HOCHSCHILD COHOMOLOGY AND HOMOLOGY OF QUANTUM COMPLETE INTERSECTIONS

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ABSTRACT. We compute Hochschild cohomology and Hochschild homology for arbitrary finite dimensional quantum complete intersections. It turns out that their behaviour varies largely, depending on the choice of commutation parameters, and we will give precise criteria when to expect what behaviour.

1. INTRODUCTION

Quantum complete intersections first appear in the work of Avramov, Gasharov, and Peeva [1]. Based on the introduction of quantized versions of polynomial rings by Manin [6] they introduced the notion of quantum regular sequences.

In this paper we restrict to finite dimensional quantum complete intersections, that is algebras of the form $k\langle x_1, \ldots, x_c\rangle /I$, where I is an ideal generated by $x_i^{n_i}$ for some $n_i \in \mathbb{N}_{\geq 2}$, and $x_j x_i - q_{ij} x_i x_j$ for some commutation parameters q_{ij} from the multiplicative group of the field.

In particular in the case of two variables it has been observed that the homological behaviour of finite dimensional quantum complete intersections varies greatly depending on the commutation parameters:

Buchweitz, Green, Madsen, and Solberg [5] have given a finite dimensional quantum complete intersection as the first example of an algebra of infinite global dimension which has finite Hochschild cohomology. This has been generalized by Bergh and Erdmann [2], who have shown that a finite dimensional quantum complete intersection of codimension 2 (that is $c = 2$ in the description above) has infinite Hochschild cohomology if and only if the commutation parameter is a root of unity.

On the other hand, Bergh and the author [3] have shown that in the situation that all commutation parameters are roots of unity, the Hochschild cohomology of a quantum complete intersection is as well behaved as in the commutative case: It is a finitely generated k algebra, and any $\mathrm{Ext}^*(M, N)$ for any finite dimensional modules M and N over the quantum complete intersection is finitely generated as a module over the Hochschild cohomology ring.

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In this paper we give a general description of Hochschild cohomology and homology of finite dimensional quantum complete intersections.

In Theorems 3.4 and 7.4 we explicitly determine a k-basis for the Hochschild cohomology and homology, respectively.

Using these results we study the size of the Hochschild cohomology and homology in the following sense: We denote by

$$\gamma(\mathrm{HH}^*(\Lambda)) = \inf\{t \in \mathbb{N} \mid \limsup \frac{\dim_k \mathrm{HH}^n(\Lambda)}{n^{t-1}} < \infty\}$$

the rate of growth of Hochschild cohomology (and similar for Hochschild homology). We obtain explicit combinatorial formulas for $\gamma(\mathrm{HH}^*(\Lambda))$ and $\gamma(\mathrm{HH}_*(\Lambda))$ in Theorems 4.5 and 8.2, respectively. In particular it will be shown (as Corollary 4.6) that whenever not all commutation parameters are roots of unity we have $\gamma(\mathrm{HH}^*(\Lambda)) \leq c - 2$. For $c = 2$ that means that the Hochschild cohomology is finite. This explains why there are essentially only two cases for $c = 2$, while we obtain more different behaviour for larger c.

We will also generalize Bergh's and Erdmann's result ([2]) in another way: It will be shown that whenever the commutation parameters are sufficiently generic the Hochschild cohomology of the quantum complete intersection is finite (see Example 6.2).

Finally we will study the multiplicative structure of the Hochschild cohomology ring. It will turn out (Theorem 5.5) that it always contains a subring $\mathcal{S}$ which is finitely generated over k, and isomorphic to the quotient of Hochschild cohomology modulo its nilpotent elements. We will give a criterion when the entire Hochschild cohomology ring is finitely generated over this subring (Theorem 5.9). We will give examples (6.4 and 6.5) that all the following behaviors occur (for $c \geq 3$):

- $\mathcal{S} = k$, but $\gamma(\mathrm{HH}^*(\Lambda)) = c - 2$,
- $\gamma(\mathcal{S}) = \gamma(\mathrm{HH}^*(\Lambda)) = c - 2$, and $\mathrm{HH}^*(\Lambda)$ is finitely generated over $\mathcal{S}$, and
- $\gamma(\mathcal{S}) = \gamma(\mathrm{HH}^*(\Lambda)) = c-2$, but $\mathrm{HH}^*(\Lambda)$ is not finitely generated over $\mathcal{S}$.

2. Notation and background

Throughout this paper we assume k to be field.

Quantum complete intersections. (see also [2, 3, 4])

A finite dimensional *quantum complete intersection* of codimension c is a k-algebra of the form

$$\Lambda_{\mathbf{q}}^{\mathbf{n}} = \frac{k\langle x_1, \ldots, x_c\rangle}{\begin{pmatrix} x_i^{n_i} & \text{for } 1 \leq i \leq c \\ x_j x_i - q_{ij} x_i x_j & \text{for } 1 \leq i < j \leq c \end{pmatrix}}$$

with $\mathbf{n} = (n_1, \dots, n_c) \in \mathbb{N}^c_{\geq 2}$ and $\mathbf{q} = (q_{ij} \mid i < j) \in (k^\times)^{\frac{n(n-1)}{2}}$, where $k^\times$ denotes the multiplicative group $k \setminus \{0\}$. For convenience of notation we also define q_{ij} for $i \geq j$: We set $q_{ii} = 1$ for any $i \in \{1, \dots, c\}$ and $q_{ij} = q_{ji}^{-1}$ for $1 \leq j < i \leq c$. Note that the relations $x_j x_i - q_{ij} x_i x_j$ for $1 \leq j \leq i \leq c$ are automatically satisfied in $\Lambda^{\mathbf{n}}_{\mathbf{q}}$.

Note that $\Lambda^{\mathbf{n}}_{\mathbf{q}}$ is a $\mathbb{Z}^c$-graded algebra by $|x_i| = \operatorname{degree}(x_i) = e_i$, the i-th unit vector. We will denote by $\leq$ the partial order on $\mathbb{Z}^c$ defined by comparing vectors component wise, and by $\mathbf{1} = \sum e_i$ the vector with 1 in every component. With this notation we have that the dimensions of the graded component of degree $\mathbf{d}$ (with $\mathbf{d} \in \mathbb{Z}^c$) is

$$\dim(\Lambda^{\mathbf{n}}_{\mathbf{q}})_{\mathbf{d}} = \begin{cases} 1 & \text{if } \mathbf{0} \leq \mathbf{d} \leq \mathbf{n} - \mathbf{1} \\ 0 & \text{otherwise.} \end{cases}$$

For $\mathbf{a} \in \mathbb{N}^c$ (here $\mathbb{N}$ denotes the non-negative integers, i.e. includes 0) we will write $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_c^{a_c}$. Note that the multiplication yields something different if we multiply in another order. In particular we do not have $\mathbf{x}^{\mathbf{a}} \mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$. By setting

$$\mathbf{q}^{\langle \mathbf{a} | \mathbf{b} \rangle} = \prod_{\substack{i,j \in \{1 \dots c\} \\ i < j}} q_{ij}^{a_j b_i}$$

we obtain the multiplication formula $\mathbf{x}^{\mathbf{a}} \mathbf{x}^{\mathbf{b}} = \mathbf{q}^{\langle \mathbf{a} | \mathbf{b} \rangle} \mathbf{x}^{\mathbf{a}+\mathbf{b}}$.

Hochschild (co)homology. Let Λ be a finite dimensional algebra. We set $\Lambda^{\mathrm{en}} = \Lambda \otimes_k \Lambda^{\mathrm{op}}$ the enveloping algebra. Then Λ^{en}-modules are Λ-Λ bimodules on which k acts centrally. In particular Λ has a natural structure of a Λ^{en}-module. Then

$$\begin{aligned} \mathrm{HH}^*(\Lambda) &= \mathrm{Ext}^*_{\Lambda^{\mathrm{en}}}(\Lambda, \Lambda) \text{ and} \\ \mathrm{HH}_*(\Lambda) &= \mathrm{Tor}^{\Lambda^{\mathrm{en}}}_*(\Lambda, \Lambda) \end{aligned}$$

are the Hochschild cohomology and Hochschild homology of Λ respectively. With the Yoneda multiplication of extensions HH^* becomes a $\mathbb{Z}$-graded k-algebra, which is graded commutative (see [7]).

Note that if Λ is graded then so is Λ^{en}, and Λ is a graded Λ^{en}-module. It follows that for any $i \in \mathbb{N}$ the Hochschild homology and cohomology groups $\mathrm{HH}_i(\Lambda)$ and $\mathrm{HH}^i(\Lambda)$ are also graded.

Projective resolutions. In order to determine the Hochschild homology and cohomology of a quantum complete intersection $\Lambda = \Lambda^{\mathbf{n}}_{\mathbf{q}}$ we need to find a projective resolution of Λ as Λ^{en}-module. Moreover we want to keep track of the $\mathbb{Z}^c$-grading, so we will need a graded projective resolution.

