pmatrix}$ and $\begin{pmatrix} K & \\ N & \Lambda \end{pmatrix}$ if K is a finite skew-field extension of our base field k, such that $K \subseteq \operatorname{End}_{\Lambda^{\operatorname{op}}}(M)$ and $K \subseteq \operatorname{End}_\Lambda(N)^{\operatorname{op}}$, respectively.

Now let Λ be a basic algebra which has a simple projective module P. We set $K_P = \operatorname{End}_\Lambda(P)^{\operatorname{op}}$. Let Q be the direct sum over the other indecomposable projective Λ-modules, that is $\Lambda = P \oplus Q$. We set $\Lambda_P := \operatorname{End}_\Lambda(Q)^{\operatorname{op}}$ and $M_P := \operatorname{Hom}_\Lambda(P, Q) \in \operatorname{mod}(K_P \otimes_k \Lambda_P^{\operatorname{op}})$. Then we have an isomorphism $\Lambda \cong \begin{pmatrix} K_P & M_P \\ & \Lambda_P \end{pmatrix}$ and P is identified with the module $\begin{pmatrix} K_P \\ 0 \end{pmatrix}$.

Theorem 3.8. *Assume Λ is a basic finite dimensional algebra with simple projective module P, and $n > 1$. Then the following are equivalent:*

   (i) *P gives rise to an n-APR tilting module,*
   (ii) *M_P has the following properties*
      • *$\operatorname{pd}_{\Lambda_P^{\operatorname{op}}} M_P = n - 1$,*
      • *$\operatorname{Ext}_{\Lambda_P^{\operatorname{op}}}^i(M_P, \Lambda_P) = 0$ for $0 \leq i \leq n - 2$,*
      • *$\operatorname{Ext}_{\Lambda_P^{\operatorname{op}}}^i(M_P, M_P) = 0$ for $1 \leq i \leq n - 2$, and*
      • *$\operatorname{End}_{\Lambda_P^{\operatorname{op}}}(M_P) = K_P$.*

Moreover, if the above conditions are satisfied and $\Gamma = \operatorname{End}_\Lambda((\tau_n^- P) \oplus Q)^{\operatorname{op}}$, then

$$\Gamma \cong \begin{pmatrix} K_P & \\ \operatorname{Tr}_{n-1} M_P & \Lambda_P \end{pmatrix}$$

Remark 3.9. The object $\operatorname{Tr}_{n-1} M_P$ is uniquely determined only up to projective summands. In this section we always understand $\operatorname{Tr}_{n-1} M_P$ to be constructed using a minimal projective resolution, or, equivalently, $\operatorname{Tr}_{n-1} M_P$ to not have any non-zero projective summands.

Proof of Theorem 3.8. Let

$$0 \longrightarrow DM_P \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

be an injective resolution of the Λ_P-module DM_P. Then the injective resolution of the Λ-module $P = \begin{pmatrix} K_P \\ 0 \end{pmatrix}$ is

$$0 \longrightarrow \begin{pmatrix} K_P \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} K_P \\ DM_P \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ I_0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ I_1 \end{pmatrix} \longrightarrow \cdots.$$

Hence $\operatorname{pd}_{\Lambda_P^{\operatorname{op}}} M_P = \operatorname{id}_{\Lambda_P} DM_P = \operatorname{id}_\Lambda P - 1$. In particular we have $\operatorname{id}_\Lambda P = n \iff \operatorname{pd}_{\Lambda_P^{\operatorname{op}}} M_P = n - 1$.

Moreover, for any $i \geq 1$ and any $I \in \operatorname{inj}\Lambda_P$ we have

$$\operatorname{Ext}^i_\Lambda(\binom{0}{I}, P) = \operatorname{Ext}^{i-1}_{\Lambda_P}(I, DM_P)$$
$$= \operatorname{Ext}^{i-1}_{\Lambda_P^{\mathrm{op}}}(M_P, DI).$$

(Note that the first equality also holds for $n = 1$, since there are no non-zero maps from $\binom{0}{I}$ to the injective Λ-module $\binom{K_P}{DM_P}$.)

Finally we look at extensions between P and the corresponding injective module. For $i > 1$ we have

$$\operatorname{Ext}^i_\Lambda(\binom{K_P}{DM_P}, P) = \operatorname{Ext}^{i-1}_\Lambda(\binom{K_P}{DM_P}, \binom{0}{DM_P}))$$
$$= \operatorname{Ext}^{i-1}_{\Lambda_P}(DM_P, DM_P)$$
$$= \operatorname{Ext}^{i-1}_{\Lambda_P^{\mathrm{op}}}(M_P, M_P).$$

For $i = 1$ we obtain

$$\operatorname{Ext}^1_\Lambda(\binom{K_P}{DM_P}, P) = \operatorname{Hom}_\Lambda(\binom{K_P}{DM_P}, \binom{0}{DM_P})) / \left(\operatorname{End}_\Lambda(\binom{K_P}{DM_P}) \cdot [\binom{K_P}{DM_P} \twoheadrightarrow \binom{0}{DM_P}] \right)$$
$$\cong \operatorname{End}_{\Lambda_P}(M_P)/K_P.$$

This proves the equivalence of (i) and (ii).

For the second claim note that by Proposition 3.3(2) we have $\operatorname{Hom}_\Lambda(\tau_n^- P, Q) = 0$. Therefore it only remains to verify $\operatorname{Hom}_\Lambda(Q, \tau_n^- P) = \operatorname{Tr}_{n-1} M_P$ and $\operatorname{End}_\Lambda(\tau_n^- P) = K_P$. This follows by looking at the injective resolution of P above and applying D to it to obtain (a projective resolution of) $\tau_n^- P$. $\qquad\square$

3.3. Quivers for 2-APR tilts.
In this subsection we give an explicit description of 2-APR tilts in terms of quivers with relations.

Remark 3.10. For comparison, recall the classical case ($n = 1$): Assume $\Lambda = kQ/(R)$, and the set of relations R is minimal ($\forall r \in R \colon r \notin (R \setminus \{r\})$). Simple projective modules correspond to sources of Q. Let P be a simple projective, $i \in Q_0$ the corresponding vertex. Then $\operatorname{id} P = 1 \iff$ no relation in R involves a path starting in i. In this situation we have

$$\Lambda_P = k[Q \setminus \{i\}]/(R), \qquad M_P = \bigoplus_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} P_j^*, \quad \text{and} \quad \Gamma = kQ'/(R),$$

where Q' is the quiver obtained from Q by reversing all arrows starting in i.

For $n = 2$ we have to take into account the second cosyzygy of P, which corresponds to relations involving the corresponding vertex of the quiver.

Let $\Lambda = kQ/(R)$ be a finite dimensional k-algebra presented by a quiver $Q = (Q_0, Q_1)$ with relations R (which is assumed to be a minimal set of relations). Let P be a simple projective Λ-module associated to a source i of Q. We define a quiver $Q' = (Q_0', Q_1')$ with relations R' as follows.

$$Q_0' = Q_0,$$
$$Q_1' = \{a \in Q_1 \mid \mathfrak{s}(a) \neq i\} \coprod \{r^* \colon \mathfrak{e}(r) \to i \mid r \in R, \ \mathfrak{s}(r) = i\}$$

where r^* is a new arrow associated to each $r \in R$ with $\mathfrak{s}(r) = i$. We write $r \in R$ with $\mathfrak{s}(r) = i$ as

$$r = \sum_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} a r_a,$$

and define $a^* \in kQ'$ for each $a \in Q_1$ with $\mathfrak{s}(a) = i$ by

$$a^* := \sum_{\substack{r \in R \\ \mathfrak{s}(r)=i}} r_a r^* \in kQ'.$$

Now we define a set R' of relations on Q' by

$$R' = \{r \in R \mid \mathfrak{s}(r) \neq i\} \coprod \{a^* \colon \mathfrak{e}(a) \to i \mid a \in Q_1, \ \mathfrak{s}(a) = i\}.$$

Theorem 3.11. *Let $\Lambda = kQ/(R)$ and P a simple projective Λ-module. Assume that P gives rise to a 2-APR tilting Λ-module T. Then $\operatorname{End}_\Lambda(T)$ is isomorphic to $kQ'/(R')$ (with Q' and R' as explained above).*

Remark 3.12. Roughly speaking, Theorem 3.11 means that arrows in Q starting in i become relations, and relations become arrows.

Let us start with the following general observation.

Observation 3.13. Let $\Delta = kQ/(R)$ be a finite dimensional k-algebra presented by a quiver Q with relations R. Let M be a Δ-module with a projective presentation

$$\bigoplus_{1 \leq n \leq N} P_{j_n} \xrightarrow{(r_{n\ell})} \bigoplus_{1 \leq \ell \leq L} P_{i_\ell} \longrightarrow M \longrightarrow 0$$

for $r_{n\ell} \in kQ$. Then the one-point coextension algebra $\left(\begin{smallmatrix} k \\ M \ \Delta \end{smallmatrix}\right)$ is isomorphic to $k\widetilde{Q}/(\widetilde{R})$ for the quiver $\widetilde{Q} = (\widetilde{Q}_0, \widetilde{Q}_1)$ with relations $\widetilde{R}$ defined by

$$\widetilde{Q}_0 = Q_0 \coprod \{i\},$$
$$\widetilde{Q}_1 = Q_1 \coprod \{a_\ell \colon i_\ell \to i \mid 1 \leq \ell \leq L\},$$
$$\widetilde{R} = R \coprod \{ \sum_{1 \leq \ell \leq L} r_{n\ell} a_\ell \mid 1 \leq n \leq N \}.$$

Now we are ready to prove Theorem 3.11.

Proof of Theorem 3.11. We can write $\Lambda = \left(\begin{smallmatrix} k & M_P \\ & \Lambda_P \end{smallmatrix}\right)$ as in Subsection 3.2. Let Q_P be the quiver obtained from Q by removing the vertex i, and $R_P := \{r \in R \mid \mathfrak{s}(r) \neq i\}$. Then we have

$$(4) \qquad \Lambda_P \cong kQ_P/(R_P).$$

By Theorem 3.8 we have

$$(5) \qquad \operatorname{End}_\Lambda(T) = \begin{pmatrix} \Lambda_P & \\ \operatorname{Tr} M_P & k \end{pmatrix}.$$

Since we have a minimal projective resolution

$$\bigoplus_{\substack{r \in R \\ \mathfrak{s}(r)=i}} P^*_{\mathfrak{e}(r)} \xrightarrow{(r_a)} \bigoplus_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} P^*_{\mathfrak{e}(a)} \longrightarrow M_P \longrightarrow 0$$

of the Λ_P^{op}-module M_P, we have a projective resolution

$$(6) \qquad \bigoplus_{\substack{a \in Q_1 \\ \mathfrak{s}(a)=i}} P_{\mathfrak{e}(a)} \xrightarrow{(r_a)} \bigoplus_{\substack{r \in R \\ \mathfrak{s}(r)=i}} P_{\mathfrak{e}(r)} \longrightarrow \operatorname{Tr} M_P \longrightarrow 0$$

of the Λ_P-module $\operatorname{Tr} M_P$. Applying Observation 3.13 to the one-point coextension (5), we have the assertion from (4) and (6). $\qquad\square$

For example we could take Q to be the Auslander Reiten quiver of A_3 and R to be the mesh relations. Then $kQ/(R)$ is the Auslander algebra. See Tables 1 (linear oriented A_3) and 2 (non-linear oriented A_3) for the iterated 2-APR tilts of these Auslander algebras. In the pictures a downward line is a 2-APR tilt. Vertices labeled T are sources that have an associated 2-APR tilt, vertices labeled C are sinks having an associated 2-APR cotilt. Sources and sinks that do not admit a 2-APR tilt or cotilt are marked X.
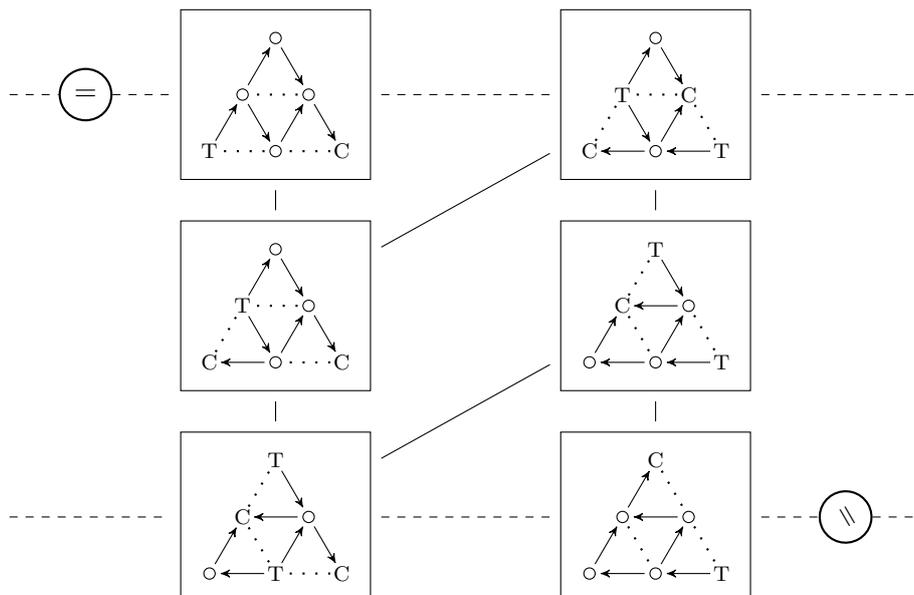
TABLE 1. Iterated 2-APR tilts of the Auslander algebra of linear oriented A_3

Note that there are no Xes occurring in Table 1. In fact, by [Iya1, Theorem 1.18] (see Theorem 5.7) the Auslander algebras of linear oriented A_n are 2-representation-finite, and hence every source and sink has an associated 2-APR tilt and cotilt, respectively. We will more closely investigate n-APR tilts on n-representation-finite algebras in Section 4, and the particular algebras coming from linear oriented A_n in Section 5.

3.4. n-**APR tilting complexes.** As in Section 2.2, throughout this section we assume Λ to be a basic finite dimensional algebra of finite global dimension. We will constantly use the functors ν and ν_n introduced in the first paragraph of Section 2.2.

Definition 3.14. Let $n \geq 1$, and let $\Lambda = P \oplus Q$ be any direct summand decomposition of the Λ-module Λ such that

   (1) $\mathrm{Hom}_\Lambda(Q, P) = 0$, and
   (2) $\mathrm{Ext}^i_\Lambda(\nu Q, P) = 0$ for any $0 < i \neq n$,

where Q is such that $\Lambda = P \oplus Q$. Clearly (1) implies $\mathrm{Hom}_\Lambda(\nu Q, P) = 0$, so (2) also holds for $i = 0$.

   We call
$$T := (\nu_n^- P) \oplus Q$$
the n-*APR tilting complex* associated with P.

   By abuse of notation (see Remark 3.15 below for a justification), we also call $\mathrm{End}_{\mathcal{D}_\Lambda}(T)^{\mathrm{op}}$ an n-*APR tilt* of Λ.

Remark 3.15.   (1) Any n-APR tilting module $(\tau_n^- P) \oplus Q$ in the sense of Definition 3.1 is an n-APR tilting comples, since in that case $\nu_n^- P = \tau_n^- P$ holds (under the assumption that Λ has finite global dimension).
   (2) Weak n-APR tilting modules are in general not n-APR tilting complexes.

