Summer school 2001: Homological conjectures for finite dimensional algebras

The subject

In the thirties and fourties parallel theories of cohomology were developed in group theory, Lie algebras, and associative algebras. A unified approach to all these theories were given by Cartan and Eilenberg in [12]. Shortly after important homological conjectures were formulated by Rosenberg-Zelinsky, Nakayama, Nunke. These conjectures have served as an inspiration and motivation to look at homological questions and to use homological techniques. Homological algebra has a central and important role in mathematics in general. In particular within the representation theory of algebras the categorical point of view and the use of homological techniques have given a greatly improved insight and progress in this field.

The homological conjectures which are formulated for finite dimensional algebras will be the main theme of this summer school. There is no unified approach towards these conjectures, neither on a theoretical level nor in the literature. However, various techniques and theories have been developed. Below we give an outline of the results and methods which we want to cover in the summer school, and we mention also the connections between the different topics.

Origin of the conjectures

Around the middle of this century, algebraists started to use homological methods in commutative algebra. Let us mention as an example the celebrated Auslander-Buchsbaum-Serre Theorem which asserts that an algebraic variety V with coordinate ring Λ is smooth if and only if the global dimension of Λ is finite. Moreover, in this case the global dimension equals the dimension of the variety. If the global dimension of a ring Λ is infinite it still makes sense to look at the big and the little finitistic dimensions respectively defined as

$$\operatorname{Fin.dim}\Lambda = \sup\{\operatorname{pd} M \mid M \in \operatorname{Mod}\Lambda,\ \operatorname{pd} M < \infty\}$$

and

$$\operatorname{fin.dim}\Lambda = \sup\{\operatorname{pd} M \mid M \in \operatorname{mod}\Lambda,\ \operatorname{pd} M < \infty\}.$$

Here, $\operatorname{Mod}\Lambda$ denotes the category of all Λ-modules, $\operatorname{mod}\Lambda$ is the full subcategory of finitely presented Λ-modules, and $\operatorname{pd} M$ denotes the projective dimension of M, i.e. the least n such that $\operatorname{Ext}^{n+1}_\Lambda(M,-) = 0$. For Λ commutative noetherian it was proved by Bass [10], Gruson and Raynaud [18] that $\operatorname{Fin.dim}\Lambda$ equals the Krull dimension of Λ. In the non-commutative situation there are conjectures which are formulated for a finite dimensional k-algebra Λ (here, k denotes a field and $D = \mathrm{Hom}_k(-, k)$ is the usual k-duality):

Finitistic dimension conjectures. (a) $\mathrm{Fin.dim}\,\Lambda = \mathrm{fin.dim}\,\Lambda$ and (b) $\mathrm{fin.dim}\,\Lambda$ is finite.

Nunke condition. For every non-zero Λ-module M there exists $n \geq 0$ such that $\mathrm{Ext}^n_\Lambda(D\Lambda, M) \neq 0$.

Generalized Nakayama conjecture. For every simple Λ-module S there exist $n \geq 0$ such that $\mathrm{Ext}^n_\Lambda(D\Lambda, S) \neq 0$.

Nakayama conjecture. If $0 \to \Lambda \to Q_0 \to Q_1 \to \dots$ is a minimal injective resolution in $\mathrm{mod}\,\Lambda$ where all terms are projective, then Λ is quasi-Frobenius.

The little finitistic dimension being finite is the most general one, and it will be referred to as the little finitistic dimension conjecture. Furthermore, the above conjectures are listed in order of generality, that is, $\mathrm{fin.dim}\,\Lambda$ finite for all finite dimensional algebras implies the Nunke condition, which in turn implies the Generalized Nakayama conjecture and so on. At present, there are only counter examples for the statement $\mathrm{Fin.dim}\,\Lambda = \mathrm{fin.dim}\,\Lambda$, see [35, 33], and all known positive results require some extra assumption on the algebra Λ.