It has been shown in [3, Lemma 4.5] that we can find such a graded projective resolution by tensoring together the projective resolutions of the $k[x_i]/(x_i^{n_i})$ as $(k[x_i]/(x_i^{n_i}))^{\mathrm{en}}$ modules. To simplify notation we set $\Lambda_i = k[x_i]/(x_i^{n_i})$. Then the graded projective resolution of Λ_i as a bimodule is

$$\mathbb{P}_i\colon \Lambda_i^{\mathrm{en}} \xleftarrow{x_i\otimes 1 - 1\otimes x_i} \Lambda_i^{\mathrm{en}}\langle 1\rangle \xleftarrow{\sum_{k=0}^{n_i-1} x_i^k \otimes x_i^{n_i-1-k}} \Lambda_i^{\mathrm{en}}\langle n_i\rangle \xleftarrow{x_i\otimes 1 - 1\otimes x_i} \Lambda_i^{\mathrm{en}}\langle n_i+1\rangle \longleftarrow \cdots,$$

where $\Lambda_i^{\mathrm{en}}\langle s\rangle$ is the graded module obtained from Λ_i^{en} by increasing the degree of all homogeneous elements by s. Note that here all the bimodules are shifted into place such that all the morphisms have degree 0.

With this notation by [3, Lemma 4.5] we have that the total complex

$$\mathrm{Tot}(\mathbb{P}_1 \otimes_k \mathbb{P}_2 \otimes_k \cdots \otimes_k \mathbb{P}_c)$$

is a graded projective resolution of Λ.

Note that the term in position $\mathbf{p} \in \mathbb{N}^c$ of the c-tuple complex $\mathbb{P}_1 \otimes_k \mathbb{P}_2 \otimes_k \cdots \otimes_k \mathbb{P}_c$ is

$$\Lambda_1^{\mathrm{en}}\left\langle \begin{array}{ll} \frac{p_1}{2}n_1 & \text{if } 2 \mid p_1 \\ \frac{p_1-1}{2}n_1+1 & \text{else} \end{array} \right\rangle \otimes \cdots \otimes \Lambda_c^{\mathrm{en}}\left\langle \begin{array}{ll} \frac{p_c}{2}n_c & \text{if } 2 \mid p_c \\ \frac{p_c-1}{2}n_c+1 & \text{else} \end{array} \right\rangle.$$

To keep notation compact define the function $\mathfrak{s}\colon \mathbb{Z}^c \longrightarrow \mathbb{Z}^c$ by

$$\mathfrak{s}(\mathbf{p})_i = \begin{cases} \frac{p_i}{2}n_i & \text{if } 2 \mid p_i \\ \frac{p_i-1}{2}n_i+1 & \text{else.} \end{cases}$$

Moreover we will also need the following left inverse of the function $\mathfrak{s}$:

$$\begin{aligned} &\mathfrak{p}\colon \mathbb{Z}^c \longrightarrow \mathbb{Z}^c \\ &\mathfrak{p}(\mathbf{s}) = \min\{\mathbf{p} \in \mathbb{Z}^c \mid \mathfrak{s}(\mathbf{p}) \geq \mathbf{s}\} \end{aligned}$$

In the c-tuple complex $\mathbb{P}_1 \otimes_k \mathbb{P}_2 \otimes_k \cdots \otimes_k \mathbb{P}_c$ all terms are of the form $\Lambda_1^{\mathrm{en}}\langle s_1\rangle \otimes_k \cdots \otimes_k \Lambda_c^{\mathrm{en}}\langle s_c\rangle$ for some $\mathbf{s} \in \mathbb{N}^c$. We have to recall how these are identified with $\Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle$.

2.1. **Lemma** ([3, Lemma 4.3]). *For $\mathbf{s} \in \mathbb{Z}^c$ we may identify*

$$\Lambda_1^{\mathrm{en}}\langle s_1\rangle \otimes_k \cdots \otimes_k \Lambda_c^{\mathrm{en}}\langle s_c\rangle = \Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle.$$

If we choose this identification such that

$$(1 \otimes 1) \otimes \cdots \otimes (1 \otimes 1) \longmapsto 1 \otimes 1,$$

then

$$(x_1^{a_1} \otimes x_1^{b_1}) \otimes \cdots \otimes (x_c^{a_c} \otimes x_c^{b_c}) \longmapsto \frac{\mathbf{q}^{\langle \mathbf{s}|\mathbf{s}\rangle}}{\mathbf{q}^{\langle \mathbf{a}+\mathbf{s}|\mathbf{b}+\mathbf{s}\rangle}} \mathbf{x}^{\mathbf{a}} \otimes \mathbf{x}^{\mathbf{b}}.$$

Of course the differentials occurring in the various directions of the c-tuple complex are of particular interest. Therefore we note that under the identification of Lemma 2.1 we have

$$(1\otimes 1)\otimes\cdots\otimes(1\otimes 1)\otimes(x_i\otimes 1-1\otimes x_i)\otimes(1\otimes 1)\otimes\cdots\otimes(1\otimes 1)$$
$$\mapsto \frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}x_i\otimes 1-\frac{1}{\mathbf{q}^{\langle \mathbf{s}|e_i\rangle}}1\otimes x_i \qquad \text{and} \tag{2.1}$$
$$(1\otimes 1)\otimes\cdots\otimes(1\otimes 1)\otimes(\sum_{j=0}^{n_i-1}x_i^j\otimes x_i^{n_i-1-j})\otimes(1\otimes 1)\otimes\cdots\otimes(1\otimes 1)$$
$$\mapsto \sum_{j=0}^{n_i-1}\frac{1}{\mathbf{q}^{\langle je_i|\mathbf{s}\rangle}\mathbf{q}^{\langle \mathbf{s}|(n_i-1-j)e_i\rangle}}x_i^j\otimes x_i^{n_i-1-j}$$
$$=\sum_{j=0}^{n_i-1}\left(\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}\right)^j\left(\frac{1}{\mathbf{q}^{\langle \mathbf{s}|e_i\rangle}}\right)^{n_i-1-j}x_i^j\otimes x_i^{n_i-1-j}. \tag{2.2}$$

Technical notation. We need the following technical definitions to keep notation short in the rest of the paper.

I – We set $Q=(q_{ij})_{ij}$, and think of Q as a (skew symmetric) matrix with entries in the abelian group $k^\times$. That is, Q represents the morphism of abelian groups

$$Q\colon \mathbb{Z}^c \longrightarrow (k^\times)^c$$
$$(d_i)_i \longmapsto (\prod_{j=1}^{c} q_{ij}^{d_j})_i.$$

As usual for matrices we will denote the image of $\mathbf{d}\in\mathbb{Z}^c$ under this map by $Q\mathbf{d}$, and its i-th component by $(Q\mathbf{d})_i$.

For $A,\ B\subseteq\{1,\dots,c\}$ we denote by $Q_{A\times B}$ the submatrix only containing the rows in A and the columns in B, that is the matrix representing the composition

$$\mathbb{Z}^B \hookrightarrow \mathbb{Z}^c \xrightarrow{Q} (k^\times)^c \twoheadrightarrow (k^\times)^A.$$

II – We set

$$\mathcal{R}_i=\begin{cases}\{\zeta\mid \zeta^{n_i}=1\} & \text{if } \operatorname{char}k \text{ divides } n_i\\ \{\zeta\mid\zeta^{n_i}=1 \text{ and } \zeta\neq 1\} & \text{else}\end{cases}.$$

III – For a $\mathbb{Z}$-submodule K of $\mathbb{Z}^a$ we denote by $\operatorname{pos.rk}K$ the rank of the $\mathbb{Z}$-submodule K' of K generated by $K\cap\mathbb{N}^a$. For example, $\operatorname{pos.rk}\left\langle \begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\-1\end{pmatrix}\right\rangle=1$.

3. Hochschild cohomology

We wish to calculate for any $\mathbf{d}\in\mathbb{Z}^c$ the degree $\mathbf{d}$ part of the Hochschild cohomology. Then we will obtain the entire Hochschild cohomology by adding up these parts.

In order to calculate the degree $\mathbf{d}$ part of cohomology we have to first understand the set

$$\operatorname{Hom}^{\mathbf{d}}_{\Lambda^{\mathrm{en}}}(\Lambda^{\mathrm{en}}\langle\mathbf{s}\rangle\,,\Lambda)$$

of degree $\mathbf{d}$ morphisms from the terms of the projective resolution to Λ.