Remark 3.16. In the setup of Definition 3.14 there is no big difference between tilting and cotilting: the n-APR tilting complex $(\nu_n^- P) \oplus Q$ associated to P, and the n-APR cotilting complex $\nu P \oplus \nu_n \nu Q$ associated to (the injective module) νQ are mapped to each other by the autoequivalence $\nu_n \nu$ of the derived category.

   In the rest of this subsection we will show that n-APR tilting complexes are indeed tilting complexes, and that they preserve gl.dim $\leq n$.
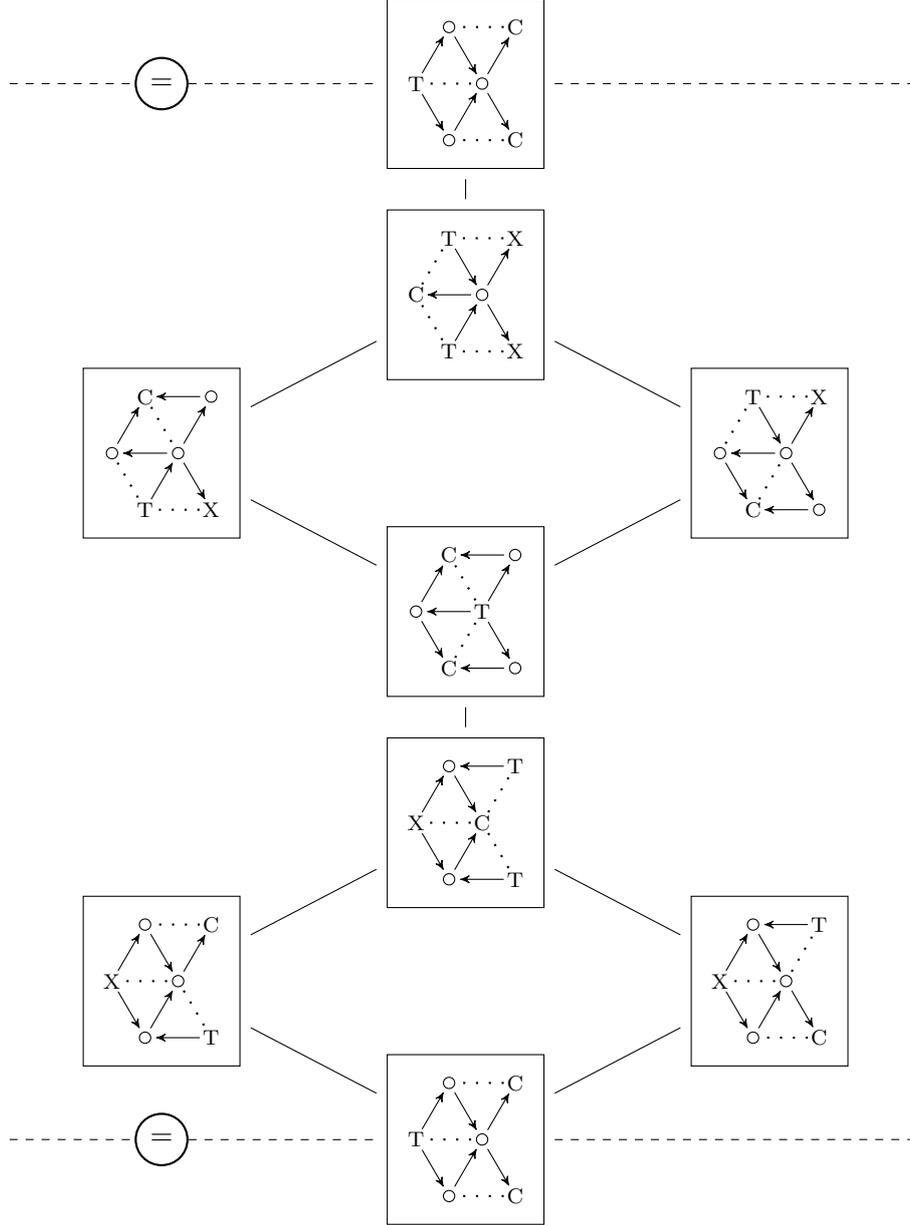
TABLE 2. Iterated 2-APR tilts of the Auslander algebra of non-linear oriented A_3

Theorem 3.17. *Let Λ be an algebra of finite global dimension, and T an n-APR tilting complex (as in Definition 3.14). Then T is a tilting complex in $\mathcal{D}_\Lambda$.*

Remark 3.18. More generally, in Theorem 3.17 it is possible to replace the assumption that Λ has finite global dimension by the weaker assumption that P has finite injective dimension. (In this case $\nu_n^- P = \mathbf{R}\operatorname{Hom}_\Lambda(D\Lambda, P)$ is still in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj}\Lambda)$, the homotopy category of complexes of finitely generated projective Λ-modules.)

Proof of Theorem 3.17. We have to check that T has no self-extensions, and that T generates the derived category $\mathcal{D}_\Lambda$. We first check that T has no self-extensions. Clearly for all $i \neq 0$ we have $\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu_n^- P, \nu_n^- P[i]) = 0$ and $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Q, Q[i]) = 0$. Moreover

$$\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu_n^- P, Q[i]) = \operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu^- P, Q[i-n]) = D\operatorname{Hom}_{\mathcal{D}_\Lambda}(Q[i-n], P)$$
$$= 0 \quad \forall i \in \mathbb{Z}.$$

Finally $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Q, \nu_n^- P[i]) = \operatorname{Ext}_\Lambda^{n+i}(\nu Q, P)$, which vanishes for $i \neq 0$ by assumption (2) of the definition.

Now we prove that T generates $\mathcal{D}_\Lambda$. Let $X \in \mathcal{D}_\Lambda$ such that $\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu^- P[i], X) = 0$ and $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Q[i], X) = 0$ for all i. By the latter property we see that the homology of X does not contain any composition factors in $\operatorname{add}(\operatorname{top} Q)$. We can assume that X is a complex

$$\cdots \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \xrightarrow{d^{i+1}} \cdots$$

in $\mathcal{K}^{\mathrm{b}}(\operatorname{proj} \Lambda)$, such that $\operatorname{Im} d^i \subseteq \operatorname{Rad} X^{i+1}$ for any i.

Assume there is an i such that $X^i \notin \operatorname{add} P$. Let i_M be the maximal i with this property. Let $Q' \in \operatorname{add} Q$ be a non-zero summand of X^{i_M}. Since $X^{i_M+1} \in \operatorname{add} P$ by our choice of i_M, we have $\operatorname{Hom}_\Lambda(Q', X^{i_M+1}) \in \operatorname{add} \operatorname{Hom}_\Lambda(Q, P) = 0$ (see Definition 3.14(1)). Hence we have $Q' \subseteq \operatorname{Ker} d^{i_M}$. Since $\operatorname{Im} d^{i_M-1} \subseteq \operatorname{Rad} X^{i_M}$, we have $Q' \not\subseteq \operatorname{Im} d^{i_M-1}$, and hence $\operatorname{Hom}_{\mathcal{D}_\Lambda}(Q', X[i_M]) \neq 0$, a contradiction to our choice of X. Consequently, we have $X \in \mathcal{K}^{\mathrm{b}}(\operatorname{add} P)$.

Now we assume $X \neq 0$. Let i_N be the minimal i such that $X^i \neq 0$. Since $X^{i_N} \in \operatorname{add} P$ we have $\operatorname{Hom}_{\mathcal{D}_\Lambda}(X[i_N], P) \neq 0$. This is a contradiction to our choice of X, since $\operatorname{Hom}_{\mathcal{D}_\Lambda}(X[i_N], P) = D \operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu_n^- P, X[n + i_N])$. $\qquad\square$

The following result generalizes Proposition 3.6 to the setup af n-APR tilting complexes.

Proposition 3.19. *If $\operatorname{gl.dim} \Lambda \leq n$ and T is an n-APR tilting complex in $\mathcal{D}_\Lambda$ then, for $\Gamma := \operatorname{End}_{\mathcal{D}_\Lambda}(T)^{\mathrm{op}}$ we have $\operatorname{gl.dim} \Gamma \leq n$.*

Proof. By [Ric] the algebra Γ has finite global dimension, and hence

$$\begin{aligned}
\operatorname{gl.dim} \Gamma &= \max\{i \mid \operatorname{Ext}_\Gamma^i(\nu\Gamma, \Gamma) \neq 0\} \\
&= \max\{i \mid \operatorname{Hom}_{\mathcal{D}_\Gamma}(\nu\Gamma, \Gamma[i]) \neq 0\} \\
&= \max\{i \mid \operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu T, T[i]) \neq 0\}.
\end{aligned}$$

Clearly $\operatorname{gl.dim} \Lambda \leq n$ implies that for $i > n$ we have $\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu\nu_n^- P, \nu_n^- P[i]) = \operatorname{Ext}_\Lambda^i(\nu P, P) = 0$, and $\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu Q, Q[i]) = \operatorname{Ext}_\Lambda^i(\nu Q, Q) = 0$. We have

$$\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu\nu_n^- P, Q[i]) = \operatorname{Hom}_{\mathcal{D}_\Lambda}(P, Q[i - n]),$$

which is non-zero only for $i = n$. Finally

$$\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu Q, \nu_n^- P[i]) = \operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu^2 Q, P[n + i]).$$

Since $\nu Q \in \operatorname{mod} \Lambda$ and $\operatorname{gl.dim} \Lambda \leq n$ it follows that $\nu^2 Q$ has non-zero homology only in degrees $-n, \ldots, 0$. Hence the above Hom-space vanishes for $i > n$, since $\operatorname{gl.dim} \Lambda \leq n$.

Summing up we obtain $\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu T, T[i]) = 0$ for $i > n$, which implies the claim of the theorem by the remark at the beginning of the proof. $\qquad\square$

Recall the definition of the subcategory

$$\mathcal{U}_\Lambda^n = \operatorname{add}\{\nu_n^i \Lambda \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}_\Lambda$$

given in Section 2.2.

Proposition 3.20. *Let Λ be n-representation-finite, and T an n-APR tilting complex in $\mathcal{D}_\Lambda$. Let $\Gamma := \operatorname{End}_{\mathcal{D}_\Lambda}(T)^{\mathrm{op}}$. Then the derived equivalence $\mathbf{R}\operatorname{Hom}_\Lambda(T, -) \colon \mathcal{D}_\Lambda \to \mathcal{D}_\Gamma$ (see [Kel2]) induces an equivalence $\mathcal{U}_\Lambda^n \to \mathcal{U}_\Gamma^n$.*

Proof. This is clear since the derived equivalence $\mathbf{R}\operatorname{Hom}_\Lambda(T, -)$ commutes with ν_n and $T \in \mathcal{U}_\Lambda^n$. $\qquad\square$

An application of Proposition 3.20 we will use in Subsection 4.4 is the following.

Proposition 3.21. *The $(n+1)$-preprojective algebra (see Definition 2.11) is invariant under n-APR tilts.*

Proof. By Propositions 3.20 and 2.12 we have

$$\widehat{\Lambda} = \operatorname{End}_{\mathcal{U}_\Lambda^n/\nu_n}(\Lambda) = \operatorname{End}_{\mathcal{U}_\Lambda^n/\nu_n}(T) = \operatorname{End}_{\mathcal{U}_\Gamma^n/\nu_n}(\Gamma) = \widehat{\Gamma}. \qquad\square$$

4. n-APR tilting for n-representation-finite algebras

In this section we study the effect of n-APR tilts on n-representation-finite algebras.

The first main result is that n-APR tilting preserves n-representation-finiteness (Theorems 4.2 and 4.7). We give two independant proofs for this fact. In Subsection 4.1 we study n-APR tilting modules for n-representation-finite algebras. We give an explicit description of a cluster tilting object in the new module category in terms of the cluster tilting object of the original algebra (Theorem 4.2). In Subsection 4.2 we give an independant proof (which is less explicit and relies heavily on a result from [IO]) that the more general procedure of tilting in n-APR tilting complexes also preserves n-representation-finiteness.

In Subsections 4.3 and 4.4 we introduce the notions of slices and admissible sets, which classify, for a given n-representation-finite algebra, all iterated n-APR tilts (see Theorem 4.23).

Throughout this Section, let Λ be an n-representation-finite algebra. For simplicity of notation we assume Λ to be basic.

4.1. n-APR tilting modules preserve n-representation-finiteness.

The following proposition shows that the setup of n-representation-finite algebras is particularly well-suited for looking at n-APR tilts.

Observation 4.1. (1) Any simple projective and non-injective Λ-module P admits an n-APR tilting Λ-module.
  (2) Any simple injective and non-projective Λ-module I admits an n-APR cotilting Λ-module.

Proof. We have $\operatorname{id} P \leq n$ by $\operatorname{gl.dim} \Lambda \leq n$. Since M is an n-rigid generator-cogenerator, we have $\operatorname{Ext}^i_\Lambda(D\Lambda, P) = 0$ for any $0 < i < n$. This proves (1), the proof of (2) is dual. $\square$

Throughout this subsection, we denote by M the unique basic n-cluster tilting object in $\operatorname{mod} \Lambda$ (see the last point of Proposition 2.3).

Now let P be a simple projective and non-injective Λ-module. We decompose $\Lambda = P \oplus Q$. Since $P \in \operatorname{add} M$ we can also decompose $M = P \oplus M'$. By Observation 4.1 we have an n-APR tilting Λ-module $T := (\tau_n^- P) \oplus Q$.

Theorem 4.2. *Under the above circumstances, we have the following.*
  (1) $T \in \operatorname{add} M$.
  (2) $\Gamma := \operatorname{End}_\Lambda(T)^{\operatorname{op}}$ *is an n-representation-finite algebra with n-cluster tilting object $N :=$* $\operatorname{Hom}_\Lambda(T, M') \oplus \operatorname{Ext}^n_\Lambda(T, P)$.

Before we prove the theorem let us note the following immediate consequence.

Corollary 4.3. *Any iterated n-APR tilt of an n-representation-finite algebra is n-representation-finite.*

In the rest we shall show Theorem 4.2. The assertion (1) follows immediately from the first part of Proposition 2.3.

Proposition 3.6 proves that $\operatorname{gl.dim} \Gamma = n$ in Theorem 4.2. We shall show that N in Theorem 4.2(2) is an n-cluster tilting object. We will use the subcategories $\mathcal{F}_i \subseteq \operatorname{mod} \Lambda$, and the functors $\mathbf{F}_i$ which were introduced in Section 3.1 (see in particular Lemma 3.5).

Lemma 4.4. $M' \in \mathcal{F}_0$.