Resolutions and syzygies

Projective resolutions have played a central role in ring theory and module theory. Up to recently they were mostly only used for theoretical purposes, but with the introduction of computers and symbolic computation into mathematics, actual computation of resolutions has become possible. At first this was only available for commutative rings, but methods and implementations have also been constructed to deal with the noncommutative case. These methods and implementations are not only important as a way of testing examples, but also some constructions of resolutions give a "theoretical" way of working with the resolutions. Some of the computer implementations use the theory of Gröbner basis as an important tool for carrying out the symbolic computations. This is the case for the program *Gröbner* by E. L. Green.

The little finitistic dimension conjecture was proved for monomial algebras by Green-Kirkman-Kuzmanonvich [16] using explicit knowledge about the minimal projective resolution of each simple module over monomial algebras (for example, [15]). This was later also proved by Igusa and Zacharia [22], studying the "syzygy pairs" given by certain maps between projectives.

Assigning combinatorial data to syzygies of modules, Green and Zimmermann-Huisgen proved that the little finitistic dimension conjecture holds for algebras with radical cube zero. Later alternative proofs have been given of this fact (Igusa-Todorov), and generalizations to the case $J^{2l+1} = (0)$ and Λ/J^l finite representation type [13], where J denotes the radical of the algebra.

Auslander formulated the following conjecture, which we here call the Auslander conjecture. Let X be a Λ-module. There exists an integer $n = n_X$ such that if $\operatorname{Ext}^i_\Lambda(X, Y) = (0)$ for i sufficiently large, then $\operatorname{Ext}^i_\Lambda(X, Y) = (0)$ for $i \geq n$. Auslander showed that if the Auslander conjecture holds for the enveloping algebra $\Lambda^e = \Lambda \otimes \Lambda^{\mathrm{op}}$, then the little finitistic dimension conjecture holds for Λ. For two Λ-modules X and Y we have the following identity involving Hochschild cohomology

$$\operatorname{Ext}^i_\Lambda(X, Y) \simeq H^i(\Lambda, \operatorname{Hom}_k(X, Y)),$$

where Λ is an algebra over k and $H^i(\Lambda, {}_\Lambda M_\Lambda) = \operatorname{Ext}^i_{\Lambda^e}(\Lambda, M)$. This explains the connection with Hochschild cohomology.

Resolutions we want to study in more detail are: the Anick-Green resolution for simple modules (generalizes Green-Happel-Zacharia resolutions for monomial algebras), Green-Solberg-Zacharia resolutions for finitely generated modules (generalize resolutions of Eilenberg, Butler, Bongartz) and minimal resolutions of Λ over Λ^e due to Bardzell, Butler, Happel.

Homologically finite subcategories

The concept of homologically finite subcategories was introduced by Auslander and Smalø in [8]. The introduction of this concept has not only had a profound impact on the representation theory of artin algebras, but has also been important for (commutative) rings of higher dimension (see [4]) and in the theory of highest weight categories. In commutative ring theory new invariants have been introduced based on the fact that the category of maximal Cohen-Macaulay modules is contravariantly finite (in a suitable category), which in many cases has been related to old invariants and notions. However, our interest is the theory developed for artin algebras.

Let Λ be an artin algebra, and let $\operatorname{mod}\Lambda$ denote the category of finitely generated left Λ-modules. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod}\Lambda$. For a given Λ-module C a map $f\colon X \to C$ with X in $\mathcal{X}$ is called a *right $\mathcal{X}$-approximation* if for all maps $f'\colon X' \to C$ with X' in $\mathcal{X}$, there exists a map $h\colon X' \to X$ such that $f \circ h = f'$. If all modules in $\operatorname{mod}\Lambda$ have a right $\mathcal{X}$-approximation, then $\mathcal{X}$ is called *contravariantly finite* in $\operatorname{mod}\Lambda$. Equivalently, this can be formulated in terms of functors as follows. A map $f\colon X \to C$ is a right $\mathcal{X}$-approximation of C if $(\ ,X)|_{\mathcal{X}} \xrightarrow{(\ ,f)} (\ ,C)|_{\mathcal{X}}$ is surjective. This explains the choice of name, and one also has the dual notion of a covariantly finite subcategory. A subcategory which is both covariantly and contravariantly finite, is called *functorially* finite. A subcategory with at least one of these properties is called *homologically finite*.