3.1. **Lemma.** *The set* $\operatorname{Hom}^{\mathbf{d}}_{\Lambda^{\mathrm{en}}}(\Lambda^{\mathrm{en}}\langle\mathbf{s}\rangle\,,\Lambda)$ *is non-zero if and only if* $\mathbf{0} \leq \mathbf{s}+\mathbf{d} \leq \mathbf{n}-\mathbf{1}$*, and then it is the one dimensional k-vector space generated by*

$$\begin{aligned} \varphi^{\mathbf{s},\mathbf{d}}\colon \Lambda^{\mathrm{en}}\langle\mathbf{s}\rangle &\longrightarrow \Lambda \\ \mathbf{x}^{\mathbf{a}}\otimes\mathbf{x}^{\mathbf{b}} &\longmapsto \mathbf{q}^{\langle\mathbf{a}+\mathbf{s}+\mathbf{d}|\mathbf{b}+\mathbf{s}+\mathbf{d}\rangle}\mathbf{x}^{\mathbf{a}+\mathbf{s}+\mathbf{d}+\mathbf{b}} \end{aligned}$$

Proof. Clearly any Λ^{en}-homomorphism from $\Lambda^{\mathrm{en}}\langle\mathbf{s}\rangle$ to any other module is uniquely determined by the image of $1\otimes 1$. If the morphism is to be of degree $\mathbf{d}$ then this image can only be a scalar multiple of $\mathbf{x}^{\mathbf{s}+\mathbf{d}}$. We choose the image of $1\otimes 1$ to be $\mathbf{q}^{\langle\mathbf{s}+\mathbf{d}|\mathbf{s}+\mathbf{d}\rangle}\mathbf{x}^{\mathbf{s}+\mathbf{d}}$ and obtain the formula of the lemma by extending Λ^{en}-linearly. $\square$

3.2. **Corollary.**

$$\dim \operatorname{Hom}^{\mathbf{d}}_{\Lambda^{\mathrm{en}}}(\Lambda^{\mathrm{en}}\langle\mathfrak{s}(\mathbf{p})\rangle\,,\Lambda) = \begin{cases} 1 & \textit{if } \mathfrak{p}(-\mathbf{d}) \leq \mathbf{p} \leq \mathfrak{p}(-\mathbf{d})+\mathbf{1} \\ 0 & \textit{otherwise.} \end{cases}$$

This means that for $\mathbf{d} \leq \mathbf{n}-\mathbf{1}$ the c-tuple complex $\operatorname{Hom}^{\mathbf{d}}_{\Lambda^{\mathrm{en}}}(\mathbb{P}_1\otimes\cdots\otimes\mathbb{P}_c,\Lambda)$ is concentrated in a cube (with sides of length 0 (in directions i with $\mathfrak{p}(-\mathbf{d})_i = -1$, i.e. $d_i = n_i - 1$) or 1), where there is a one-dimensional space in each corner of the cube.

Since by formulas (2.1) and (2.2) these are the terms occurring in the projective resolution, we are in particular interested in what the maps $\varphi^{\mathbf{s},\mathbf{d}}$ of Lemma 3.1 do to terms of the form

$$\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}x_i\otimes 1 - \frac{1}{\mathbf{q}^{\langle\mathbf{s}|e_i\rangle}}1\otimes x_i \qquad \text{and}$$
$$\sum_{j=0}^{n_i-1}\left(\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}\right)^{j}\left(\frac{1}{\mathbf{q}^{\langle\mathbf{s}|e_i\rangle}}\right)^{n_i-1-j} x_i^j\otimes x_i^{n_i-1-j}.$$

3.3. **Lemma.** *Let* $\mathbf{s}$ *and* $\mathbf{d}$ *be such that* $\mathbf{0} \leq \mathbf{s}+\mathbf{d} \leq \mathbf{n}-\mathbf{1}$*, and let* $i \in \{1\ldots c\}$*.*

(1) *Assume further that* $s_i + d_i + 1 < n_i$*. Then*

$$\varphi^{\mathbf{s},\mathbf{d}}\left(\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}x_i\otimes 1 - \frac{1}{\mathbf{q}^{\langle\mathbf{s}|e_i\rangle}}1\otimes x_i\right) = 0$$

if and only if $(Q\mathbf{d})_i = 1$ *(for the definition of Q see (I) at the end of Section 2).*

(2) *Assume further that* $s_i + d_i = 0$. *Then*

$$\varphi^{\mathbf{s},\mathbf{d}}\Big(\sum_{j=0}^{n_i-1}\left(\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}\right)^j\left(\frac{1}{\mathbf{q}^{\langle \mathbf{s}|e_i\rangle}}\right)^{n_i-1-j} x_i^j \otimes x_i^{n_i-1-j}\Big) = 0$$

if and only if $(Q\mathbf{d})_i \in \mathcal{R}_i$ *(for the definition of* $\mathcal{R}_i$ *see (II) at the end of Section 2).*

Proof. We only prove (2), the proof of (1) is a similar and simpler straightforward calculation using the formula of Lemma 3.1. By Lemma 3.1 we have

$$\begin{aligned}
&\varphi^{\mathbf{s},\mathbf{d}}\Big(\sum_{j=0}^{n_i-1}\left(\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}\right)^j\left(\frac{1}{\mathbf{q}^{\langle \mathbf{s}|e_i\rangle}}\right)^{n_i-1-j} x_i^j \otimes x_i^{n_i-1-j}\Big)\\
&= \sum_{j=0}^{n_i-1}\left(\frac{1}{\mathbf{q}^{\langle e_i|\mathbf{s}\rangle}}\right)^j\left(\frac{1}{\mathbf{q}^{\langle \mathbf{s}|e_i\rangle}}\right)^{n_i-1-j} \mathbf{q}^{\langle je_i+\mathbf{s}+\mathbf{d}|(n_i-1-j)e_i+\mathbf{s}+\mathbf{d}\rangle}\mathbf{x}^{\mathbf{s}+\mathbf{d}+(n_i-1)e_i}\\
&= \mathbf{q}^{\langle \mathbf{s}+\mathbf{d}|\mathbf{s}+\mathbf{d}\rangle}\sum_{j=0}^{n_i-1}\left(\mathbf{q}^{\langle e_i|\mathbf{d}\rangle}\right)^j\left(\mathbf{q}^{\langle \mathbf{d}|e_i\rangle}\right)^{n_i-1-j}\mathbf{x}^{\mathbf{s}+\mathbf{d}+(n_i-1)e_i}\\
&= \underbrace{\mathbf{q}^{\langle \mathbf{s}+\mathbf{d}|\mathbf{s}+\mathbf{d}\rangle}}_{\neq 0}\mathbf{x}^{\mathbf{s}+\mathbf{d}+(n_i-1)e_i}\cdot\begin{cases} n_i\underbrace{\left(\mathbf{q}^{\langle e_i|\mathbf{d}\rangle}\right)^{n_i-1}}_{\neq 0} & \text{if } \mathbf{q}^{\langle e_i|\mathbf{d}\rangle} = \mathbf{q}^{\langle \mathbf{d}|e_i\rangle}\\[2ex] \frac{\left(\mathbf{q}^{\langle e_i|\mathbf{d}\rangle}\right)^{n_i}-\left(\mathbf{q}^{\langle \mathbf{d}|e_i\rangle}\right)^{n_i}}{\mathbf{q}^{\langle e_i|\mathbf{d}\rangle}-\mathbf{q}^{\langle \mathbf{d}|e_i\rangle}} & \text{otherwise}\end{cases}
\end{aligned}$$

Now the claim follows from the fact that $\frac{\mathbf{q}^{\langle \mathbf{d}|e_i\rangle}}{\mathbf{q}^{\langle e_i|\mathbf{d}\rangle}} = \prod_{j=1}^{c} q_{ij}^{d_j} = (Q\mathbf{d})_i$. □

We have shown that the vanishing of the maps on the edges in direction i of the cube $\mathrm{Hom}^{\mathbf{d}}_{\Lambda^{\mathrm{en}}}(\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c, \Lambda)$ does not depend on $\mathbf{s}$, that is, if one edge in direction i vanishes, then all vanish. Note also that if all the edges in one direction are isomorphisms, then the total complex is acyclic. Hence we have shown

3.4. **Theorem.** *Let* $\Lambda = \Lambda^{\mathbf{n}}_{\mathbf{q}}$ *be a quantum complete intersection, and let* $\mathbf{d} \leq \mathbf{n}-\mathbf{1}$. *We divide the set* $\{1\dots c\}$ *into the following three parts:*

$$\begin{aligned}
I_{max} &= \{i \in \{1\dots c\} \colon d_i = n_i - 1\}\\
I_1 &= \{i \in \{1\dots c\} \colon n_i \mid d_i + 1\} \setminus I_{max}\\
I_2 &= \{i \in \{1\dots c\} \colon n_i \nmid d_i + 1\}
\end{aligned}$$

Then $\mathrm{HH}^{*,\mathbf{d}}(\Lambda) \neq 0$ *if and only if the following hold:*

- *for any* $i \in I_1$ *we have* $(Q\mathbf{d})_i \in \mathcal{R}_i$, *and*
- *for any* $i \in I_2$ *we have* $(Q\mathbf{d})_i = 1$.