Proof. By Theorem 4.2(1) we know that $T \in \operatorname{add} M$. Hence, since M is an n-rigid Λ-module, we have $\operatorname{Ext}^i_\Lambda(T, M) = 0$ for any $0 < i < n$. Since $\operatorname{gl.dim} \Lambda \leq n$, we only have to check $\operatorname{Ext}^n_\Lambda(T, M') = 0$. Of course, we have $\operatorname{Ext}^n_\Lambda(Q, M') = 0$ since Q is projective. By Proposition 2.3(2), we have $\operatorname{Ext}^n_\Lambda(\tau_n^- P, M') \cong \overline{\operatorname{Hom}}_\Lambda(M', P)$, and the latter Hom-space vanishes since P is simple projective. $\square$

Lemma 4.5. $N = \mathbf{F}_0 M' \oplus \mathbf{F}_n P$ *is an n-rigid Γ-module.*

Proof. We have $\operatorname{Ext}^i_\Gamma(-, \mathbf{F}_n P) = 0$ for any $i > 0$ since $\mathbf{F}_n P$ is injective (Lemma 3.7(2)). Since $M' \in \mathcal{F}_0$ and $P \in \mathcal{F}_n$ by (1) and Lemma 3.7(1) respectively, we can check the assertion as follows by using Lemma 3.5:

$$
\begin{aligned}
\operatorname{Ext}^i_\Gamma(\mathbf{F}_0 M', \mathbf{F}_0 M') &= \operatorname{Hom}_{\mathcal{D}_\Gamma}(\mathbf{F}M', \mathbf{F}M'[i]) \\
&\cong \operatorname{Hom}_{\mathcal{D}_\Lambda}(M', M'[i]) \\
&= \operatorname{Ext}^i_\Lambda(M', M'), \\
\operatorname{Ext}^i_\Gamma(\mathbf{F}_n P, \mathbf{F}_0 M') &= \operatorname{Hom}_{\mathcal{D}_\Gamma}(\mathbf{F}P[n], \mathbf{F}M'[i]) \\
&\cong \operatorname{Hom}_{\mathcal{D}_\Lambda}(P, M'[i-n]) \\
&= \operatorname{Ext}^{i-n}_\Lambda(P, M').
\end{aligned}
$$

For $0 < i < n$ both of the above vanish, since M is n-rigid. $\qquad\square$

We now complete the proof of Theorem 4.2.

Proof of Theorem 4.2(2). By Lemma 4.5 we know that N is n-rigid, and hence we may apply Proposition 2.4. We will show that N is n-cluster tilting by checking the third of the equivalent conditions in Proposition 2.4(3).

(i) First we consider $\mathbf{F}_n P$. Take a minimal injective resolution

$$
0 \longrightarrow P \longrightarrow I_0 \longrightarrow \cdots \longrightarrow I_n \longrightarrow 0.
$$

By Proposition 3.3 we have $\operatorname{Ext}^i_\Lambda(T, P) = 0$ for $0 \le i < n$. Hence, applying $\operatorname{Hom}_\Lambda(T, -)$, we have an exact sequence

$$
0 \longrightarrow \mathbf{F}_0 I_0 \longrightarrow \cdots \longrightarrow \mathbf{F}_0 I_n \overset{f}{\longrightarrow} \mathbf{F}_n P \longrightarrow 0
$$

with $\mathbf{F}_0 I_i \in \operatorname{add} N$. We shall show that f is a right almost split map in $\operatorname{add} N$.

By Lemma 3.5, we have $\operatorname{Ext}^j_\Gamma(\mathbf{F}_0 M', \mathbf{F}_0 I_i) \cong \operatorname{Ext}^j_\Lambda(M', I_i) = 0$ for any i and any $j > 0$. Using this, we see that the map

$$
\operatorname{Hom}_\Gamma(\mathbf{F}_0 M', \mathbf{F}_0 I_n) \overset{f}{\longrightarrow} \operatorname{Hom}_\Gamma(\mathbf{F}_0 M', \mathbf{F}_n P)
$$

is surjective. Since $\mathbf{F}_n P$ is a simple injective Λ-module by Lemma 3.7, any non-zero endomorphism of $\mathbf{F}_n P$ is an automorphism. Thus f is a right almost split map in $\operatorname{add} N$.

(ii) Next we consider $\mathbf{F}_0 X$ for any indecomposable $X \in \operatorname{add} M'$. Since M is an n-cluster tilting object in $\operatorname{mod} \Lambda$, we have an exact sequence

$$
0 \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_0 \overset{f}{\longrightarrow} X
$$

with $M_i \in \operatorname{add} M$ and a right almost split map f in $\operatorname{add} M$ by Proposition 2.4. Applying $\mathbf{F}_0$, we have an exact sequence

$$
0 \longrightarrow \mathbf{F}_0 M_n \longrightarrow \cdots \longrightarrow \mathbf{F}_0 M_0 \overset{\mathbf{F}_0 f}{\longrightarrow} \mathbf{F}_0 X
$$

since we have $\operatorname{Ext}^i_\Lambda(T, M) = 0$ for any $0 < i < n$. Since $\mathbf{F}_0 M_i \in \operatorname{add} N$, we only have to show that $\mathbf{F}_0 f$ is a right almost split map in $\operatorname{add} N$.

Since $\mathbf{F}_n P$ is a simple injective Λ-module by Lemma 3.7, there is no non-zero map from $\mathbf{F}_n P$ to $\mathbf{F}_0 X$. Thus we only have to show that any morphism $g \colon \mathbf{F}_0 M' \to \mathbf{F}_0 X$ which is not a split epimorphism factors through f. By Lemma 3.5, we can put $g = \mathbf{F}_0 h$ for some $h \colon M' \to X$ which is not a split epimorphism. Since h factors through f, we have that $g = \mathbf{F}_0 h$ factors through $\mathbf{F}_0 f$. Thus we have shown that $\mathbf{F}_0 f$ is a right almost split map in $\operatorname{add} N$. $\qquad\square$

4.2. n-**APR tilting complexes preserve n-representation-finiteness.** Similar to Observation 4.1 we have the following result for n-representation finite algebras.

Observation 4.6. Let $\Lambda = P \oplus Q$ as Λ-modules, such that $\operatorname{Hom}_\Lambda(Q, P) = 0$. Then P has an associated n-APR tilting complex.

We have the following result.

Theorem 4.7. *Let Λ be n-representation-finite, and T be an n-APR tilting complex in $\mathcal{D}_\Lambda$. Then $\operatorname{End}_{\mathcal{D}_\Lambda}(T)^{\mathrm{op}}$ is also n-representation-finite.*

Proof. We set $\Gamma = \operatorname{End}_{\mathcal{D}_\Lambda}(T)^{\mathrm{op}}$. By Proposition 3.19 we know that, since $\operatorname{gl.dim}\Lambda \leq n$ we also have $\operatorname{gl.dim}\Gamma \leq n$.

By Proposition 3.20 we know that the derived equivalence $\mathcal{D}_\Lambda \to \mathcal{D}_\Gamma$ induces an equivalence $\mathcal{U}_\Lambda \to \mathcal{U}_\Gamma$. Hence, by Theorem 2.6 we have

$$\Lambda \text{ is } n\text{-representation finite} \iff \nu\mathcal{U}_\Lambda = \mathcal{U}_\Lambda$$
$$\iff \nu\mathcal{U}_\Gamma = \mathcal{U}_\Gamma$$
$$\iff \Gamma \text{ is } n\text{-representation finite} \qquad \square$$

4.3. **Slices.** In this subsection we introduce the notion of slices in the n-cluster tilting subcategory $\mathcal{U}$ (see Definition 4.8). The aim is to provide a bijection between these slices and the iterated n-APR tilting complexes of Λ (Theorem 4.15). This will be done by introducing a notion of mutation of slices (Definition 4.12), and by proving that this mutation coincides with n-APR tilts.

Throughout, let Λ be an n-representation-finite algebra. We consider the n-cluster tilting subcategory $\mathcal{U} = \mathcal{U}_\Lambda^n \subseteq \mathcal{D}_\Lambda$ given in Section 2.2.

Definition 4.8. An object $S \in \mathcal{U}$ is called *slice* if

(1) for any indecomposable projective module P there is exactly one i such that $\nu_n^i P \in \operatorname{add} S$, and

(2) $\operatorname{add} S$ is convex, that means any path (that is, sequence of non-zero maps) in $\operatorname{ind}\mathcal{U}$, which starts and ends in $\operatorname{add} S$, lies entirely in $\operatorname{add} S$.

The following two observations give us the slices we are interested in here.

Observation 4.9. In the setup above, $\Lambda \in \mathcal{U}$ is a slice, since we have $\operatorname{Hom}_{\mathcal{D}_\Lambda}(\nu_n^i\Lambda, \nu_n^j\Lambda) = H^0(\nu_n^{j-i}\Lambda) = 0$ if $i < j$.

Similarly, any iterated n-APR tilting complex of Λ is a slice in $\mathcal{U}$, by Theorem 4.7 and Proposition 3.20.

Proposition 4.10. *Let S be a slice. Then $\operatorname{Hom}_{\mathcal{D}_\Lambda}(S, \nu_n^i S) = 0$ for any $i > 0$.*

For the proof we will need the following observation:

Lemma 4.11. *Assume Λ is indecomposable (as a ring) and not semi-simple. For any indecomposable $X \in \mathcal{U}$ there is a path $\nu_n X \rightsquigarrow X$ in $\mathcal{U}$.*

Proof. Assume first that X is a non-projective Λ-module. By [Iya1, Theorem 2.2] there is an n-almost split sequence

$$\nu_n X = \tau_n X \rightarrowtail X_{n-1} \longrightarrow \cdots \longrightarrow X_1 \twoheadrightarrow X$$

with $X_i \in \mathcal{U} \cap \operatorname{mod}\Lambda$. This sequence gives rise to the desired path $\nu_n X \rightsquigarrow X$ in $\mathcal{U}$.

Now let $X \in \mathcal{U}$ be arbitrary indecomposable. By [IO, Lemma 4.9] there exists $i \in \mathbb{Z}$ such that $\nu^i X$ is a non-projective Λ-module. Then there exists a path $\nu_n \nu^i X \rightsquigarrow \nu^i X$ in $\mathcal{U}$. Since ν is an autoequivalence of $\mathcal{U}$ by Theorem 2.6, we have a path $\nu_n X \rightsquigarrow X$ in $\mathcal{U}$. $\qquad \square$

Proof of 4.10. We may assume Λ to be connected, and not semi-simple. Then, by the above lemma, for any indecomposable $S' \in \operatorname{add} S$ there is a path $\nu_n S' \rightsquigarrow S'$ in $\mathcal{U}$. Hence there are also a paths $\nu_n^i S' \rightsquigarrow S'$ for $i > 0$. If $\operatorname{Hom}_{\mathcal{D}_\Lambda}(S, \nu_n^i S') \neq 0$ for some $i > 0$, then we have $\nu_n^i S' \in \operatorname{add} S$ by Definition 4.8(2), contradicting 4.8(1). $\qquad \square$

Definition 4.12. Let S be a slice, $S = S' \oplus S''$ a direct summand decomposition of S, such that $\mathrm{Hom}_{\mathcal{D}_\Lambda}(S'', S') = 0$. We set

$$\mu_{S'}^+(S) = (\nu_n^- S') \oplus S'' \text{ and}$$
$$\mu_{S''}^-(S) = S' \oplus (\nu_n S'').$$

We call them *mutations* of S.

Lemma 4.13. *In the setup of Definition 4.12, $\mu_{S'}^+(S)$ and $\mu_{S''}^-(S)$ are slices again.*

Proof. We restrict to the case of $\mu_{S'}^+(S)$. It is clear that it satisfies Condition (1) of Definition 4.8. To see that $\mu_{S'}^+(S)$ is convex, let p be a path in $\mathrm{ind}\,\mathcal{U}$ starting and ending in $\mu_{S'}^+(S)$. We have the following four cases with respect to where p starts and ends:

(1) If p starts and ends in S'' then it lies entirely in S. Since $\mathrm{Hom}_{\mathcal{D}_\Lambda}(S'', S') = 0$ it lies entirely in S''.
(2) Similarly, if p starts and ends in $\nu_n^- S'$ then it lies entirely in $\nu_n^- S'$.
(3) By Proposition 4.10 p cannot start in $\nu_n^- S'$ and end in S''.
(4) Assume that p starts in S'' and ends in $\nu_n^- S'$. Hence, by Proposition 4.10 the path p to lies entirely in $S \oplus \nu_n^- S$. Then, since $\mathrm{Hom}_{\mathcal{D}_\Lambda}(S'', S') = 0$, it can pass neither through S' nor through $\nu_n^- S''$. Therefore it lies entirely in our slice.

Thus also Condition (2) of Definition 4.8 is satisfied. $\qquad\square$

Lemma 4.14.   (1) *Any two slices in $\mathcal{U}$ are iterated mutations of each other.*
   (2) *If moreover the quiver of Λ contains no oriented cycles, then any two slices are iterated mutations with respect to sinks or sources of each other.*

Proof. Let $\Lambda = \bigoplus P_i$ be a decomposition into indecomposable projectives. We choose d_i and e_i such that the two slices are $\bigoplus \nu_n^{d_i} P_i$ and $\bigoplus \nu_n^{e_i} P_i$, respectively. Since $\mu_S^+(S) = \nu_n^- S$, we can assume $e_i > d_i$ for all i. We set $I = \{i \mid e_i - d_i \text{ is maximal}\}$,

$$S' = \bigoplus_{i \in I} \nu_n^{e_i} P_i \text{ and } S'' = \bigoplus_{j \notin I} \nu_n^{e_j} P_j.$$

Now for $i \in I$ and $j \notin I$ we have

$$\mathrm{Hom}_{\mathcal{D}_\Lambda}(\nu_n^{e_j} P_j, \nu_n^{e_i} P_i) = \mathrm{Hom}_{\mathcal{D}_\Lambda}(\nu_n^{d_j} P_j, \nu_n^{(e_i - d_i) - (e_j - d_j)} \nu_n^{d_i} P_i).$$

Since by our choice of I we have $(e_i - d_i) - (e_j - d_j) > 0$, the above space vanishes by Proposition 4.10. Hence we may mutate, and obtain

$$\mu_{S'}^+\left(\bigoplus \nu_n^{e_i} P_i\right) = \left(\bigoplus_{i \in I} \nu_n^{e_i - 1} P_i\right) \oplus \left(\bigoplus_{j \notin I} \nu_n^{e_j} P_j\right).$$

Repeating this procedure we see that any two slices are iterated mutations of each other.

For the proof of the second claim first note that if the quiver of Λ containes no oriented cycles, then neither does the quiver of $\mathcal{U}$. So we can number the indecomposable direct summands of S' as $S' = S_1' \oplus \cdots \oplus S_\ell'$ such that $\mathrm{Hom}_{\mathcal{D}_\Lambda}(S_i', S_j') = 0$ for any $i > j$. Then we have $\mu_{S'}^+(S) = \mu_{S_\ell'}^+ \circ \cdots \circ \mu_{S_1'}^+(S)$ by Proposition 4.10. $\qquad\square$

Theorem 4.15. *Assume that Λ is n-representation-finite.*

(1) *The iterated n-APR tilting complexes of Λ are exactly the slices in $\mathcal{U}$.*
(2) *If moreover the quiver of Λ contains no oriented cycles, then any iterated n-APR tilting complex can be obtained by a sequence of n-APR (co)tilts in the sense of Definition 3.1.*

Proof. (1) By Observation 4.9 any iterated n-APR tilt comes from a slice. The converse follows from Lemma 4.14(1) and Observation 4.6.

(2) Follows similarly using Lemma 4.14(2) and Remark 3.15. $\qquad\square$

4.4. **Admissible sets.** In this subsection we will see that all the endomorphism rings of slices, and hence all the iterated n-APR tilts, of an n-representation-finite algebra are obtained as quotients of the $(n + 1)$-preprojective algebra (see Definition 2.11).