Let $\mathcal{P}^\infty(\Lambda)$ denote the subcategory of $\operatorname{mod}\Lambda$ consisting of all modules with finite projective dimension. This category has been studied by Auslander and Reiten in [5], and they have shown the following striking connection with the little finitistic dimension conjecture. If $\mathcal{P}^\infty(\Lambda)$ is contravariantly finite in $\operatorname{mod}\Lambda$, then the little finitistic dimension conjecture is true for Λ. The converse is known not to be true by an example of Igusa, Smalø and Todorov.

Homologically finite subcategories also play a central role in tilting/cotilting theory. To explain this we first need to recall some notions. A Λ-module T is called a *cotilting module* if $\operatorname{Ext}^i_\Lambda(T,T) = (0)$ for all $i > 0$, $\operatorname{id}_\Lambda T < \infty$ (finite injective dimension) and $D(\Lambda^{\mathrm{op}})$ has a finite resolution in $\operatorname{add} T$. A subcategory $\mathcal{X}$ of $\operatorname{mod}\Lambda$ is called *resolving* if $\mathcal{X}$ contains the projective Λ-modules and is closed under extensions and kernels of epimorphisms. In [5] Auslander and Reiten showed that there is an one-one correspondence between isomorphism classes of basic cotilting modules and contravariantly finite resolving subcategories of $\operatorname{mod}\Lambda$ such that every module has a finite resolution in the subcategory. Furthermore, if T is a Λ-cotilting module, then Happel has shown that the little finitistic dimension of Λ is finite if and only if the little finitistic dimension of $\operatorname{End}_\Lambda(T)$ is finite. Hence, satisfying the little finitistic dimension conjecture is an invariant under cotilting.

Let T be a Λ-module with $n-1$ nonisomorphic indecomposable summands, where n is the number of nonisomorphic simple Λ-modules. If T in addition satisfies the conditions (i) $\operatorname{Ext}^i_\Lambda(T,T) = (0)$ for $i > 0$, (ii) $\operatorname{id}_\Lambda T < \infty$, and (iii) it can be completed to a cotilting module, then T is called an *almost complete cotilting module*. A rich theory for completing an almost complete cotilting module has been developed by Coelho, Happel and Unger. It was observed by Happel and Unger that if the little finitistic dimension conjecture is true for Λ, then an almost complete cotilting module can only be completed to a cotilting module in a finite number of ways up to additivity. It was proved independently by Buan-Solberg and Happel-Unger that the Generalized Nakayama conjecture is true for Λ if and only if the injective almost complete cotilting modules over Λ only have a finite number of complements.

Representation Dimension

In [3], Auslander established a bijective correspondence between artin algebras of finite representation type and algebras having global dimension at most 2 and dominant dimension at least 2. This correspondence sends an algebra Λ of finite type to $\operatorname{End}_\Lambda(M)$ where M is the direct sum of a representative set of indecomposable Λ-modules. Note that M is a generator-cogenerator for $\operatorname{mod}\Lambda$. This motivated Auslander to define the *representation dimension* of Λ to be

$$\operatorname{rep.dim}\Lambda = \inf\{\operatorname{gl.dim}\operatorname{End}_\Lambda(M) \mid M \text{ is a generator-cogenerator}\}.$$

Note that M in $\operatorname{mod}\Lambda$ is a generator-cogenerator if and only if every indecomposable projective and every indecomposable injective Λ-module arises as a direct summand of M. The representation dimension measures in some sense how far Λ is from being of finite representation type. However, it is still an open question whether this dimension is always finite.