In this situation $\mathrm{HH}^{*,\mathbf{d}}(\Lambda)$ *has the* k*-vector space basis*

$$\{E^{\mathbf{d}}_{\mathbf{p}} \mid \mathbf{0} \leq \mathbf{p} \text{ and } \mathfrak{p}(-\mathbf{d}) \leq \mathbf{p} \leq \mathfrak{p}(-\mathbf{d}) + \mathbf{1}\},$$

where $E^{\mathbf{d}}_{\mathbf{p}}$ *is represented by the (degree* $\mathbf{d}$*) map from the c-tuple complex* $\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c$ *to* Λ *(shifted to position* $\mathbf{p}$*) sending* $1 \otimes 1$ *to* $\mathbf{x}^{\mathbf{d}+\mathfrak{s}(\mathbf{p})}$ *in position* $\mathbf{p}$*. In particular* $E^{\mathbf{p}}_{\mathbf{d}}$ *has extension degree* $\sum_{i=1}^{c} p_i$*. Note that the assumptions on* $\mathbf{p}$ *just make sure that* $\mathbf{0} \leq \mathbf{d} + \mathfrak{s}(\mathbf{p}) \leq \mathbf{n} - \mathbf{1}$*, or, in other words, that we are in the cube where* $\operatorname{Hom}^{\mathbf{d}}_{\Lambda^{\mathrm{en}}}(\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c, \Lambda)$ *does not vanish.*

Let us now compare this result to the description of $\operatorname{Ext}^*_\Lambda(k,k)$ obtained in [3]. More precisely: tensoring over Λ with k yields a map from Hochschild cohomology to the Ext-algebra of the Λ-module k. Our aim now is to determine its image. By [3, Theorem 5.3] the latter ring has the following form:

$$\operatorname{Ext}^*_\Lambda(k,k) = \frac{k\langle y_1, \ldots, y_c, z_1, \ldots, z_c\rangle}{\begin{pmatrix} y_j y_i + q_{ij} y_i y_j & \text{for } i \neq j \\ y_j z_i - q_{ij}^{n_i} z_i y_j & \\ z_j z_i - q_{ij}^{n_i n_j} z_i z_j & \\ y_i^2 - z_i & \text{if } n_i = 2 \\ y_i^2 & \text{if } n_i \neq 2 \end{pmatrix}},$$

where $|y_i| = (1, -e_i)$ and $|z_i| = (2, -n_i e_i)$.

3.5. **Corollary.** *With the above notation the image of the map* $(- \otimes_\Lambda k)_*\colon \mathrm{HH}^*(\Lambda) \longrightarrow \operatorname{Ext}^*_\Lambda(k,k)$ *is*

$$\bigoplus_{\substack{\mathbf{d} \in \mathbb{Z}^c \text{ such that} \\ \forall i \text{ with } n_i \mid d_i + 1\colon (Q\mathbf{d})_i \in \mathcal{R}_i \\ \forall i \text{ with } n_i \nmid d_i + 1\colon (Q\mathbf{d})_i = 1}} \operatorname{Ext}^{*,\mathbf{d}}(k,k).$$

That is the sum runs over exactly those graded pieces, where the corresponding graded piece of Hochschild cohomology does not vanish.

Proof. By construction the image cannot be bigger than the sum of the corollary. To see that any $\operatorname{Ext}^{*,\mathbf{d}}(k,k)$ with $\mathbf{d}$ as specified under the sum is contained in the image first note that

$$\dim_k \operatorname{Ext}^{*,\mathbf{d}}(k,k) = \begin{cases} 1 & \text{if } \forall i\colon d_i \leq 0 \text{ and } n_i \mid d_i \vee n_i \mid d_i + 1 \\ 0 & \text{else.} \end{cases}$$

Note that the condition for $\operatorname{Ext}^{*,\mathbf{d}}(k,k)$ not vanishing is equivalent to asking that $\mathbf{d} = -\mathfrak{s}(\mathbf{p})$ for some $\mathbf{p} \in \mathbb{N}^c$. Now by definition $E^{\mathbf{d}}_{\mathbf{p}}$ is represented by a map sending $1 \otimes 1$ to 1 in position $\mathbf{p}$, and hence it does not vanish when being tensored over Λ by k. Therefore the image is at least one dimensional in degree $\mathbf{d}$. $\square$

4. The rate of growth of Hochschild cohomology

In this section we study how big the Hochschild cohomology of a finite dimensional quantum complete intersection is. Our way to measure the size is the rate of growth as explained in the following definition.

4.1. **Definition.** Let $X = \coprod_{i=0}^{\infty} X_i$ be an $\mathbb{N}$-graded k-module, such that the X_i have finite k-dimension. The rate of growth of X, denoted $\gamma(X)$, is defined as

$$\gamma(X) = \inf\{t \in \mathbb{N} \mid \exists a \in \mathbb{N} \text{ such that } \dim_k X_i \leq a i^t \forall i\}.$$

Note that if X is a graded commutative ring which is finitely generated over k, then $\gamma(X) = \text{Krull.dim}\, X$. However this assumption is not always satisfied for the Hochschild cohomology ring of quantum complete intersections (see Sections 5 and 6).

We first decompose Hochschild cohomology as follows:

4.2. **Construction.** For $G \subseteq \{1, \ldots, c\}$ we denote by HH^*_G the k-span of the $E^{\mathbf{d}}_{\mathbf{p}}$ with

$$G = \{i \in \{1, \ldots, c\} \mid d_i < n_i - 1 \text{ or } (Q\mathbf{d})_i \in \mathcal{R}_i\}.$$

That is we take all those $E^{\mathbf{d}}_{\mathbf{p}}$ from Theorem 3.4 such that the indices in G are exactly the ones not in $I_{\max}$, plus those in $I_{\max}$ which fulfill the requirements for elements of I_1 anyway.

Clearly this yields a decomposition $\mathrm{HH}^*(\Lambda) = \bigoplus_{G \subseteq \{1,\ldots,c\}} \mathrm{HH}^*_G$, and hence

$$\gamma(\mathrm{HH}^*(\Lambda)) = \max_{G \subseteq \{1,\ldots,c\}} \gamma(\mathrm{HH}^*_G).$$

4.3. **Proposition.** *For $G \subseteq \{1, \ldots, c\}$ the rate of growth of HH^*_G is*

$$\gamma(\mathrm{HH}^*_G) = \begin{cases} 0 & \textit{if } \mathrm{HH}^*_G = 0 \\ \text{pos.rk}\, \mathrm{Ker}\, Q_{G \times G} & \textit{else} \end{cases}.$$

(For the definition of pos.rk *see (III) at the end of Section 2.) In particular we always have $\gamma(\mathrm{HH}^*_G) \leq |G|$.*

For the proof we will need the following observation.

4.4. **Observation.** *Let $K \leq \mathbb{Z}^a$ be a submodule. The k-module with basis $K \cap \mathbb{N}^a$ is $\mathbb{Z}$-graded by $|\mathbf{x}| = \sum_{i=1}^a x_i$ for $\mathbf{x} \in K$. With this grading, its rate of growth is $\gamma(k(K \cap \mathbb{N}^a)) = \text{pos.rk}\, K$.*

Proof of 4.3. We write $\overline{G} = \{1, \ldots, c\} \setminus G$. By construction HH^*_G has the k-basis

$$\begin{aligned} \{E^{\mathbf{d}}_{\mathbf{p}} \mid & \mathbf{p} \geq \mathbf{0},\ \mathbf{d} \leq \mathbf{n} - \mathbf{1},\ \mathfrak{p}(-\mathbf{d}) \leq \mathbf{p} \leq \mathfrak{p}(-\mathbf{d}) + \mathbf{1}, \\ & \forall i \in \overline{G}\colon d_i = n_i - 1 \text{ and } (Q\mathbf{d})_i \notin \mathcal{R}_i, \\ & \forall i \in G \text{ with } n_i \mid d_i + 1\colon (Q\mathbf{d})_i \in \mathcal{R}_i, \\ & \forall i \in G \text{ with } n_i \nmid d_i + 1\colon (Q\mathbf{d})_i = 1\}, \end{aligned}$$

and the extension degree of $E^{\mathbf{d}}_{\mathbf{p}}$ is $\sum_{i=1}^c p_i$.

Note that the map $\mathfrak{p}$ is linear up to some rounding. Hence we may calculate the rate of growth with respect to the grading given by $-\sum_{i=1}^c d_i$.

Since for any $\mathbf{d}$ there are at least one and at most 2^c values of $\mathbf{p}$ satisfying the conditions of the set above, we may disregard the number of different choices for $\mathbf{p}$ for a given $\mathbf{d}$.

Finally, since $\mathbf{d}$ is fixed outside G, we may restrict our attention to the G part of the indices. That is, we need to understand the rate of growth of the k-module with basis $\mathcal{B}_{\{1,\dots,c\}}$, where for $G \subseteq G' \subseteq \{1,\dots,c\}$ we set

$$\begin{aligned}\mathcal{B}_{G'} = \{\mathbf{d}_G \in \mathbb{Z}^G \mid \ & \mathbf{d}_G \leq \mathbf{n}_G - \mathbf{1} \\ & \forall i \in G' \cap \overline{G} \colon Q_{\{i\}\times G}\mathbf{d}_G \cdot Q_{\{i\}\times\overline{G}}(\mathbf{n}_{\overline{G}} - \mathbf{1}) \notin \mathcal{R}_i \\ & \forall i \in G \text{ with } n_i \mid d_i + 1 \colon Q_{\{i\}\times G}\mathbf{d}_G \cdot Q_{\{i\}\times\overline{G}}(\mathbf{n}_{\overline{G}} - \mathbf{1}) \in \mathcal{R}_i \\ & \forall i \in G \text{ with } n_i \nmid d_i + 1 \colon Q_{\{i\}\times G}\mathbf{d}_G \cdot Q_{\{i\}\times\overline{G}}(\mathbf{n}_{\overline{G}} - \mathbf{1}) = 1\}\end{aligned}$$

Note that for $G' \subseteq G''$ we have $\mathcal{B}_{G'} \supseteq \mathcal{B}_{G''}$. In particular $\mathcal{B}_{\{1,\dots,c\}} \subseteq \mathcal{B}_G$.