Lemma 4.16. *Let S be a slice in $\mathcal{U}$. Then*

$$\operatorname{Hom}_{\mathcal{U}}(S, \nu_n^{-i} S) \subseteq \operatorname{Rad}_{\mathcal{U}}^i(S, \nu_n^{-i} S).$$

Proof. By Theorem 4.15 we may assume S to be the slice Λ. Then the claim follows from Proposition 2.12. $\qquad\square$

Construction 4.17. For $P, Q \in \operatorname{add} \Lambda$ indecomposable we choose

$\quad C_0(P, Q) \subseteq \operatorname{Rad}_{\mathcal{U}}(P, \nu_n^- Q)$ such that $C_0(P, Q)$ is a minimal generating set of

$\quad\quad \operatorname{Rad}_{\mathcal{U}}(P, \nu_n^- Q) / \operatorname{Rad}_{\mathcal{U}}^2(P, \nu_n^- Q)$ as $\frac{\operatorname{End}_{\mathcal{U}}(P)^{\mathrm{op}}}{\operatorname{Rad}\operatorname{End}_{\mathcal{U}}(P)^{\mathrm{op}}}$-$\frac{\operatorname{End}_{\mathcal{U}}(Q)^{\mathrm{op}}}{\operatorname{Rad}\operatorname{End}_{\mathcal{U}}(Q)^{\mathrm{op}}}$-bimodule, and

$H(P, Q) \subseteq \operatorname{Rad}_{\mathcal{U}}(P, Q)$ such that $H(P, Q)$ is a minimal generating set of

$\quad\quad \operatorname{Rad}_{\mathcal{U}}(P, Q) / \operatorname{Rad}_{\mathcal{U}}^2(P, Q)$ as $\frac{\operatorname{End}_{\mathcal{U}}(P)^{\mathrm{op}}}{\operatorname{Rad}\operatorname{End}_{\mathcal{U}}(P)^{\mathrm{op}}}$-$\frac{\operatorname{End}_{\mathcal{U}}(Q)^{\mathrm{op}}}{\operatorname{Rad}\operatorname{End}_{\mathcal{U}}(Q)^{\mathrm{op}}}$-bimodule.

We set

$$A(P, Q) = C_0(P, Q) \coprod H(P, Q) \subseteq \operatorname{Hom}_{\mathcal{C}_\Lambda^n}(P, Q)$$

We write $C_0 = \coprod_{P,Q} C_0(P, Q)$ and $A = \coprod_{P,Q} A(P, Q)$. Note that by Definition 2.11 the set $A(P, Q)$ generates $\operatorname{Rad}_{\mathcal{C}_\Lambda^n}(P, Q) / \operatorname{Rad}_{\mathcal{C}_\Lambda^n}^2(P, Q)$.

If k is algebraically closed, then H consists of the arrows in the quiver of Λ, and C_0 consist of the additional arrows in the quiver of $\widehat{\Lambda}$. Thus A consist of all arrows in the quiver of $\widehat{\Lambda}$.

Lemma 4.18.

$$\Lambda \cong \widehat{\Lambda}/(C_0).$$

Proof. This follows from Proposition 2.12 and the definition of C_0 above. $\qquad\square$

Definition 4.19. (1) We call C_0 as above the *standard admissible set*.
  (2) For $C \subset A$ and a decomposition $\Lambda = \Lambda' \oplus \Lambda''$ (as modules) with
    (a) $\operatorname{add} \Lambda' \cap \operatorname{add} \Lambda'' = 0$,
    (b) for $P \in \operatorname{add} \Lambda'$ and $Q \in \operatorname{add} \Lambda''$ indecomposable we have $C(P, Q) = \emptyset$
    (c) for $P \in \operatorname{add} \Lambda''$ and $Q \in \operatorname{add} \Lambda'$ indecomposable we have $C(P, Q) = A(P, Q)$
    we define a new subset $\mu_{\Lambda'}^+(C) = \mu_{\Lambda''}^-(C) \subseteq A$ by

$$\mu_{\Lambda'}^+(C)(P, Q) = \begin{cases} C(P, Q) & \text{if } P \oplus Q \in \operatorname{add} \Lambda' \\ C(P, Q) & \text{if } P \oplus Q \in \operatorname{add} \Lambda'' \\ A(P, Q) & \text{if } P \in \operatorname{add} \Lambda' \text{ and } Q \in \operatorname{add} \Lambda'' \\ \emptyset & \text{if } P \in \operatorname{add} \Lambda'' \text{ and } Q \in \operatorname{add} \Lambda' \end{cases}$$

    That is, we remove from C all arrows $\operatorname{add} \Lambda'' \rightarrow \operatorname{add} \Lambda'$, and add all arrows $\operatorname{add} \Lambda' \rightarrow \operatorname{add} \Lambda''$ in A.
      We call this set a *mutation* of C.
  (3) An *admissible set* is a subset of A which is an iterated mutation of the standard admissible set.

We will now investigate the relation of slices in $\mathcal{U}$ and admissible sets.

Construction 4.20. Let $S = \bigoplus \nu_n^{s_i} P_i$ be a slice in $\mathcal{U}$. We set

$$C_S(P_i, P_j) = \{\varphi \in A(P_i, P_j) \mid \varphi \text{ is a map } P_i \rightarrow \nu_n^{s_j - s_i - r} P_j \text{ for some } r > 0\}.$$

Proposition 4.21. *For any slice S in $\mathcal{U}$ we have*

$$\operatorname{End}_{\mathcal{D}_\Lambda}(S)^{\mathrm{op}} \cong \widehat{\Lambda}/(C_S).$$

Proof. We have

$$\mathrm{End}_{\mathcal{D}_\Lambda}(S)^{\mathrm{op}} = \mathrm{Hom}_{\mathcal{D}_\Lambda}(S, \bigoplus \nu_n^i S)/(\text{maps } S \to \nu_n^- S) \qquad \text{(by 4.10 and 4.16)}$$

$$= \widehat{\Lambda}/(C_S) \qquad\qquad\qquad \text{(by definition of } C_S) \qquad \square$$

Proposition 4.22.   (1) *The map $C_? \colon S \mapsto C_S$ sends slices in $\mathcal{U}$ to admissible sets. Moreover any admissible set is of the form C_S for some slice S.*

  (2) *$C_?$ commutes with mutations in the following way:*

$$C_{\mu_{S'}^+(S)} = \mu_{\Lambda'}^+(C_S), \text{ and}$$

$$C_{\mu_{S''}^-(S)} = \mu_{\Lambda''}^-(C_S)$$

whenever $S = S' \oplus S''$ and $\Lambda = \Lambda' \oplus \Lambda''$ such that $\pi(S') \cong \pi(\Lambda')$ and $\pi(S'') \cong \pi(\Lambda'')$ (recall that π denotes the map from the derived category to the n-Amiot cluster category – see Definition 2.7). In particular the mutations of slices are defined if and only if the mutations of admissible sets are.

Proof. By definition Λ is a slice, and $C_\Lambda = C_0$ is the standard admissible set. We now proceed by checking that all these properties are preserved under mutation.

Assume we are in the setup of (2), that is $S = S' \oplus S''$ is a slice, $\Lambda = \Lambda' \oplus \Lambda''$, such that $\pi(S') \cong \pi(\Lambda')$ and $\pi(S'') \cong \pi(\Lambda'')$. We may further inductively assume that C_S is an admissible set.

$$\text{all maps } \widehat{\Lambda}'' \to \widehat{\Lambda}' \text{ in } A \text{ lie in } C_S$$

$$\Longleftrightarrow \mathrm{Hom}_{\mathcal{D}_\Lambda}(S'', S') = 0 \qquad\qquad \text{(by Proposition 4.21)}$$

$$\Longleftrightarrow S' \text{ admits a mutation} \qquad\qquad \text{(by Definition 4.12)}$$

$$\Longrightarrow \mu_{S'}^+(S) = \nu_m^- S' \oplus S'' \text{ is also a slice} \qquad \text{(by Lemma 4.13)}$$

$$\Longrightarrow \mathrm{Hom}_{\mathcal{U}}(S', \nu_n^- S'') \subseteq \mathrm{Rad}_{\mathcal{U}}^2(S', \nu_n^- S'') \qquad \text{(by Lemma 4.16)}$$

$$\Longrightarrow \text{no maps } \widehat{\Lambda}' \to \widehat{\Lambda}'' \text{ in } A \text{ lie in } C_S \qquad \text{(by Proposition 4.21)}$$

Therefore, the "in particular"-part of (2) holds. Similar to the arguments above one sees that $C_{\mu_{S'}^+(S)} = \mu_{\Lambda'}^+(C_S)$.

Now the surjectivity in (1) follows from the fact that, by definition, any admissible set is an iterated mutation of the standard admissible set. $\qquad \square$

Theorem 4.23. *Let Λ be n-representation-finite. Then the iterated n-APR tilts of Λ are precisely the algebras of the form $\widehat{\Lambda}/(C)$, where C is an admissible set.*

In particular all these algebras are also n-representation-finite.

Proof. The first part follows from Propositions 4.21, 4.22 and Theorem 4.15. The second part then follows by Theorem 4.7. $\qquad \square$

5. n-REPRESENTATION-FINITE ALGEBRAS OF TYPE A

The aim of this section is to construct n-representation-finite algebras of 'type A'. The starting point (and the reason we call these algebras type A) is the construction of higher Auslander algebras of type A_s in [Iya1] (we recall this in Theorem 5.7 here). The main result of this section is Theorem 5.6, which gives an explicit combinatorial description of all iterated n-APR tilts of these higher Auslander algebras by removing certain arrows from a given quiver (see also Definitions 5.1 and 5.3 for the notation used in that theorem).

Definition 5.1.   (1) For $n \geq 1$ and $s \geq 1$, let $Q^{(n,s)}$ be the quiver with vertices

$$Q_0^{(n,s)} = \{(\ell_1, \ell_2, \cdots, \ell_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1} \mid \sum_{i=1}^{n+1} \ell_i = s - 1\}$$

and arrows

$$Q_1^{(n,s)} = \{x \xrightarrow{i} x + f_i \mid i \in \{1, \ldots, n+1\}, x, x + f_i \in Q_0^{(n,s)}\},$$

where f_i denotes the vector

$$f_i = (0, \ldots, 0, \overset{i}{-1}, \overset{i+1}{1}, 0, \ldots, 0) \in \mathbb{Z}^{n+1}$$

(cyclically, that is $f_{n+1} = (\overset{1}{1}, 0, \ldots, 0, \overset{n+1}{-1})$).

(2) For $n \geq 1$ and $s \geq 1$, we define the k-algebra $\widehat{\Lambda}^{(n,s)}$ to be the quiver algebra of $Q^{(n,s)}$ with the following relations:

For any $x \in Q_0^{(n,s)}$ and $i, j \in \{1, \ldots, n+1\}$ satisfying $x + f_i, x + f_i + f_j \in Q_0^{(n,s)}$,

$$(x \xrightarrow{i} x + f_i \xrightarrow{j} x + f_i + f_j) = \begin{cases} (x \xrightarrow{j} x + f_j \xrightarrow{i} x + f_i + f_j) & \text{if } x + f_j \in Q_0^{(n,s)}, \\ 0 & \text{otherwise.} \end{cases}$$
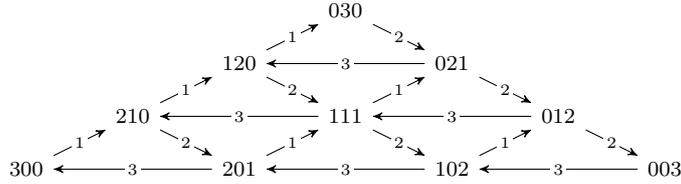
(We will show later that this notation is justified: In Subsection 5.1 we construct algebras $\Lambda^{(n,s)}$, such that $\widehat{\Lambda}^{(n,s)}$ is the $(n+1)$-preprojective algebra of $\Lambda^{(n,s)}$ – see also Proposition 5.48.)

Example 5.2.   (1) The quiver $Q^{(1,s)}$ is the following.

$$(s-1,0) \underset{2}{\overset{1}{\rightleftarrows}} (s-2,1) \underset{2}{\overset{1}{\rightleftarrows}} \cdots \underset{2}{\overset{1}{\rightleftarrows}} (1,s-2) \underset{2}{\overset{1}{\rightleftarrows}} (0,s-1)$$
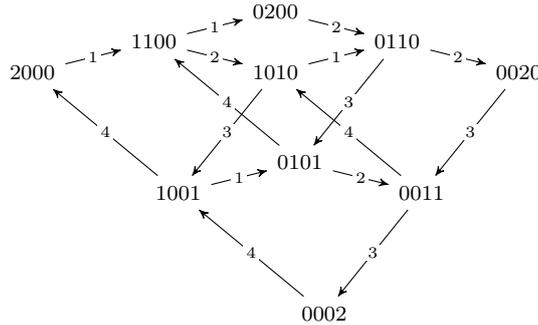
The algebra $\widehat{\Lambda}^{(1,s)}$ is the preprojective algebra of type A_s.

(2) The quiver $Q^{(2,4)}$ is

The algebras $\widehat{\Lambda}^{(2,s)}$ appeared in the work of Geiss, Leclerc, and Schröer [GLS1, GLS2].

(3) The quiver $Q^{(3,3)}$ is

Definition 5.3. We call a subset $C \subseteq Q_1^{(n,s)}$ of the arrows of $Q^{(n,s)}$ *cut*, if it contains exactly one arrow from each $(n+1)$-cycle (see [BMR, BRS, BFP$^+$] for similar constructions).

Remark 5.4.   (1) We will show later (see Remark 5.13) that cuts coincide with admissible sets (as introduced in Definition 4.19).

(2) Clearly, in Definition 5.3, any $(n+1)$-cycle is of the form

$$x \xrightarrow{\sigma(1)} x + f_{\sigma(1)} \xrightarrow{\sigma(2)} x + f_{\sigma(1)} + f_{\sigma(2)} \xrightarrow{\sigma(3)} \cdots \xrightarrow{\sigma(n)} x + f_{\sigma(1)} + \cdots + f_{\sigma(n)} \xrightarrow{\sigma(n+1)} x,$$

for some $\sigma \in \mathfrak{S}_{n+1}$.

Example 5.5.   (1) Clearly the cuts of $Q^{(1,s)}$ correspond bijectively to orientations of the Dynkin diagram A_s.

(2) See Tables 1, 3 and 4 for the cuts of $Q^{(2,3)}$, $Q^{(2,4)}$, and $Q^{(3,3)}$, respectively.

We are now ready to state the main result of this section.

Theorem 5.6. (1) *Let $Q^{(n,s)}$ as in Definition 5.1, and let C be a cut. Then the algebra*

$$\Lambda_C^{(n,s)} := \widehat{\Lambda}^{(n,s)}/(C)$$

is n-representation-finite.
(2) *All these algebras (for fixed (n,s)) are iterated n-APR tilts of one another.*

We call the algebras of the form $\Lambda_C^{(n,s)}$ as in the theorem above *n-representation-finite of type A*. Note that 1-representation-finite algebras of type A are exactly path algebras of Dynkin quivers of type A. See Tables 1, 3, and 4 for the examples $(n, s) = (2, 3)$, $(2, 4)$, and $(3, 3)$, respectively.