Now $\mathcal{B}_G$ is invariant under adding elements of the set

$$-(\prod_{i\in G} n_i\mathbb{N}) \cap \operatorname{Ker} Q_{G\times G},$$

and contains only finitely many elements which are not obtained from another element by such an addition. Hence, if $\mathcal{B}_G$ is non-empty, the rate of growth of the k-module with basis $\mathcal{B}_G$ is identical to the rate of growth of the k-module with basis $\mathbb{N}^G \cap \operatorname{Ker} Q_{G\times G}$, which, by Observation 4.4, is $\operatorname{pos.rk} \operatorname{Ker} Q_{G\times G}$.

It follows that $\gamma(\mathrm{HH}^*_G) \leq \operatorname{pos.rk} \operatorname{Ker} Q_{G\times G}$.

Now we let $\widehat{G}$ be maximal with $G \subseteq \widehat{G} \subseteq \{1,\dots,c\}$ such that $\operatorname{pos.rk} \operatorname{Ker} Q_{\widehat{G}\times G} = \operatorname{pos.rk} \operatorname{Ker} Q_{G\times G}$. It follows as in the discussion above that if $\mathcal{B}_{\widehat{G}} \neq \emptyset$ then the rate of growth of the k-module with basis $\mathcal{B}_{\widehat{G}}$ is $\operatorname{pos.rk} \operatorname{Ker} Q_{G\times G}$.

Finally let $i \notin \widehat{G}$. Using similar arguments as above one sees that the rate of growth of the free module with basis $\mathcal{B}_G \setminus \mathcal{B}_{G\cup\{i\}}$ is strictly smaller than $\operatorname{pos.rk} \operatorname{Ker} Q_{G\times G}$.

Since

$$\mathcal{B}_{\{1,\dots,c\}} = \mathcal{B}_{\widehat{G}} \setminus (\bigcup_{i\notin\widehat{G}} (\mathcal{B}_G \setminus \mathcal{B}_{G\cup\{i\}}))$$

it follows that, provided $\mathcal{B}_{\{1,\dots,c\}} \neq \emptyset$, the rate of growth of the k-module with basis $\mathcal{B}_{\{1,\dots,c\}}$ is $\operatorname{pos.rk} \operatorname{Ker} Q_{G\times G}$. □

Summing up the results for the HH^*_G we have shown

4.5. **Theorem.** *The rate of growth of the Hochschild cohomology of a finite dimensional quantum complete intersection is*

$$\begin{aligned} \max\{\operatorname{pos.rk}\operatorname{Ker} Q_{G\times G} \mid & G = \{i \in \{1,\dots,c\} \mid d_i < n_i - 1 \text{ or } (Q\mathbf{d})_i \in \mathcal{R}_i\} \\ & \text{for some } \mathbf{d} \in \mathbb{Z}^c \text{ with } \mathbf{d} \le \mathbf{n} - \mathbf{1}, \\ & \forall i \text{ with } n_i \mid d_i + 1 \text{ and } d_i < 0\colon (Q\mathbf{d}_i) \in \mathcal{R}_i, \text{ and} \\ & \forall i \text{ with } n_i \nmid d_i + 1\colon (Q\mathbf{d}_i) = 1\}. \end{aligned}$$

4.6. **Corollary.** *For a finite quantum complete intersection either all q_{ij} are roots of unity, or the rate of growth of Hochschild cohomology is at most $c-2$.*

Proof. Assume not all q_{ij} are roots of unity. Then we have $\operatorname{rk}\operatorname{Ker} Q \le c-2$, since Q is skew symmetric. Hence $\operatorname{pos.rk}\operatorname{Ker} Q \le c-2$. Now we consider G with $|G| = c-1$, that is $G = \{1,\dots,c\} \setminus \{h\}$ for some h. If $\operatorname{rk}\operatorname{Ker} Q_{G\times G} \le c-2$ there is nothing to show, so assume $Q_{G\times G}$ only contains roots of unity. Since Q does not only contain roots of unity there is $i \in G$ such that q_{ih} is not a root of unity. But then $(Q\mathbf{d})_i$ cannot be a root of unity for any $\mathbf{d} \in \mathbb{Z}^c$ with $d_h = n_h - 1 \neq 0$. Hence this G is not to be considered in the maximum of Theorem 4.5. □

5. On the multiplicative structure of Hochschild cohomology

In this section we will identify a subring $\mathcal{S}$ of the Hochschild cohomology ring, which is a finitely generated commutative k-algebra without zero divisors, and is isomorphic to Hochschild cohomology modulo nilpotent objects. We will completely describe $\mathcal{S}$, determine its Krull dimension, and determine when the entire Hochschild cohomology ring is finitely generated as a module over $\mathcal{S}$.

By Theorem 3.4 we know that Hochschild cohomology has a k-vector space basis

$$\begin{aligned} \{E_{\mathbf{p}}^{\mathbf{d}} \mid \mathbf{d} \text{ such that } \forall i\colon\ & n_i \mid d_i + 1 > 0 \Rightarrow (Q\mathbf{d})_i \in \mathcal{R}_i, \\ & n_i \nmid d_i + 1 \Rightarrow (Q\mathbf{d})_i = 1, \\ \mathbf{p} \ge \mathbf{0}, \text{ and } & \mathfrak{p}(-\mathbf{d}) \le \mathbf{p} \le \mathfrak{p}(-\mathbf{d}) + \mathbf{1}\}. \end{aligned}$$

For simplicity of notation we set $E_{\mathbf{p}}^{\mathbf{d}} = 0$ whenever $\mathbf{d}$ and $\mathbf{p}$ do not satisfy the conditions above. Then we always have

$$E_{\mathbf{p}}^{\mathbf{d}} \cdot E_{\mathbf{p}'}^{\mathbf{d}'} \in kE_{\mathbf{p}+\mathbf{p}'}^{\mathbf{d}+\mathbf{d}'}.$$

5.1. **Lemma.** *Assume $\mathfrak{s}(\mathbf{p}) \neq -\mathbf{d}$. Then $E_{\mathbf{p}}^{\mathbf{d}}$ is nilpotent.*

Proof. Let i be such that $\mathfrak{s}(\mathbf{p})_i > -d_i$. Then

$$\mathfrak{s}(n_i\mathbf{p})_i \ge n_i\mathfrak{s}(\mathbf{p})_i \ge n_i(1-d_i) \ge n_i - n_i d_i,$$

and hence $(E_{\mathbf{p}}^{\mathbf{d}})^{n_i} \in kE_{n_i\mathbf{p}}^{n_i\mathbf{d}} = 0$. □

We are particularly interested in the non-nilpotent elements of the Hochschild cohomology ring. For simplicity of notation, we give the remaining candidates a new name:

$$s_{\mathbf{p}} := E_{\mathbf{p}}^{-\mathfrak{s}(\mathbf{p})}$$

5.2. **Lemma.** *Let $\mathbf{p} \in \mathbb{N}^c$ such that there is $i \in \{1, \dots, c\}$ with $n_i > 2$ and p_i is odd. Then $s_{\mathbf{p}}$ is nilpotent.*

Proof. Straightforward calculation shows that $(s_{\mathbf{p}})^2$ satisfies the assumption of Lemma 5.1. □

Now we set

$$\begin{aligned} \mathcal{S} = {}_k\langle s_{\mathbf{p}} \,|\, &\forall i \text{ with } p_i \text{ even: } (Q\mathfrak{s}(\mathbf{p}))_i = 1 \\ &\forall i \text{ with } p_i \text{ odd: } n_i = 2 \text{ and } (Q\mathfrak{s}(\mathbf{p}))_i = -1 \rangle \end{aligned}$$

By the above two lemmas the composition $\mathcal{S} \hookrightarrow \mathrm{HH}^*(\Lambda) \twoheadrightarrow \frac{\mathrm{HH}^*(\Lambda)}{(\text{nilpotence})}$ is onto.

Our next aim is to understand how the elements of $\mathcal{S}$ are multiplied with each other and with the other $E_{\mathbf{p}}^{\mathbf{d}}$. To do so we lift the map representing $s_{\mathbf{p}}$, with $\mathbf{p}$ as in the definition of $\mathcal{S}$, to a map of c-tuple complexes.

5.3. **Lemma.** *The element $s_{\mathbf{p}}$ with $\mathbf{p}$ as in the definition of $\mathcal{S}$ is represented by the map of c-tuple complexes $\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c \longrightarrow (\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c)[\mathbf{p}]$, which sends $1 \otimes 1$ to $\frac{1}{\mathbf{q}^{\langle \mathfrak{s}(\mathbf{r}) | \mathfrak{s}(\mathbf{p}) \rangle}} \cdot 1 \otimes 1$ in position $\mathbf{p} + \mathbf{r}$.*

Proof. It suffices to verify that the map given in the lemma is a map of c-tuple complexes, since then it clearly does the right thing in position $\mathbf{p}$. This amount to checking that various squares commute. Doing so is a (somewhat tiresome) straightforward calculation, with four different cases according to the parity of the p_i and r_i. □

Note that when passing from c-tuple complexes to their total complexes some maps need to be multiplied by -1. One choice of doing so is to multiply the map in direction i from position $\mathbf{p} + e_i$ to $\mathbf{p}$ by $\prod_{j<i}(-1)^{p_i}$. With this convention the following is an immediate consequence of Lemma 5.3.

5.4. **Corollary.** *We have*

$$s_{\mathbf{p}} E_{\mathbf{p}'}^{\mathbf{d}} = \frac{\prod_{j<i}(-1)^{p_j p'_i}}{\mathbf{q}^{\langle \mathfrak{s}(\mathbf{p}') | \mathfrak{s}(\mathbf{p}) \rangle}} E_{\mathbf{p}+\mathbf{p}'}^{\mathbf{d}-\mathfrak{s}(\mathbf{p})}$$

and in particular

$$s_{\mathbf{p}} s_{\mathbf{p}'} = \frac{\prod_{j<i}(-1)^{p_j p'_i}}{\mathbf{q}^{\langle \mathfrak{s}(\mathbf{p}') | \mathfrak{s}(\mathbf{p}) \rangle}} s_{\mathbf{p}+\mathbf{p}'}.$$

From these results we obtain the following theorem.

5.5. **Theorem.** *The Hochschild cohomology ring of a quantum complete intersection has a subring $\mathcal{S}$ which is isomorphic to*

$$_k\Big\langle y_1^{\frac{p_1 n_1}{2}} \cdots y_c^{\frac{p_c n_c}{2}} \in k[y_1, \dots y_c] \;|\forall i \text{ with } p_i \text{ even: } (Q\mathfrak{s}(\mathbf{p}))_i = 1, \text{ and}$$
$$\forall i \text{ with } p_i \text{ odd: } n_i = 2 \text{ and } (Q\mathfrak{s}(\mathbf{p}))_i = -1\Big\rangle.$$

In particular $\mathcal{S}$ is a finitely generated k-algebra without zero-divisors.

Moreover the composition $\mathcal{S} \hookrightarrow \mathrm{HH}^*(\Lambda) \twoheadrightarrow \frac{\mathrm{HH}^*(\Lambda)}{(nilpotence)}$ is an isomorphism. Hence $\frac{\mathrm{HH}^*(\Lambda)}{(nilpotence)}$ is a split quotient of $\mathrm{HH}^*(\Lambda)$, which is isomorphic to $\mathcal{S}$.

Proof. The fact that the $s_{\mathbf{p}}$ commute can be check directly, using the formula of Corollary 5.4. Alternatively note that, since by that formula $s_{\mathbf{p}}^2 \neq 0$, we have that $s_{\mathbf{p}}$ lies in the even part of Hochschild cohomology, or $\operatorname{char} k = 2$. In both cases it follows from general theory that $s_{\mathbf{p}}$ lies in the center of the Hochschild cohomology ring.

Thus $\mathcal{S}$ has the form described in the theorem.

To see that $\mathcal{S}$ is finitely generated as a k-algebra we partially order the set $\{y_1^{\frac{p_1 n_1}{2}} \cdots y_c^{\frac{p_c n_c}{2}} \in \mathcal{S}\}$ by comparing the exponents component wise. Since the ideal in $k[y_1, \dots, y_c]$ generated by this set is finitely generated it follows that there are only finitely many minimal elements with respect to this partial order. We claim that these generate $\mathcal{S}$ as k-algebra. Assume that $y_1^{\frac{p_1 n_1}{2}} \cdots y_c^{\frac{p_c n_c}{2}} \in \mathcal{S}$ is not minimal. Then one easily sees that $y_1^{\frac{p_1 n_1}{2}} \cdots y_c^{\frac{p_c n_c}{2}}$ is the product of two smaller elements of this form (for instance one of them could be chosen minimal). Iterating this we see that any $y_1^{\frac{p_1 n_1}{2}} \cdots y_c^{\frac{p_c n_c}{2}} \in \mathcal{S}$ is a product of minimal ones.

The final part of the theorem follows from the comment below the definition of $\mathcal{S}$, and Corollary 5.4. □

We conclude this section by giving a precise criterion of when the entire Hochschild cohomology ring is finitely generated over $\mathcal{S}$.

5.6. **Lemma.** *The decomposition $\mathrm{HH}^*(\Lambda) = \bigoplus_{G \subseteq \{1,\dots,c\}} \mathrm{HH}_G^*$ from Construction 4.2 respects the $\mathcal{S}$-module structure.*

Proof. This follows immediately from the definition of the HH_G^*, and from the multiplication formula in Corollary 5.4. □

5.7. **Proposition.** *The module HH_G^* is finitely generated over $\mathcal{S}$ if and only if one of the following holds.*

(1) $\mathrm{HH}_G^* = 0$, *or*
(2) $\operatorname{pos.rk} \operatorname{Ker} Q_{G \times G} = \operatorname{pos.rk} \operatorname{Ker} Q_{\{1,\dots,c\} \times G}$.

Proof. Clearly we may assume $\mathrm{HH}_G^* \neq 0$. Note that the $s_{\mathbf{p}}$ with $p_i \neq 0$ for some $i \in \{1, \dots, c\} \setminus G$ annihilate HH_G^*, and hence that $\mathrm{HH}_G^* \neq 0$ is actually a module over the split quotient

$$\mathcal{S}_G := {}_k\langle s_{\mathbf{p}} \in \mathcal{S} \mid \forall i \colon i \in G \vee p_i = 0\rangle.$$

Now note that $\gamma(\mathcal{S}_G) = \text{pos.rk}\,\text{Ker}\,Q_{\{1,\dots,c\}\times G}$ by Observation 4.4.

Moreover $\mathcal{S}_G$ acts on HH^*_G without zero-divisors: Since both the $\mathcal{S}_G$ and HH^*_G are $\mathbb{Z}^c$-graded it suffices to look at graded parts. For those this is immediate from the multiplication formula in Corollary 5.4.

Now the claim follows. □

5.8. **Corollary.** *For any finite dimensional quantum complete intersection* $\text{HH}^*_{\{1,\dots,c\}}$ *is a finitely generated* $\mathcal{S}$*-module.*

5.9. **Theorem.** *The Hochschild cohomology ring is finitely generated as a module over* $\mathcal{S}$ *if and only if*

$$\forall G \subseteq \{1,\dots,c\} \text{ such that there is } \mathbf{d} \in \mathbb{Z}^c \text{ with } \mathbf{d} \le \mathbf{n}-\mathbf{1},$$
$$\forall i \in \{1,\dots,c\} \setminus G\colon d_i = n_i - 1,$$
$$\forall i \in G \text{ with } n_i \mid d_i + 1\colon (Q\mathbf{d}_i) \in \mathcal{R}_i, \text{ and}$$
$$\forall i \in G \text{ with } n_i \nmid d_i + 1\colon (Q\mathbf{d}_i) = 1$$
$$\text{the equality } \text{pos.rk}\,\text{Ker}\,Q_{G\times G} = \text{pos.rk}\,\text{Ker}\,Q_{\{1,\dots,c\}\times G} \text{ holds.}$$

6. Examples

The following result has been obtained in [2].