5.1. **Outline of the proof of Theorem 5.6.**

STEP 1. Let C_0 be the set of all arrows of type $n + 1$. This is clearly a cut. We set

$$\Lambda^{(n,s)} := \Lambda_{C_0}^{(n,s)}.$$

For example, $\Lambda^{(1,s)}$ is a path algebra of the linearly oriented Dynkin quiver A_s, and $\Lambda^{(2,s)}$ is the Auslander algebra of $\Lambda^{(1,s)}$. More generally, the following result is shown in [Iya1].

Theorem 5.7 (see [Iya1]). *The algebra $\Lambda^{(n,s)}$ is n-representation-finite. In particular* $\mathrm{mod}\,\Lambda^{(n,s)}$ *has a unique basic n-cluster tilting object $M^{(n,s)}$. We have*

$$\Lambda^{(n+1,s)} \cong \mathrm{End}_{\Lambda^{(n,s)}}(M^{(n,s)})^{\mathrm{op}},$$

that is $\Lambda^{(n+1,s)}$ is the n-Auslander algebra of $\Lambda^{(n,s)}$.

STEP 2. We now introduce mutation on cuts.

For simplicity of notation, we fix n and s for the rest of this section, and omit all superscripts $-^{(n,s)}$ whenever there is no danger of confusion. (That is, by Q we mean $Q^{(n,s)}$, by Λ we mean $\Lambda^{(n,s)}$ and similar.)

Definition 5.8. Let C be a cut of Q.
   (1) We denote by Q_C the quiver obtained by removing all arrows in C from Q.
   (2) Let x be a source of the quiver Q_C. Define a subset $\mu_x^+(C)$ of Q_1 by removing all arrows in Q ending at x from C and adding all arrows in Q starting at x to C.
   (3) Dually, for each sink x of Q_C, we get another subset $\mu_x^-(C)$ of Q_1.

We call the process of replacing a cut C by $\mu_x^+(C)$ or $\mu_x^-(C)$, when the conditions of (2) or (3) above are satisfied, *mutation* of cuts.

We will show in Proposition 5.14 in Subsection 5.2 that mutations of cuts are again cuts.

Observation 5.9. The quiver Q_C is the quiver of the algebra $kQ/(C)$.

The following remark explains the relationship between cuts and admissible sets.

Remark 5.10. (1) Whenever we mention admissible sets, it is implicitly understood that we choose $A = Q_1$ the set of arrows in Q in Definition 4.19. (It is shown in Subsection 4.4 that the choice of A does not matter there, but with this choice we can easier compare admissible sets and cuts.)
   (2) When C is a cut and an admissible set, and x is a source of Q_C, then the mutations $\mu_x^+(C)$ of C as cut and as admissible set coincide.
   (3) The standard admissible set C_0, as defined in Construction 4.17 and Definition 4.19, is identical to the set C_0 defined in STEP 1. In particular it is a cut.
   (4) By (3) and (2) we know that any admissible set is a cut. The converse follows when we have shown that all cuts are iterated mutations of one another (see Theorem 5.11 and Remark 5.13).
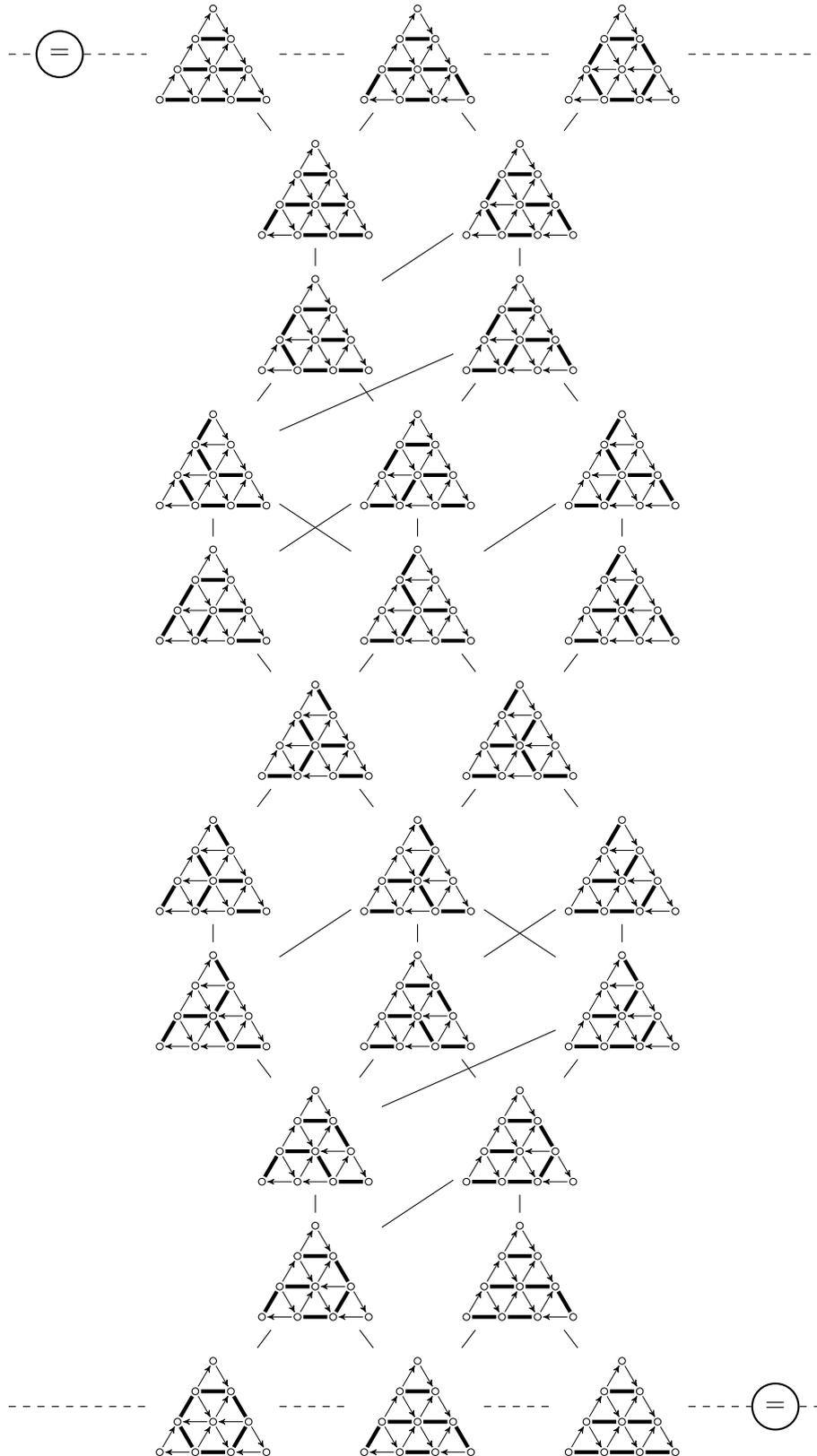
TABLE 3. Iterated 2-APR tilts of the Auslander algebra of linear oriented A_4 (thick lines indicate cuts)
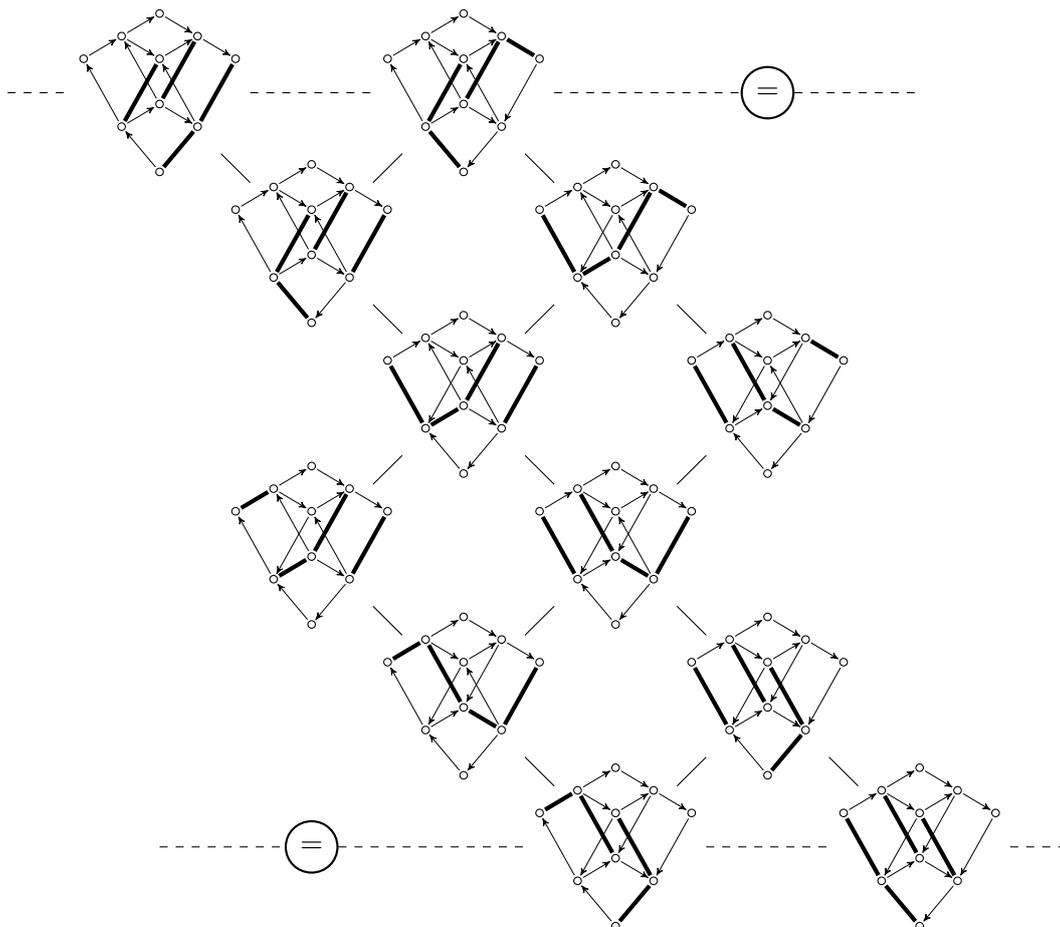
TABLE 4. Iterated 3-APR tilts of the higher Auslander algebra of linear oriented A_3 (thick lines indicate cuts)

We need the following purely combinatorial result, which will be proven in Subsections 5.3 to 5.5.

Theorem 5.11. *All cuts of Q are successive mutations of one another.*

STEP 3. Finally, we need the following result which will also be shown in Subsections 5.3 to 5.5.

Proposition 5.12.   (1) $\Lambda_{\mu_x^+(C)}$ *is an n-APR tilt of Λ_C.*
   (2) $\Lambda_{\mu_x^+(C)}$ *is n-representation-finite if and only if so is Λ_C.*

Now Theorem 5.6 follows:

Proof of Theorem 5.6. By Theorem 5.7, there is a cut C_0 such that Λ_{C_0} is strictly n-representation-finite. By Proposition 5.12 this property is preserved under mutation of cuts, and by Theorem 5.11 all cuts are iterated mutations of C_0.   □

Remark 5.13. Theorem 5.11, together with Remark 5.10(2) and (3), shows that in the setup of Definition 5.1 the set of cuts (as defined in 5.3) and the set of admissible sets (as defined in 4.19), coincide.

5.2. **Mutation of cuts.** In this subsection we show that the mutations $\mu_x^+(C)$ (or $\mu_x^-(C)$) as in Definition 5.8 for a cut C are cuts again.

Proposition 5.14. *In the setup of Definition 5.8 we have the following:*
   (1) *Any arrow in Q ending at x belongs to C, and any arrow in Q starting at x does not belong to C.*

(2) $\mu_x^+(C)$ *is again a cut.*

(3) x *is a sink of the quiver* $Q_{\mu_x^+(C)}$.

For the proof we need the following observation, which tells us that any sequence of arrows of pairwise different type may be completed to an $(n+1)$-cycle.

Lemma 5.15. *Let* $x \in \mathbb{Z}^{n+1}$ *and let* $\sigma \colon \{1, \ldots, \ell\} \to \{1, \ldots, n+1\}$ *be an injective map. Assume that* $x + \sum_{j=1}^{i} f_{\sigma(j)}$ *belongs to* Q_0 *for any* $0 \le i \le \ell$. *Then* σ *extends to an element* $\sigma \in \mathfrak{S}_{n+1}$ *such that* $x + \sum_{j=1}^{i} f_{\sigma(j)}$ *belongs to* Q_0 *for any* $0 \le i \le n+1$.

Proof. The statement makes sense only for $s \ge 2$. We set $I := \{0, \ldots, s-1\}$. For any $i \in \{1, \ldots, n+1\}$ we have $x_i + 1 \in I$ or $x_i - 1 \in I$.

We can assume $\ell < n+1$. We will define $\sigma(\ell+1) \in \{1, \ldots, n+1\}$ such that $x + \sum_{j=1}^{\ell+1} f_{\sigma(j)}$ belongs to Q_0. Without loss of generality, we assume that i_0 and $i_1 (\neq i_0, i_0 + 1)$ belong to $\operatorname{Im} \sigma$, but none of $i_0 + 1, i_0 + 2, \cdots, i_1 - 1$ belong to $\operatorname{Im} \sigma$. Since x and $x + \sum_{j=1}^{\ell} f_{\sigma(j)}$ belong to Q_0, we have

$$x_{i_0+1} \in I, \ x_{i_0+1} + 1 \in I, \ x_{i_1} \in I, \ \text{and } x_{i_1} - 1 \in I.$$

If $i_1 = i_0 + 2$, then $\sigma(\ell+1) := i_0 + 1$ satisfies the desired condition. In the rest, we assume $i_1 \neq i_0 + 2$. We divide into three cases.

(i) If $x_{i_0+2} + 1 \in I$, then $\sigma(\ell+1) := i_0 + 1$ satisfies the desired condition.

(ii) If $x_{i_1-1} - 1 \in I$, then $\sigma(\ell+1) := i_1 - 1$ satisfies the desired condition.

(iii) By (i) and (ii), we can assume $x_{i_0+2} - 1 \in I$, $x_{i_1-1} + 1 \in I$ and $i_1 \neq i_0 + 3$. Then there exists $i_0 + 2 \le i_2 < i_1 - 1$ satisfying

$$x_{i_2} - 1 \in I \quad \text{and} \quad x_{i_2+1} + 1 \in I.$$

Then $\sigma(\ell+1) := i_2$ satisfies the desired condition. $\square$

Proof of Proposition 5.14. (1) The former condition is clear since x is a source of Q_C. Assume that an arrow a starting at x belongs to C. By Lemma 5.15 we know that a is part of an $(n+1)$-cycle c. Then c contains at least two arrows which belong to C, a contradiction.

(2) Let c be an $(n+1)$-cycle. We only have to check that exactly one of the $(n+1)$ arrows in c is contained in $\mu_x^+(C)$. This is clear if x is not contained in c. Assume that x is contained in c, and let a and b be the arrows in c ending and starting in x, respectively. Since C is a cut a is the unique arrow in c contained in C. Thus b is the unique arrow in c contained in $\mu_x^+(C)$.