6.1. **Example.** Let $\Lambda = \Lambda^{n_1,n_2}_{q_{12}}$ be a codimension 2 quantum complete intersection, such that q_{12} is not a root of unity. Let $\mathbf{d} = (d_1,d_2) \le (n_1-1,n_2-1)$. Then $\text{HH}^{*,\mathbf{d}}(\Lambda)$ does not vanish if and only if for any $i \in \{1,2\}$ we have $d_i = n_i - 1$ or $n_i \nmid d_i + 1$ and $q_{12}^{d_{3-i}} = 1$. Since q_{12} is not a root of unity this means that whenever one of the d_i is not $n_i - 1$, then d_{3-i} has to be 0. Therefore the only $\mathbf{d}$ which contribute to Hochschild cohomology are (n_1-1,n_2-1) and $(0,0)$. For $\mathbf{d} = (n_1-1,n_2-1)$ we obtain

$$I_{\max} = \{1,2\} \qquad I_1 = \emptyset \qquad I_2 = \emptyset \qquad \mathfrak{p}(-\mathbf{d}) = (-1,-1)$$

and hence

$$\text{HH}^{*,\mathbf{d}}(\Lambda) = {}_k\Big\langle E^{(n_1-1,n_2-1)}_{(0,0)} \Big\rangle.$$

For $\mathbf{d} = (0,0)$ we obtain

$$I_{\max} = \emptyset \qquad I_1 = \emptyset \qquad I_2 = \{1,2\} \qquad \mathfrak{p}(-\mathbf{d}) = (0,0)$$

and hence

$$\text{HH}^{*,\mathbf{d}}(\Lambda) = {}_k\Big\langle E^{(0,0)}_{(0,0)}, E^{(0,0)}_{(0,1)}, E^{(0,0)}_{(1,0)}, E^{(0,0)}_{(1,1)} \Big\rangle.$$

Summing up we obtain

$$\text{HH}^{*,\mathbf{d}}(\Lambda) = {}_k\Big\langle E^{(n_1-1,n_2-1)}_{(0,0)}, E^{(0,0)}_{(0,0)}, E^{(0,0)}_{(0,1)}, E^{(0,0)}_{(1,0)}, E^{(0,0)}_{(1,1)} \Big\rangle,$$

and hence

$$\dim \text{HH}^*(\Lambda) = (2,2,1,0,\dots).$$

We generalize this example to arbitrary codimension:

6.2. **Example.** Let $c \geq 2$ and the q_{ij} generic (that means, $(Q\mathbf{d})_i$ is a root of unity only if $d_j = 0$ for all $j \neq i$). Then $\mathrm{HH}^{*,\mathbf{d}}(\Lambda) \neq 0$ only for $\mathbf{d} = \mathbf{n} - \mathbf{1}$ or $\mathbf{d} = \mathbf{0}$. Similar to Example 6.1 we obtain

$$\mathrm{HH}^{*,\mathbf{n-1}}(\Lambda) = {}_k\langle E_{\mathbf{0}}^{\mathbf{n-1}} \rangle,$$

and

$$\mathrm{HH}^{*,\mathbf{0}}(\Lambda) = {}_k\langle E_{\mathbf{p}}^{\mathbf{0}} \mid \mathbf{0} \leq \mathbf{p} \leq \mathbf{1} \rangle.$$

In particular

$$\dim \mathrm{HH}^*(\Lambda) = (1 + \tbinom{c}{0}, \tbinom{c}{1}, \tbinom{c}{2}, \tbinom{c}{c}, \ldots).$$

Since the total dimension is finite, the rate of growth $\gamma(\mathrm{HH}^*(\Lambda)) = 0$, and $\mathcal{S} = k$.

Now let us look at the other extreme case. This case has already been studied in [3].

6.3. **Example.** Let $c \geq 2$ and let all q_{ij} be roots of unity. Then

$$\operatorname{pos.rk} \operatorname{Ker} Q_{G' \times G} = \operatorname{rk} \operatorname{Ker} Q_{G' \times G} = |G|$$

for any $G, G' \subseteq \{1, \ldots, c\}$. Hence $\mathrm{HH}^*(\Lambda)$ is finitely generated over $\mathcal{S}$, and

$$\operatorname{Krull.dim} \mathcal{S} = \gamma(\mathrm{HH}^*(\Lambda)) = c.$$

The final two examples illustrate that in the case $\gamma(\mathrm{HH}^*(\Lambda)) = c - 2$ very different kinds of behaviour can occur.

6.4. **Example.** Let $q \in k^\times$ not a root of unity, and $c \in \mathbb{N}_{\geq 3}$. Let Λ be a codimension c quantum complete intersection with

$$\begin{aligned} &q_{ij} = 1 \quad \text{for } i, j < c-1, && q_{i,c-1} = q \quad \text{for } i < c-1, \\ &q_{ic} = q^{-1} \quad \text{for } i < c-1, && q_{c-1,c} = q^{-1}. \end{aligned}$$

One easily sees that $\mathcal{S} = k$. A case by case study (according to for which i we have $d_i = n_i - 1$) shows that the subspace $\mathrm{HH}^*_{\{1,\ldots,c-2\}}$ has a finite dimensional complement in $\mathrm{HH}^*(\Lambda)$. It is non-empty if and only if $n_{c-1} = n_c$, and in that case

$$\begin{aligned} \gamma(\mathrm{HH}^*(\Lambda)) &= \gamma(\mathrm{HH}^*_{\{1,\ldots,c-2\}}) \\ &= \operatorname{pos.rk} \operatorname{Ker} \underbrace{Q_{\{1,\ldots,c-2\} \times \{1,\ldots,c-2\}}}_{=0} \\ &= c - 2. \end{aligned}$$

6.5. **Example.** Let $q \in k^\times$ not a root of unity, and $c \in \mathbb{N}_{\geq 3}$ and (for simplicity) $\operatorname{char} k \neq 2$. Let Λ be a codimension c quantum complete intersection with

$$\begin{aligned} &q_{ij} = 1 \quad \text{for } i, j < c-1, && q_{i,c-1} = q \quad \text{for } i < c-1, \\ &q_{ic} = q^{-1} \quad \text{for } i < c-1, && q_{c-1,c} = q. \end{aligned}$$

Then we have

$$\mathcal{S} = {}_k\left\langle s_{\mathbf{p}} \mid \forall i\colon p_i \text{ even, and } (Q\left(\frac{p_j n_j}{2}\right)_j)_i = 1 \right\rangle$$
$$= {}_k\left\langle s_{\mathbf{p}} \mid \forall i\colon p_i \text{ even, and } \sum_{j=0}^{c-2} p_j n_j = p_{c-1}n_{c-1} = p_c n_c \right\rangle.$$

In particular

$$\operatorname{Krull.dim} \mathcal{S} = c - 2,$$

and hence (by Corollary 4.6) also $\gamma(\mathrm{HH}^*(\Lambda)) = c - 2$.

Similar to Example 6.4 one sees that $\mathrm{HH}^*_{\{1,\ldots,c-2\}} + \mathrm{HH}^*_{\{1,\ldots,c\}}$ form a subspace of $\mathrm{HH}^*(\Lambda)$ which has a finite dimensional complement. Since by Corollary 5.8 $\mathrm{HH}^*_{\{1,\ldots,c\}}$ is always finitely generated over $\mathcal{S}$, we only have to look at $\mathrm{HH}^*_{\{1,\ldots,c-2\}}$. As in Example 6.4, one sees that $\mathrm{HH}^*_{\{1,\ldots,c-2\}} \neq 0$ if and only if $n_{c-1} = n_c$. Since

$$\operatorname{pos.rk} \operatorname{Ker} Q_{\{1,\ldots,c-2\}\times\{1,\ldots,c-2\}} = c-2 \neq 0 = \operatorname{pos.rk} \operatorname{Ker} Q_{\{1,\ldots,c\}\times\{1,\ldots,c-2\}}$$

it follows that $\mathrm{HH}^*(\Lambda)$ is finitely generated over $\mathcal{S}$ if and only if $n_{c-1} \neq n_c$.

7. Hochschild homology

To calculate Hochschild homology, we proceed as for Hochschild cohomology. That is we calculate for any $\mathbf{d} \in \mathbb{Z}^c$ the degree $\mathbf{d}$ part of the Hochschild homology. The actual calculations are very similar to the corresponding ones in Section 3, and will therefore be omitted here.

7.1. **Observation.** *The degree* $\mathbf{d}$*-part* $(\Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle \otimes_{\Lambda^{\mathrm{en}}} \Lambda)_{\mathbf{d}}$ *is non-zero if and only if* $\mathbf{s} \leq \mathbf{d} \leq \mathbf{s} + \mathbf{n} - \mathbf{1}$. *Moreover in that case it is one dimensional.*

As in the case of cohomology, it follows that the c-tuple complex $(\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c \otimes_{\Lambda^{\mathrm{en}}} \Lambda)_{\mathbf{d}}$ is concentrated in a cube (with sides of length 0 or 1), where there is a one-dimensional space in each corner of the cube.

Next we need to understand what happens to a map $f\colon \Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle \longrightarrow \Lambda^{\mathrm{en}}\langle \mathbf{s}'\rangle$ when it is tensored over Λ^{en} with Λ.

7.2. **Lemma.** *Let* $f\colon \Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle \longrightarrow \Lambda^{\mathrm{en}}\langle \mathbf{s}'\rangle$. *Then*

$$\begin{aligned} f \otimes_{\Lambda^{\mathrm{en}}} \Lambda\colon \Lambda\langle \mathbf{s}\rangle &\longrightarrow \Lambda\langle \mathbf{s}'\rangle \\ \mathbf{x}^{\mathbf{a}} &\longmapsto \sum_i f_2^i \mathbf{x}^{\mathbf{a}} f_1^i, \end{aligned}$$

where $f(1 \otimes 1) = \sum_i f_1^i \otimes f_2^i$.

Now we are ready to calculate what tensoring over Λ^{en} with Λ does to the maps occurring in the c-tuple complex $\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c$.

7.3. **Lemma.** *Let* $\mathbf{s}$ *and* $\mathbf{d}$ *such that* $\mathbf{s} \leq \mathbf{d} \leq \mathbf{s} + \mathbf{n} - \mathbf{1}$, *and let* $i \in \{1, \ldots, c\}$.