(3) Clear from (1). $\square$

5.3. n-cluster tilting in derived categories. This and the following two subsections are dedicated to the proofs of Theorem 5.11 and Proposition 5.12.

We consider a covering $\widetilde{Q}$ of Q, then introduce the notion of slices (see Definition 5.20) in $\widetilde{Q}$, and their mutation. Then we construct a correspondence between cuts and ν_n-orbits of slices (Theorem 5.24) and show that slices are transitive under mutations (Theorem 5.27). These results are the key steps of the proofs of Theorem 5.11 and Proposition 5.12.

We give the conceptual part of the proof in this subsection, and postpone the proof of the combinatorial parts (Theorems 5.24 and 5.27) to Subsection 5.4.

We recall the subcategory

$$\mathcal{U} = \operatorname{add}\{\nu_n^i \Lambda \mid i \in \mathbb{Z}\}$$

of $\mathcal{D}_\Lambda$ (see Subsection 2.2).

Definition 5.16. We denote by $\widetilde{Q} = \widetilde{Q}^{(n,s)}$ the quiver with

$$\widetilde{Q}_0 = \{(\ell_1, \ell_2, \cdots, \ell_{n+1} \colon i) \in \mathbb{Z}_{\geq 0}^{n+1} \times \mathbb{Z} \mid \sum_{j=1}^{n+1} \ell_j = s - 1\},$$

(we separate the last entry of the vector to emphasize its special role)

$$\widetilde{Q}_1 = \{\widetilde{a}_{x,i}\colon x \xrightarrow{i} x + g_i \mid 1 \le i \le n+1, \ x, x+g_i \in \widetilde{Q}_0\},$$

where g_i denotes the vector

$$g_i = \begin{cases} (0,\ldots,0,\overset{i}{-1},\overset{i+1}{1},0,\ldots,0:0) & 1 \le i \le n \\ (\overset{1}{1},0,\ldots,0,\overset{n+1}{-1}:1) & i = n+1 \end{cases}.$$

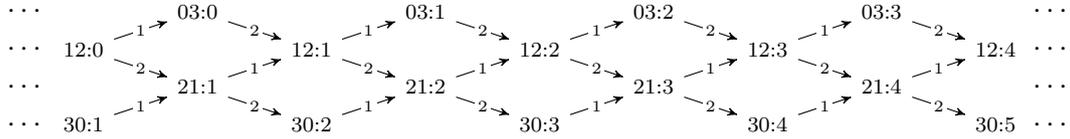We consider the category obtained from the quiver $\widetilde{Q}$ by factoring out the relations

$$[x \xrightarrow{i} x+g_i \xrightarrow{j} x+g_i+g_j] = [x \xrightarrow{j} x+g_j \xrightarrow{i} x+g_i+g_j]$$
$$\text{if } x, x+g_i, x+g_j, x+g_i+g_j \in \widetilde{Q}_0$$
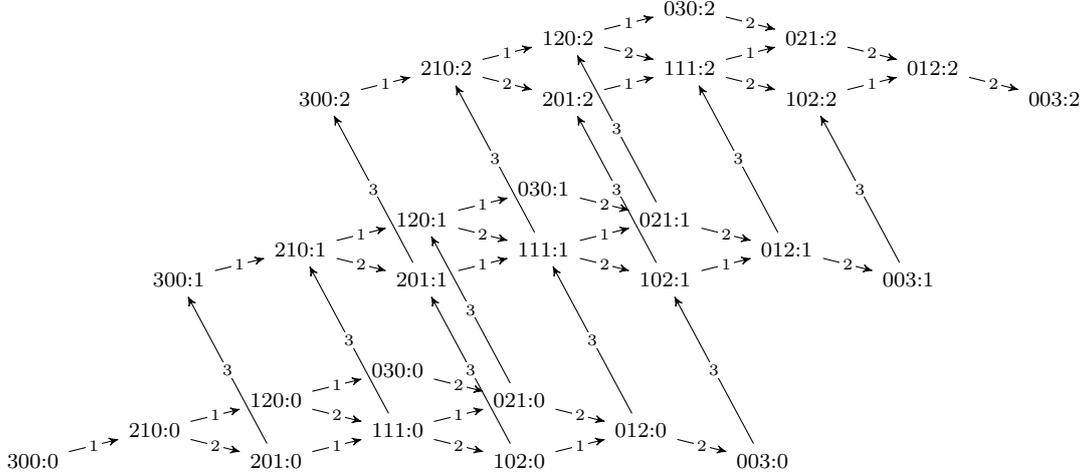
$$[x \xrightarrow{i} x+g_i \xrightarrow{j} x+g_i+g_j] = 0$$
$$\text{if } x, x+g_i, x+g_i+g_j \in \widetilde{Q}_0 \text{ and } x+g_j \notin \widetilde{Q}_0$$

Example 5.17. The quiver $\widetilde{Q}^{(1,4)}$ is the following.

The quiver $\widetilde{Q}^{(2,4)}$ is the following.

Remark 5.18. By abuse of notation we also denote the automorphism of $\widetilde{Q}$ induced by sending $(\ell_1, \ell_2, \cdots, \ell_{n+1} : i)$ to $(\ell_1, \ell_2, \cdots, \ell_{n+1} : i-1)$ by ν_n, and the map $\widetilde{Q} \to Q$ induced by sending $(\ell_1, \ell_2, \cdots, \ell_{n+1} : i)$ to $(\ell_1, \ell_2, \cdots, \ell_{n+1})$ by π.

The following result is shown in [Iya1, Theorem 6.10] (see Theorem 2.5).

Theorem 5.19. (1) *The n-cluster tilting subcategory $\mathcal{U}$ of $\mathcal{D}_\Lambda$ is presented by the quiver $\widetilde{Q}$ with relations as in 5.16.*

(2) *In this presentation the indecomposable projective Λ-modules correspond to the vertices $(\ell_1, \ldots, \ell_{n+1} : 0)$, and the indecomposable injective Λ-modules correspond to the vertices $(\ell_1, \ldots, \ell_{n+1} : \ell_1)$.*

(3) *The n-cluster tilting Λ-module is given by the direct sum of all objects corresponding to the vertices between projective and injective Λ-modules.*

We now carry over the concept of slices to the quiver-setup.

Definition 5.20. A *slice* of $\widetilde{Q}$ is a full subquiver S of $\widetilde{Q}$ satisfying the following conditions.

(1) Any ν_n-orbit in $\widetilde{Q}$ contains precisely one vertex which belongs to S.
(2) S is convex, i.e. for any path p in $\widetilde{Q}$ connecting two vertices in S, all vertices appearing in p belong to S.

Remark 5.21. Definition 5.20 is just a "quiver version" of Definition 4.8. In particular it is clear that slices in $\widetilde{Q}$ and slices in $\mathcal{U}$ are in natural bijection.

Next we carry over Construction 4.20 to this combinatorial situation, that is, we produce from any slice in $\widetilde{Q}$ a cut C_S.

Proposition 5.22. (1) *For any slice S in $\widetilde{Q}$, we have a cut*

$$C_S := Q_1 \setminus \pi(S_1)$$

in Q.
(2) *π gives an isomorphism $S \to Q_{C_S}$ of quivers.*

Proof. (1) Let

$$x_1 \xrightarrow{a_1} x_2 \xrightarrow{a_2} \cdots \xrightarrow{a_n} x_{n+1} \xrightarrow{a_{n+1}} x_1$$

be an $(n+1)$-cycle in Q. We only have to show that there exists precisely one $i \in \{1, \ldots, n+1\}$ such that the arrow a_i does not lie in $\pi(S_1)$.

Let $\widetilde{Q}'$ be the full subquiver of $\widetilde{Q}$ defined by $\widetilde{Q}'_0 := \pi^{-1}(\{x_1, \cdots, x_{n+1}\})$. Then $\widetilde{Q}'$ is isomorphic to the A_∞^∞ quiver

$$\cdots \longrightarrow y_{-1} \longrightarrow y_0 \longrightarrow y_1 \longrightarrow y_2 \longrightarrow \cdots$$

where $\pi(y_{i+(n+1)j}) = \{x_i\}$ holds for any $i \in \{1, \ldots, n+1\}, j \in \mathbb{Z}$. Since S is a slice, there exists $k \in \mathbb{Z}$ such that the $n+1$ vertices $y_k, y_{k+1}, \cdots, y_{k+n}$ belong to S_0 and any other y_i does not belong to S_0. Take $k' \in \{1, \ldots, n+1\}$ such that $k - k' \in (n+1)\mathbb{Z}$. Then the n arrows

$$x_{k'} \xrightarrow{a_{k'}} x_{k'+1} \xrightarrow{a_{k'+1}} \cdots \xrightarrow{a_n} x_{n+1} \xrightarrow{a_{n+1}} x_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{k'-2}} x_{k'-2}$$

belong to $\pi(S_1)$, and $x_{k'-1} \xrightarrow{a_{k'-1}} x_{k'}$ does not belong to $\pi(S_1)$.

(2) By Definition 5.20(1), $\pi \colon S_0 \to (Q_{C_S})_0 = Q_0$ is bijective and $\pi \colon S_1 \to (Q_{C_S})_1$ is injective. Since $(Q_{C_S})_1 = \pi(S_1)$ by our construction, we have that π is an isomorphism. □

Example 5.23. Two slices and the corresponding cuts for $n = 1$ and $s = 4$:

slices   corresponding cuts
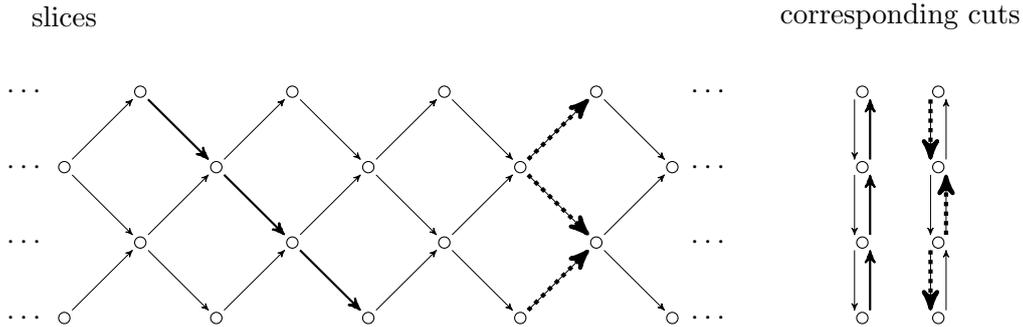
Some slices and corresponding cuts for $n = 2$ and $s = 3$ can be found in Table 5.

Now we state the first main assertion of this subsection, which will be proven in the next one.

Theorem 5.24. *The correspondence $S \mapsto C_S$ in Proposition 5.22 gives a bijection between ν_n-orbits of slices in $\widetilde{Q}$ and cuts in Q.*

Let us introduce the following notion.

Definition 5.25. Let S be a slice in $\widetilde{Q}$.
(1) Let x be a source of S. Define a full subquiver $\mu_x^+(S)$ of $\widetilde{Q}$ by removing x from S and adding $\nu_n^- x$.
(2) Dually, for each sink x of S, we define $\mu_x^-(S)$.

slices: corresponding cuts:
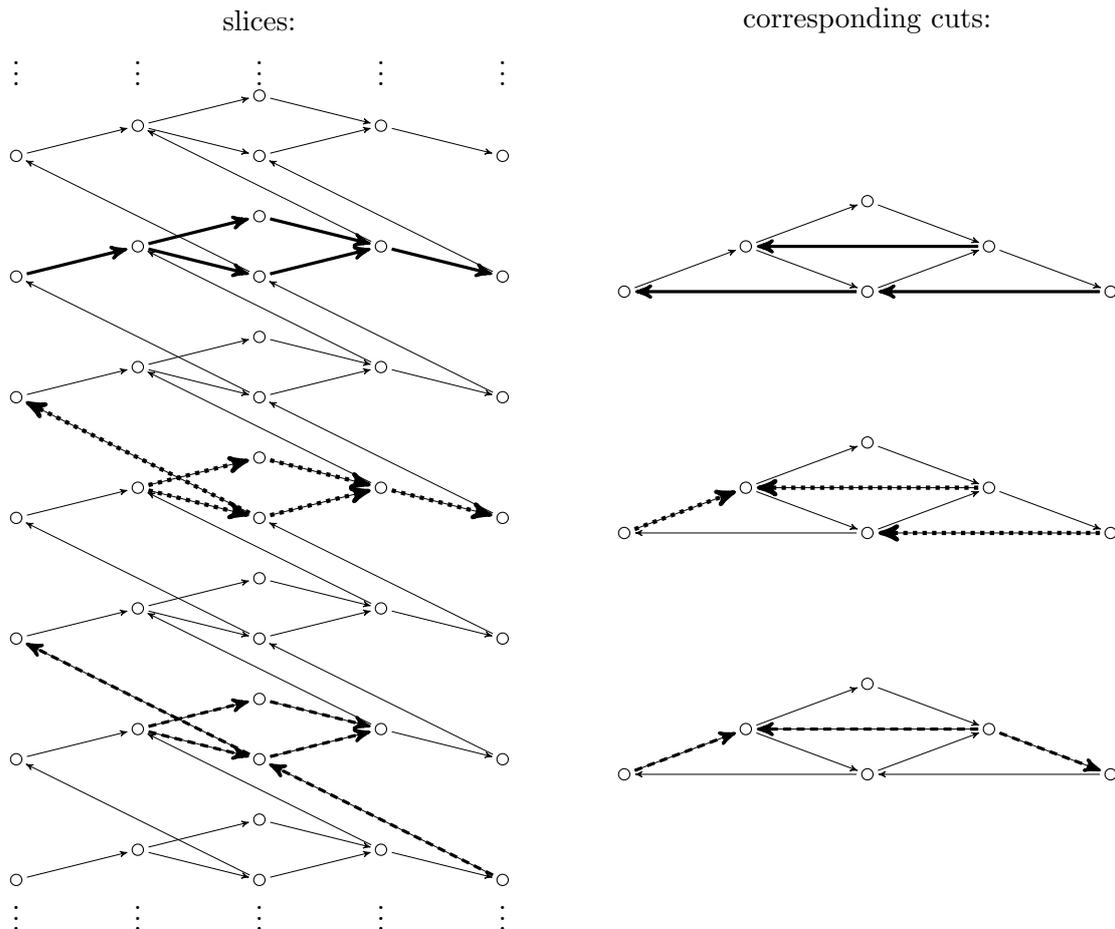
TABLE 5. Some slices and corresponding cuts for $n = 2$ and $s = 3$

We call the process of replacing a slice S by $\mu_x^+(S)$ or $\mu_x^-(S)$ *mutation* of slices.

Proposition 5.26. *In the setup of Definition 5.25(1) we have the following.*

(1) *Any successor of x in $\widetilde{Q}$ belongs to S, and any predecessor of x in $\widetilde{Q}$ does not belong to S.*

(2) *Any successor of $\nu_n^- x$ in $\widetilde{Q}$ does not belong to $\mu_x^+(S)$, and any predecessor of $\nu_n^- x$ in $\widetilde{Q}$ belongs to $\mu_x^+(S)$.*

(3) *$\mu_x^+(S)$ is again a slice, and $\nu_n^- x$ is a sink of $\mu_x^+(S)$.*

(4) *We have $C_{\mu_x^+(S)} = \mu_{\pi(x)}^+(C_S)$.*

Proof. (1) Let C_S be the cut given in Proposition 5.22. Then x is a source of Q_{C_S}. By Remark 5.10(1), we have the assertion.

(2) The former assertion follows from the former assertion in (1) and the definition of slice.

Take a predecessor y of $\nu_n^- x$ and an integer i such that $\nu_n^i y \in S_0$. If $i > 0$, then we have $i = 1$ since there exists a path from $\nu_n^i y$ to $\nu_n^- x$ passing through $\nu_n y$. This is a contradiction to the latter assertion of (1) since $\nu_n y$ is a predecessor of x. Thus we have $i \leq 0$. Since there exists a path from x to $\nu_n^i y$ passing through y, we have $y \in S_0$.

(3) By (2), $\nu_n^- x$ is a sink of $\mu_x^+(S)$. We only have to show that $\mu_x^+(S)$ is convex. We only have to consider paths p in $\widetilde{Q}$ starting at a vertex in $\mu_x^+(S)$ and ending at $\nu_n^- x$. Since any predecessor of $\nu_n^- x$ in $\widetilde{Q}$ belongs to S by (2) and S is convex, any vertex appearing in p belongs to $\mu_x^+(S)$.

(4) This is clear from (1) and (2).   □

The following is the second main statement in this section, which will be proven in the next one.

Theorem 5.27. *The slices in $\widetilde{Q}$ are transitive under successive mutation.*

Remark 5.28. Note that one can prove Theorem 5.27 by using the categorical argument in Lemma 4.14. But we will give a purely combinatorial proof in the next subsection since it has its own interest.

Clearly Theorem 5.11 is an immediate consequence of Proposition 5.26(4) and Theorems 5.24 and 5.27 above.

We now work towards a proof of Proposition 5.12. We identify a slice S in $\widetilde{Q}$ with the direct sum of all objects in $\mathcal{D}_\Lambda$ corresponding to vertices in S.

Lemma 5.29.   (1) $\mathrm{End}_{\mathcal{D}_\Lambda}(S) \cong \Lambda_{C_S}$.
   (2) *Let x be a source of S. If S is a tilting complex in $\mathcal{D}_\Lambda$, then $\mu_x^+(S)$ is an n-APR tilting Λ_{C_S}-module.*

Proof. (1) π gives an isomorphism $S \twoheadrightarrow C_S$. It is easily checked that the relations for $\mathcal{U}$ correspond to those for $\widehat{\Lambda}$.
   (2) This is clear from the definition. □

Proposition 5.30. *For any slice S in $\widetilde{Q}$, the corresponding object $S \in \mathcal{D}_\Lambda$ is an iterated n-APR tilting complex.*

Proof. This is clear for the slice consisting of the vertices of the form $(\ell_1, \ldots \ell_{n+1} : 0)$ by Theorem 5.19(2). We have the assertion by Theorem 5.27 and Lemma 5.29(2). □

Proof of Proposition 5.12. By Theorem 5.24 there exists a slice S in $\widetilde{Q}$ such that $C = C_S$. Take a source y of S such that $x = \pi(y)$. By Lemma 5.29(1) we can identify Λ_C with S. By Lemma 5.29(2) and Proposition 5.30, $\mu_x^+(S)$ is an n-APR tilting Λ_C-module with

$$\mathrm{End}_{\mathcal{D}_\Lambda}(\mu_x^+(S)) \cong \Lambda_{C_{\mu_x^+(S)}} = \Lambda_{\mu_x^+(C)}.$$

Thus the assertion follows. □

5.4. **Proof of Theorems 5.24 and 5.27.** In this subsection we give the proofs of Theorems 5.24 and 5.27 which were postponed in Subsection 5.3. We postpone further (to Subsection 5.5) the proof of Proposition 5.33, a technical classification result needed in the proofs here.

We need the following preparation.

Definition 5.31.   (1) We denote by $\mathrm{walk}(Q)$ the set of *walks* in Q (that is, finite sequences of arrows and inverse arrows such that consecutive entries involve matching vertices). For a walk p we denote by $\mathfrak{s}(p)$ and $\mathfrak{e}(p)$ the starting and ending vertex of p, respectively. A walk p is called *cyclic* if $\mathfrak{s}(p) = \mathfrak{e}(p)$.
   (2) We define an equivalence relation $\sim$ on $\mathrm{walk}(Q)$ as the transitive closure of the following relations:
      (a) $aa^{-1} \sim e_{\mathfrak{s}(a)}$ and $a^{-1}a \sim e_{\mathfrak{e}(a)}$ for any $a \in Q_1$.
      (b) If $p \sim q$, then $rpr' \sim rqr'$ for any r and r'.
Similarly we define $\mathrm{walk}(\widetilde{Q})$ and the equivalence relation $\sim$ on $\mathrm{walk}(\widetilde{Q})$.

For a walk $p = a_1 \cdots a_n$ we denote by $p^{-1} := a_n^{-1} \cdots a_1^{-1}$ the inverse walk.
Any map $\omega \colon Q_1 \to A$ with an abelian group A is naturally extended to a map $\omega \colon \mathrm{walk}(Q) \to A$ by putting $\omega(a^{-1}) := -\omega(a)$ for any $a \in Q_1$ and

$$\omega(p) := \sum_{i=1}^{\ell} \omega(b_i)$$

for any walk $p = b_1 \cdots b_\ell$. We define $\omega \colon \mathrm{walk}(\widetilde{Q}) \to A$ by $\omega(p) := \omega(\pi(p))$. Clearly these maps $\omega \colon \mathrm{walk}(Q) \to A$ and $\omega \colon \mathrm{walk}(\widetilde{Q}) \to A$ are invariant under the equivalence relation $\sim$.
In particular, we define maps

$$\phi_i \colon \mathrm{walk}(Q) \to \mathbb{Z} \quad \text{and} \quad \Phi = (\phi_1, \ldots, \phi_{n+1}) \colon \mathrm{walk}(Q) \to \mathbb{Z}^{n+1}$$

by setting $\phi_i(a) := \delta_{ij}$ for any arrow a of type j in Q.

Definition 5.32. We denote by G be the set of cyclic walks satisfying

$$p \sim (q_1 c_1^{\pm 1} q_1^{-1})(q_2 c_2^{\pm 1} q_2^{-1}) \cdots (q_\ell c_\ell^{\pm 1} q_\ell^{-1})$$

for some walks q_i and $(n+1)$-cycles c_i.

We will prove Theorems 5.24 and 5.27 by using the following result, which will be shown in the next subsection.

Proposition 5.33. *Any cyclic walk on Q belongs to G.*

Using this, we will now prove the following proposition, telling us that on Q_C the value $\Phi(p)$ depends only on $\mathfrak{s}(p)$ and $\mathfrak{e}(p)$.

Proposition 5.34. *Let C be a cut of Q.*

    (1) *For any cyclic walk p on Q_C, we have $\Phi(p) = 0$.*
    (2) *For any walks p and q on Q_C satisfying $\mathfrak{s}(p) = \mathfrak{s}(q)$ and $\mathfrak{e}(p) = \mathfrak{e}(q)$, we have $\Phi(p) = \Phi(q)$.*

To prove Proposition 5.34, we define a map

$$\phi_C \colon \operatorname{walk}(Q) \to \mathbb{Z}$$

by setting

$$\phi_C(a) := \begin{cases} 1 & \text{if } a \notin C \\ -n & \text{if } a \in C \end{cases}$$

for any arrow $a \in Q_1$.

Lemma 5.35. *For any cyclic walk p on Q, we have $\phi_C(p) = 0$.*

Proof. Any $(n+1)$-cycle C satisfies $\phi_C(c) = 0$. By Proposition 5.33 we have the assertion. $\square$

We define a map

$$\ell_C \colon \operatorname{walk}(Q) \to \mathbb{Z}$$

by putting

$$\ell_C(a) := \begin{cases} 0 & \text{if } a \notin C \\ 1 & \text{if } a \in C \end{cases}$$

for any arrow $a \in Q_1$.

The following result is clear.

Lemma 5.36. *For any $p \in \operatorname{walk}(Q)$ we have $\sum_{i=1}^{n+1} \phi_i(p) = \phi_C(p) + (n+1)\ell_C(p)$.*

Now we are ready to prove Proposition 5.34.

Proof of Proposition 5.34. (1) Since p is a cyclic walk, we have $\sum_{i=1}^{n+1} \phi_i(p) f_i = 0$ (with f_i as in Definition 5.1). This implies $\phi_1(p) = \cdots = \phi_{n+1}(p)$.

Since p is a cyclic walk on Q_C, we have

$$\sum_{i=1}^{n+1} \phi_i(p) = \phi_C(p) + (n+1)\ell_C(p) = 0 + (n+1) \cdot 0 = 0$$

by Lemmas 5.35 and 5.36. Thus we have $\phi_1(p) = \cdots = \phi_{n+1}(p) = 0$.

(2) We have $\Phi(p) - \Phi(q) = \Phi(pq^{-1}) = 0$ by (1). $\square$

The fact that $\widetilde{Q} \to Q$ is a Galois covering is reflected by the following lemma on lifting of walks.

Lemma 5.37. *Fix $x_0 \in Q_0$ and $\widetilde{x}_0 \in \widetilde{Q}_0$ such that $\pi(\widetilde{x}_0) = x_0$. For any walk p in Q with $\mathfrak{s}(p) = x_0$, there exists a unique walk $\widetilde{p}$ in $\widetilde{Q}$ such that $\mathfrak{s}(\widetilde{p}) = \widetilde{x}_0$ and $\pi(\widetilde{p}) = p$.*

Proof. For any $x \in Q_0$ and $y \in \widetilde{Q}_0$ such that $\pi(y) = x$, the morphism $\pi \colon \widetilde{Q} \to Q$ gives a bijection from the set of arrows starting (respectively, ending) at y to the set of arrows starting (respectively, ending) at x. Thus the assertion follows. $\square$

We have the following key observation.

Lemma 5.38. *Fix $x_0 \in Q_0$ and $\widetilde{x}_0 \in \widetilde{Q}_0$ such that $\pi(\widetilde{x}_0) = x_0$. For any walks p and q in Q_C satisfying $\mathfrak{s}(p) = \mathfrak{s}(q) = x_0$ and $\mathfrak{e}(p) = \mathfrak{e}(q)$, then $\widetilde{p}$ and $\widetilde{q}$ as given in Lemma 5.37 satisfy $\mathfrak{e}(\widetilde{p}) = \mathfrak{e}(\widetilde{q})$.*

Proof. By our definition of Φ, we have that $\phi_i(\widetilde{p})$ counts the number of arrows of type i appearing in $\widetilde{p}$. Since we have $\phi_i(\widetilde{p}) = \phi_i(\widetilde{q})$ by Proposition 5.34, we have that the number of arrows of type i appearing in $\widetilde{p}$ is equal to that in $\widetilde{q}$. Since $\mathfrak{s}(\widetilde{p}) = \mathfrak{s}(\widetilde{q})$, we have $\mathfrak{e}(\widetilde{p}) = \mathfrak{e}(\widetilde{q})$. $\square$

Now Theorem 5.24 follows from the following result, which allows us to construct slices from cuts.

Proposition 5.39. *Let C be a cut in Q. Fix a vertex $x_0 \in Q_0$ and $\widetilde{x}_0 \in \pi^{-1}(x_0)$.*

(1) *There exists a unique morphism $\iota \colon Q_C \rightarrow \widetilde{Q}$ of quivers satisfying the following conditions.*
   - *$\iota(x_0) = \widetilde{x}_0$,*
   - *the composition $\pi \circ \iota \colon Q_C \rightarrow Q$ is the identity on Q_C.*

(2) *$\iota(Q_C)$ is a slice in $\widetilde{Q}$.*

Proof. (1) To give the desired morphism $\iota \colon Q_C \rightarrow \widetilde{Q}$ of quivers, we only have to give a map $\iota \colon Q_0 \rightarrow \widetilde{Q}_0$ between the sets of vertices, satisfying the following conditions.

   - $\iota(x_0) = \widetilde{x}_0$,
   - the composition $\pi \circ \iota \colon Q_C \rightarrow Q$ is the identity on Q_0,
   - for any arrow $a \colon x \rightarrow y$ in Q_C, there is an arrow $\iota(x) \rightarrow \iota(y)$ in $\widetilde{Q}$.

We define $\iota \colon Q_0 \rightarrow \widetilde{Q}_0$ as follows: Fix any $x \in Q_0$. We take any walk p in Q_C from x_0 to x. By Lemma 5.37, there exists a unique walk $\widetilde{p}$ in $\widetilde{Q}$ such that $\mathfrak{s}(\widetilde{p}) = \widetilde{x}_0$ and $\pi(\widetilde{p}) = p$. Then we put $\iota(x) := \mathfrak{e}(\widetilde{p})$. By Lemma 5.38, $\iota(x)$ does not depend on the choice of the walk p.

We only have to check the third condition above. Fix an arrow $a \colon x \rightarrow y$ in Q_C. Take any walk p in Q_C from x_0 to x. The walk $pa \colon x_0 \rightsquigarrow y$ in Q_C gives the corresponding walk $\widetilde{pa} \colon \widetilde{x}_0 \rightsquigarrow \iota(y)$ in $\widetilde{Q}$. Then $\widetilde{pa}$ has the form $\widetilde{p}b$ for an arrow $b \colon \iota(x) \rightarrow \iota(y)$ and a walk $\widetilde{p} \colon \widetilde{x}_0 \rightsquigarrow \iota(x)$ in $\widetilde{Q}$. Thus the third condition is satisfied.

The uniqueness of ι is clear.

(2) Fix vertices $x, y \in \iota(Q_C)_0$ and a path p in $\widetilde{Q}$ from x to y. We only have to show that p is a path in $\iota(Q_C)$.

Since Q_C is connected, we can take a walk q on $\iota(Q_C)$ from x to y. Then we have $\Phi(\pi(p)) = \Phi(\pi(q))$. We have

$$\phi_C(p) + (n+1)\ell_C(p) = \sum_{i=1}^{n+1} \phi_i(p) = \sum_{i=1}^{n+1} \phi_i(q) = \phi_C(q) + (n+1)\ell_C(q) = \phi_C(q)$$

by Lemma 5.36. Since we have $\phi_C(p) = \phi_C(q)$ by Lemma 5.35, we have $\ell_C(p) = 0$. Thus any arrow appearing in p belongs to $\iota(Q_C)$. $\square$

This completes the proof of Theorem 5.24.

In the remainder of this subsection we give a purely combinatorial proof of Theorem 5.27.

For a slice S, we denote by S_0^+ the subset of $\widetilde{Q}_0$ consisting of sources in S.

Lemma 5.40. *The correspondence $S \mapsto S_0^+$ is injective.*

Proof. We denote by S_0' be the set of vertices x of $\widetilde{Q}$ satisfying the following conditions.

   - there exists a path in $\widetilde{Q}$ from some vertex in S_0^+ to x,
   - there does not exist a path in $\widetilde{Q}$ from any vertex in S_0^+ to $\nu_n x$.

To prove the assertion, we only have to show $S_0 = S_0'$. It is easily seen from the definition of S_0' that each ν_n-orbit in $\widetilde{Q}_0$ contains at most one vertex in S_0'. Since S_0 is a slice, we only have to show $S_0 \subset S_0'$.