(1) *Assume further that $d_i > s_i$. The map $(\Lambda\langle \mathbf{s} + e_i\rangle)_{\mathbf{d}} \longrightarrow (\Lambda\langle \mathbf{s}\rangle)_{\mathbf{d}}$ induced obtained by tensoring the map*

$$\begin{aligned} \Lambda^{\mathrm{en}}\langle \mathbf{s} + e_i\rangle &\longrightarrow \Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle \\ 1 \otimes 1 &\longmapsto \frac{1}{\mathbf{q}^{\langle e_i | \mathbf{s}\rangle}} x_i \otimes 1 - \frac{1}{\mathbf{q}^{\langle \mathbf{s} | e_i\rangle}} 1 \otimes x_i \end{aligned}$$

over Λ^{en} with Λ and taking the degree $\mathbf{d}$ part vanishes if and only if $(Q\mathbf{d})_i = 1$.

(2) *Assume further that $d_i = s_i + n_i - 1$. The map $(\Lambda\langle \mathbf{s} + (n_i - 1)e_i\rangle)_{\mathbf{d}} \longrightarrow (\Lambda\langle \mathbf{s}\rangle)_{\mathbf{d}}$ induced obtained by tensoring the map*

$$\begin{aligned} \Lambda^{\mathrm{en}}\langle \mathbf{s} + e_i\rangle &\longrightarrow \Lambda^{\mathrm{en}}\langle \mathbf{s}\rangle \\ 1 \otimes 1 &\longmapsto \sum_{j=0}^{n_i - 1} \left(\frac{1}{\mathbf{q}^{\langle e_i | \mathbf{s}\rangle}}\right)^{j} \left(\frac{1}{\mathbf{q}^{\langle \mathbf{s} | e_i\rangle}}\right)^{n_i - 1 - j} x_i^j \otimes x_i^{n_i - 1 - j} \end{aligned}$$

over Λ^{en} with Λ and taking the degree $\mathbf{d}$ part vanishes if and only if $(Q\mathbf{d})_i \in \mathcal{R}_i$.

As for cohomology, it follows that if the map on one edge of the cube $(\mathbb{P}_1 \otimes \cdots \otimes \mathbb{P}_c \otimes_{\Lambda^{\mathrm{en}}} \Lambda)_{\mathbf{d}}$ vanishes then all parallel maps also vanish.

7.4. **Theorem.** *Let $\Lambda = \Lambda^{\mathbf{n}}_{\mathbf{q}}$ be a quantum complete intersection, and let $\mathbf{d} \in \mathbb{N}^c$. We divide the set $\{1 \dots c\}$ into the following three parts:*

$$\begin{aligned} I_0 &= \{i \in \{1 \dots c\} \colon d_i = 0\} \\ I_1 &= \{i \in \{1 \dots c\} \colon n_i \mid d_i\} \setminus I_0 \\ I_2 &= \{i \in \{1 \dots c\} \colon n_i \nmid d_i\} \end{aligned}$$

Then $\mathrm{HH}_{*,\mathbf{d}}(\Lambda) \neq 0$ if and only if the following hold:

- *for any $i \in I_1$ the product $(Q\mathbf{d})_i \in \mathcal{R}_i$, and*
- *for any $i \in I_2$ we have $(Q\mathbf{d})_i = 1$.*

In this situation $\mathrm{HH}_{*,\mathbf{d}}(\Lambda)$ has a k-vector space basis

$$\{T^{\mathbf{p}}_{\mathbf{d}} \mid \mathbf{p} \geq \mathbf{0} \text{ and } \mathfrak{p}(\mathbf{d} + \mathbf{1}) - \mathbf{2} \leq \mathbf{p} \leq \mathfrak{p}(\mathbf{d} + \mathbf{1}) - \mathbf{1}\}.$$

Here $T^{\mathbf{p}}_{\mathbf{d}}$ is represented by $\mathbf{x}^{\mathbf{d} - \mathfrak{s}(\mathbf{p})}$ in position $\mathbf{p}$. In particular $T^{\mathbf{p}}_{\mathbf{d}}$ has torsion degree $\sum_{i=1}^{c} p_i$.

8. The rate of growth of Hochschild Homology

To study the rate of growth of Hochschild homology, we decompose it similar to our decomposition of Hochschild cohomology in 4.2.

8.1. **Construction.** For $G \subseteq \{1, \dots, c\}$ we denote by HH^G_* the k-span of the $T^{\mathbf{p}}_{\mathbf{d}}$ with

$$G = \{i \in \{1, \dots, c\} \mid d_i > 0 \text{ or } (Q\mathbf{d})_i \in \mathcal{R}_i\}.$$

This yields a decomposition $\mathrm{HH}_*(\Lambda) = \bigoplus_{G \subseteq \{1,\dots,c\}} \mathrm{HH}^G_*$, and hence

$$\gamma(\mathrm{HH}_*(\Lambda)) = \max_{G \subseteq \{1,\dots,c\}} \gamma(\mathrm{HH}^G_*).$$

As in the proof of Theorem 4.5 one obtains the following result.

8.2. **Theorem.** *The rate of growth of the Hochschild homology of a finite dimensional quantum complete intersection is*

$$\begin{aligned}\max\{\operatorname{pos.rk}\operatorname{Ker} Q_{G\times G}\ |G = \{i\in\{1,\dots,c\} \mid d_i>0 \text{ or } (Q\mathbf{d})_i\in\mathcal{R}_i\}\\ \text{for some } \mathbf{d}\in\mathbb{N}^c \text{ with}\\ \forall i \text{ with } n_i\mid d_i \text{ and } d_i>0\colon (Q\mathbf{d}_i)\in\mathcal{R}_i, \text{ and}\\ \forall i \text{ with } n_i\nmid d_i\colon (Q\mathbf{d}_i)=1\}.\end{aligned}$$

We conclude this paper by showing that the Hochschild homology of Λ is closely related to the Hochschild homologies of certain subalgebras.

Let Λ_I denote the subalgebra of Λ generated by the x_i with $i\in I$ for some $I\subset\{1,\dots,c\}$. Note that Λ_I is a split quotient of Λ (that is we have algebra homomorphisms $\Lambda_I\to\Lambda\to\Lambda_I$ whose composition is the identity on Λ_I). Therefore it follows from the functoriality of Hochschild homology that $\mathrm{HH}_*(\Lambda_I)$ can be embedded into $\mathrm{HH}_*(\Lambda)$.

The following theorem shows that the Hochschild homologies of these subalgebras determine the Hochschild homology of Λ to a large extent.

8.3. **Theorem.** *Let M be the maximum of the rates of growth of $\mathrm{HH}_*(\Lambda_{\overline{\{i\}}})$, where $i\in\{1,\dots,c\}$ and $\overline{\{i\}}=\{1,\dots,c\}\setminus\{i\}$. Then the rate of growth of $\mathrm{HH}_*(\Lambda)$ is*

$$\begin{array}{ll} M & \text{if } \mathrm{HH}_*^{\{1,\dots,c\}}=0\\ \max\{M,\operatorname{pos.rk}\operatorname{Ker} Q\} & \text{if } \mathrm{HH}_*^{\{1,\dots,c\}}\neq 0.\end{array}$$

Proof. We will need to look at the sets HH_*^G as well as their analogs for $\mathrm{HH}_*(\Lambda_{\overline{\{i\}}})$. To avoid confusion we write $\mathrm{HH}_*^G(\Lambda)$ and $\mathrm{HH}_*^G(\Lambda_{\overline{\{i\}}})$, respectively, for these vector spaces.

Let $i_0\in\{1,\dots,c\}$ and $G\subseteq\overline{\{i_0\}}$. Then if follows from the explicit description of bases in Theorem 7.4 (and Construction 8.1) that $\mathrm{HH}_*^G(\Lambda)$ can be identified with a subspace of $\mathrm{HH}_*^G(\Lambda_{\overline{\{i_0\}}})$, and moreover that the set

$$\begin{aligned}\{T_{\mathbf{d}}^{\mathbf{p}}\ |G=\{i\in\overline{\{i_0\}}\mid d_i>0 \text{ or } (Q\mathbf{d})_i\in\mathcal{R}_i\}\\ G\neq\{i\in\{1,\dots,c\}\mid d_i>0 \text{ or } (Q\mathbf{d})_i\in\mathcal{R}_i\}\}\end{aligned}$$

is a basis of the quotient space. This clearly means that

$$\{i\in\{1,\dots,c\}\mid d_i>0 \text{ or } (Q\mathbf{d})_i\in\mathcal{R}_i\}=G\cup\{i_0\},$$

so the quotient embeds naturally into $\mathrm{HH}_*^{G\cup\{i_0\}}$.

It follows that

$$\gamma(\mathrm{HH}_*^G(\Lambda))\le\gamma(\mathrm{HH}_*^G(\Lambda_{\overline{\{i_0\}}}))\le\max\{\gamma(\mathrm{HH}_*^G(\Lambda)),\gamma(\mathrm{HH}_*^{G\cup\{i_0\}}(\Lambda))\}.$$

Taking the maximum over all G and i_0 the claim follows. □

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