For any $x \in S_0$, there exists a path in S from some vertex in S_0^+ to x since S is a finite acyclic quiver. Assume that there exists a path p in $\widetilde{Q}$ from $y \in S_0^+$ to $\nu_n x$. Since there exists a path

q in $\widetilde{Q}$ from $\nu_n x$ to x, we have a path pq from x to y. Since S is convex, we have $\nu_n x \in S_0$, a contradiction to $x \in S_0$.   □

For a slice S of $\widetilde{Q}$, define the full subquiver $\widetilde{Q}_S^{\geq 0}$ by

$$(\widetilde{Q}_S^{\geq 0})_0 := \bigcup_{\ell \geq 0} \nu_n^\ell S_0.$$

Clearly we have $(\widetilde{Q}_{\mu_x^+(S)}^{\geq 0})_0 = (\widetilde{Q}_S^{\geq 0})_0 \cup \{\nu_n^- x\}$.

Lemma 5.41. *Let S be a slice in $\widetilde{Q}$. Then there exists a numbering $S_0 = \{x_1, \cdots, x_N\}$ of vertices of S such that the following conditions are satisfied.*

    (1) *x_{i+1} is a source in $\mu_{x_i}^+ \circ \cdots \circ \mu_{x_1}^+(S)$ for any $0 \leq i < N$.*
    (2) *We have $\mu_{x_N}^+ \circ \cdots \circ \mu_{x_1}^+(S) = \nu_n^- S$.*

Proof. When we have $x_1, \cdots, x_{i-1} \in S_0$, then we define x_i as a source of the quiver $S \setminus \{x_0, \cdots, x_{i-1}\}$. It is easily checked that the desired conditions are satisfied.   □

For slices S and T in $\widetilde{Q}$, we write $S \leq T$ if $(\widetilde{Q}_S^{\geq 0})_0 \subseteq (\widetilde{Q}_T^{\geq 0})_0$. In this case, we put

$$d(S, T) := \#((\widetilde{Q}_T^{\geq 0})_0 \setminus (\widetilde{Q}_S^{\geq 0})_0).$$

Now we are ready to prove Theorem 5.27.

Let S and T be slices. We can assume $S \leq T$ by Lemma 5.41. We use the induction on $d(S, T)$. If $d(S, T) = 0$, then we have $S = T$. Assume $d(S, T) > 0$. By Lemma 5.40, there exists a source x of S such that $x \notin T_0$. Then we have $\mu_x^+(S) \leq T$ and $d(\mu_x^+(S), T) = d(S, T) - 1$. By our assumption on induction, $\mu_x^+(S)$ is obtained from T by a successive mutation. Thus S is obtained from T by a successive mutation.   □

5.5. **Proof of Proposition 5.33.** We complete the proof of Theorem 5.6 by filling the remaining gap, that is by proving Proposition 5.33.

For a walk p, we denote by $|p|$ the length of p. For $x, y \in Q_0$, we denote by $d(x, y)$ the minimum of the length of walks on Q from x to y.

It is easily checked (similarly to the proof of Lemma 5.15) that $d(x, y) = d(x', y')$ whenever $x - y = x' - y'$.

Lemma 5.42. *Let p be a cyclic walk. Assume that, for any decomposition $p = p_1 p_2 p_3$ of p,*

$$d(\mathfrak{s}(p_2), \mathfrak{e}(p_2)) = \min\{|p_2|, |p_3 p_1|\}$$

holds. Then one of the following conditions holds.

    (1) *p or p^{-1} is an $(n+1)$-cycle.*
    (2) *p has the form $p = a_1^{\epsilon_1} \cdots a_\ell^{\epsilon_\ell} b_1^{-\epsilon_1} \cdots b_\ell^{-\epsilon_\ell}$ with an injective map $\sigma \colon \{1, \cdots, \ell\} \to \{1, \cdots, n\}$, arrows a_i and b_i of type $\sigma(i)$ and $\epsilon_i \in \{\pm 1\}$.*

Proof. (i) Assume that p contains an arrow of type i and an inverse arrow of type i at the same time. Take any decomposition $p = q_1 a q_2 b^{-1}$ with arrows a, b of type i and walks q_1 and q_2. If $|q_2| < |q_1|$, then we have

$$d(\mathfrak{s}(q_1), \mathfrak{e}(q_1)) = d(\mathfrak{e}(q_2), \mathfrak{s}(q_2)) < \min\{|q_1|, |a q_2 b^{-1}|\},$$

a contradiction. Similarly, $|q_1| < |q_2|$ cannot occur. Consequently, we have $|q_1| = |q_2|$. This equality also implies that q_1 and q_2 do not contain arrows or inverse arrows of type i. Consequently, p satisfies Condition (2).

(ii) In the rest, we assume that p does not satisfy Condition (2). By (i), we have that p does not contain an arrow of type i and an inverse arrow of type i at the same time. Without loss of generality we may assume $\phi_i(p) > 0$. Then p contains exactly $\phi_i(p)$ arrows of type i for each i, and does not contain inverse arrows.

Since p is a cyclic walk, we have an equality $\sum_{i=1}^{n+1} \phi_i(p) f_i = 0$. This implies $\phi := \phi_1(p) = \cdots = \phi_{n+1}(p)$. We shall show that $\phi = 1$. Then Condition (1) is satisfied.

Assume that $\phi > 1$ holds.

Assume that $|p|$ is odd, so $n+1$ is also odd. We write $p = ap_1p_2$ with an arrow a and $|p_1| = |p_2|$. By our assumption, we have $d(\mathfrak{s}(p_1), \mathfrak{e}(p_1)) = |p_1| = |p_2| = d(\mathfrak{s}(p_2), \mathfrak{e}(p_2))$. This implies that less than $\frac{n+1}{2}$ types of arrows appear in p_1 (respectively, p_2). Since $\phi > 1$, either p_1 or p_2 contains an arrow of same type with a. Hence p contains less than $\frac{n+1}{2} + \frac{n+1}{2} = n+1$ kinds of arrows, a contradiction.

Assume that $|p|$ is even. We write $p = ap_1bp_2$ with arrows a, b and $|p_1| = |p_2|$. By our assumption, we have

$$d(\mathfrak{s}(ap_1), \mathfrak{e}(ap_1)) = |ap_1| = |bp_2| = d(\mathfrak{s}(bp_2), \mathfrak{e}(bp_2)).$$

This implies that at most $\frac{n+1}{2}$ types of arrows appear in ap_1 (respectively, bp_2). Since all kinds of arrows appear in p, we have that ap_1 and bp_2 contain exactly $\frac{n+1}{2}$ types of arrows, and there is no common type of arrows in ap_1 and bp_2. By the same argument, we have that p_1b and p_2a contain exactly $\frac{n+1}{2}$ types of arrows, and there is no common type of arrows in p_1b and p_2a.

Since $\phi > 1$, either p_1 or p_2 contains an arrow of same type with a. Assume that p_1 contains an arrow of same type with a. Then p_1b contains more than $\frac{n+1}{2}$ types of arrows, a contradiction. Similarly, p_2 does not contain an arrow of same type with a, a contradiction. $\square$

Lemma 5.43. *The cyclic walk in Lemma 5.42(2) belongs to G if $\epsilon_1 = \cdots = \epsilon_\ell = 1$.*

Proof. By Lemma 5.15, $a_1 \cdots a_\ell$ extends to an $(n+1)$-cycle $a_1 \cdots a_{n+1}$ in Q with a_i an arrow of type $\sigma(i)$ ($\sigma \in \mathfrak{S}_{n+1}$ an extending the original σ). Since

$$a_1 \cdots a_\ell b_1^{-1} \cdots b_\ell^{-1} \sim (a_1 \cdots a_{n+1})(b_\ell \cdots b_1 a_{\ell+1} \cdots a_{n+1})^{-1} \in G,$$

we have the assertion. $\square$

Lemma 5.44. *Let $pa^\epsilon b^{\epsilon'} q$ and $pc^{\epsilon'} d^\epsilon q$ be cyclic walks on Q, with $\epsilon, \epsilon' \in \{\pm 1\}$, such that a and d are arrows of the same type, and b and c are arrows of the same type. Then one of them belongs to G if and only if the other does.*

Proof. We have the equivalences

$$
\begin{aligned}
pabq &\sim (p(abd^{-1}c^{-1})p^{-1})(pcdq) & (\epsilon = \epsilon' = 1), \\
pab^{-1}q &\sim (pc^{-1}(cab^{-1}d^{-1})cp^{-1})(pc^{-1}dq) & (\epsilon = 1, \epsilon' = -1),
\end{aligned}
$$

and similar for the remaining cases. The claim now follows from Lemma 5.43. $\square$

Lemma 5.45. *Let $x \in Q_0$, $\sigma \colon \{1, \ldots, \ell\} \to \{1, \ldots, n+1\}$ be an injective map and $\epsilon_i \in \{\pm 1\}$ for any $1 \le i \le \ell$. Assume that $x + \sum_{j=1}^i \epsilon_j f_{\sigma(j)}$ and $x + \sum_{j=i}^\ell \epsilon_j f_{\sigma(j)}$ belong to Q_0 for any $0 \le i \le \ell$. Then, for any subset I of $\{1, \ldots, \ell\}$, we have that $x + \sum_{j \in I} \epsilon_j f_{\sigma(j)}$ belongs to Q_0.*

Proof. We only have to show that

$$0 \le x_{\sigma(i)} - \epsilon_i < s \text{ and } 0 \le x_{\sigma(i)+1} + \epsilon_i < s$$

hold for any $i \in \{1, 2, \cdots, \ell\}$.

If $\sigma(i) - 1 \notin \{\sigma(1), \cdots, \sigma(i-1)\}$, then the $\sigma(i)$-th entry of $x + \sum_{j=1}^i \epsilon_j f_{\sigma(j)}$ is equal to $x_{\sigma(i)} - \epsilon_i$. If $\sigma(i) - 1 \notin \{\sigma(i+1), \cdots, \sigma(\ell)\}$, then the $\sigma(i)$-th entry of $x + \sum_{j=i}^\ell \epsilon_j f_{\sigma(j)}$ is equal to $x_{\sigma(i)} - \epsilon_i$. In each case we have the former inequality.

The latter inequality can be shown similarly. $\square$

We now look at the following special case of Proposition 5.33.

Lemma 5.46. *Any cyclic walk satisfying the condition in Lemma 5.42(2) belongs to G.*

Proof. Let p be the cyclic walk in Lemma 5.42(2), and $x = \mathfrak{s}(p)$. It follows from Lemma 5.45 that for any $\varrho \in \mathfrak{S}_\ell$, $\widetilde{Q}$ contains the cyclic walk

$$p_\varrho := ({}_\varrho a_1^{\epsilon_{\varrho(1)}}) \cdots ({}_\varrho a_\ell^{\epsilon_{\varrho(\ell)}}) b_1^{-\epsilon_1} \cdots b_\ell^{-\epsilon_\ell}$$

starting from x, where ${}_\varrho a_i$ is an arrow of type $\sigma(\varrho(i))$. When ϱ is given by $\varrho(i) = \ell + 1 - i$, the cyclic walk p_ϱ is

$$p_\varrho = b_\ell^{\epsilon_\ell} \cdots b_1^{\epsilon_1} b_1^{-\epsilon_1} \cdots b_\ell^{-\epsilon_\ell},$$

which clearly belongs to G. Using Lemma 5.44 repeatedly, we see that all p_ϱ lie in G, so in particular $p = p_{\mathrm{id}} \in G$. $\square$

Now we are ready to prove Proposition 5.33.

Proof of Proposition 5.33. We use the induction on $|p|$. Assume that p does not satisfy the conditions (1) and (2) in Lemma 5.42. Then we can write $p = p_1 p_2 p_3$ with

$$d(\mathfrak{s}(p_2), \mathfrak{e}(p_2)) < \min\{|p_2|, |p_3 p_1|\}.$$

Take a walk q from $\mathfrak{s}(p_2)$ to $\mathfrak{e}(p_2)$ with $|q| = d(\mathfrak{s}(p_2), \mathfrak{e}(p_2))$. Then we have

$$p \sim (p_1 q p_3)(p_3^{-1}(q^{-1} p_2) p_3)$$

and $|p_1 q p_3| = |p_1| + |q| + |p_3| < |p|$ and $|q^{-1} p_2| = |q| + |p_2| < |p|$. By our assumption of induction, $p_1 q p_3$ and $q^{-1} p_2$ belong to G. Thus p also belongs to G. $\square$

5.6. $(n+1)$-preprojective algebras. We end this paper by showing that the algebras $\widehat{\Lambda}^{(n,s)}$ have the following properties:

Theorem 5.47. *$\widehat{\Lambda}^{(n,s)}$ is self-injective weakly $(n+1)$-representation-finite, and we have a triangle equivalence $\underline{\mathrm{mod}}\,\widehat{\Lambda}^{(n,s)} \approx \mathcal{C}^{n+1}_{\Lambda^{(n+1,s-1)}}$.*

We remark that this proof relies heavily on a results from [IO] (also see Remark 4.17 in that paper). We need the following observation.

Proposition 5.48. *For any cut C of $Q^{(n,s)}$, the $(n+1)$-preprojective algebra of the n-representation-finite algebra $\Lambda^{(n,s)}_C$ is $\widehat{\Lambda}^{(n,s)}$.*

Proof. The quiver morphism $\pi\colon \widetilde{Q}^{(n,s)} \to Q^{(n,s)}$ gives an equivalence $\mathcal{U}/\nu_n \approx \mathrm{proj}\,\widehat{\Lambda}^{(n,s)}$ of categories, which sends $\Lambda^{(n,s)}_C$ to $\widehat{\Lambda}^{(n,s)}$. Thus the $(n+1)$-preprojective algebra of $\Lambda^{(n,s)}_C$ is

$$\mathrm{End}_{\mathcal{U}/\nu_n}(\Lambda^{(n,s)}_C)^{\mathrm{op}} \cong \widehat{\Lambda}^{(n,s)}.$$ $\square$

Proof of Theorem 5.47. By Proposition 5.48 the algebra $\widehat{\Lambda}^{(n,s)}$ is the $(n+1)$-preprojective algebra of the n-representation finite algebra $\Lambda^{(n,s)}_C$ for any cut C. Thus, by [IO, Corollary 3.4], $\widehat{\Lambda}^{(n,s)}$ is self-injective.

Moreover, by [IO, Theorem 1.1], we have

$$\underline{\mathrm{mod}}\,\widehat{\Lambda}^{(n,s)} \approx \mathcal{C}^{n+1}_\Gamma,$$

where Γ is the stable n-Auslander algebra of $\Lambda^{(n,s)}_C$. In particular for $C = C_0$ we have Γ is the stable n-Auslander algebra $\Lambda^{(n,s)}$, which is $\underline{\mathrm{End}}_{\Lambda^{(n,s)}}(M^{(n,s)}) \cong \Lambda^{(n+1,s-1)}$.

The fact that $\widehat{\Lambda}^{(n,s)}$ is weakly $(n+1)$-representation finite now follows from the existence of an $(n+1)$-cluster tilting object in $\mathcal{C}^{n+1}_{\Lambda^{(n+1,s-1)}}$ by work of Amiot ([Ami1, Ami2] – also see [IO, Corollary 4.16]). $\square$

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