Geometric aspects

Geometrical arguments are used frequently in representation theory of finite dimensional algebras in order to study the representation type. There are also some results on homological properties. In order to state them fix an algebraically closed field k and consider the affine scheme $\operatorname{Algmod}_{d,n}(k)$ of n-dimensional modules over algebras of dimension d. It has been shown by Jensen and Lenzing [25] that for fixed $r \geq 0$ the modules of projective dimension at most r form an open subscheme of $\operatorname{Algmod}_{d,n}(k)$. This has various interesting consequences. We mention just one of them which is due to Schofield [32]: There exists a function $\gamma \colon \mathbb{N} \to \mathbb{N}$ such that the finite global dimensions of all d-dimensional algebras are bounded by $\gamma(d)$.

Infinitely generated modules

It has already been mentioned that there are examples of finite dimensional algebras where the big and the little finitistic dimension are different. In fact, the difference $\operatorname{Fin.dim}\Lambda - \operatorname{fin.dim}\Lambda$ can take any positive value [33]. However, there are also positive results. For instance, Huisgen-Zimmermann and Smalø have shown that $\operatorname{Fin.dim}\Lambda = \operatorname{fin.dim}\Lambda$ if $\mathcal{P}^\infty(\Lambda)$ is contravariantly finite in $\operatorname{mod}\Lambda$ [21]. Their result follows from the fact that under this assumption on Λ every module of finite projective dimension is a direct limit of finitely generated modules of finite projective dimension. This is not true in general and there is some related work which characterizes the algebras having this property [26]. There are also results on $\operatorname{Fin.dim}\Lambda$ which involve the so-called *Ziegler spectrum*. The isomorphism classes of indecomposable pure-injective Λ-modules form the points of the Ziegler spectrum of Λ, and $\operatorname{Fin.dim}\Lambda$ is already obtained by taking the supremum over all $\operatorname{pd} M < \infty$ with M being a point of this spectrum [26].

The category $\mathcal{P}^\infty(\operatorname{mod}\Lambda)$ of finitely generated modules having finite projective dimension is not necessarily contravariantly finite in $\operatorname{mod}\Lambda$. On the other hand, the category $\mathcal{P}^\infty(\operatorname{Mod}\Lambda)$ of all modules having finite projective dimension is contravariantly finite in $\operatorname{Mod}\Lambda$ if and only if $\operatorname{Fin.dim}\Lambda < \infty$. This follows from a recent result of Aldrich, Enochs, Jenda, and Oyonarte, which says that $\mathcal{P}^n(\operatorname{Mod}\Lambda)$ is contravariantly finite in $\operatorname{Mod}\Lambda$ for all $n \geq 0$ [1]. An alternative proof shows that the right $\mathcal{P}^n(\operatorname{Mod}\Lambda)$-approximation $P_n \to \Lambda/J$ of Λ/J plays some special role: a Λ-module X has projective dimension at most n if and only if it is a direct factor of some X' having a filtration $X' = X_0 \supseteq X_1 \supseteq \ldots \supseteq X_n = 0$ such that X_i/X_{i+1} is isomorphic to a product of copies of P_n [27]. In particular, $\operatorname{Fin.dim}\Lambda = \sup\{\operatorname{pd} P_n \mid n \geq 0\}$.

References

[1] S.T. Aldrich, E.E. Enochs, O.M.G. Jenda, L. Oyonarte, *Envelopes and covers by modules of finite injective and projective dimension*, preprint.

[2] Anick, D. J., Green, E. L., *On the homology of quotients of path algebras*, Comm. Algebra 15 (1987), no. 1-2, 309–341.

[3] M. Auslander, *Representation dimension of artin algebras*, Queen Mary College, Mathematics Notes, University of London (1971). Also in Selected works of Maurice Auslander, part 1, Amer. Math. Soc.

[4] Auslander, M., Buchweitz, R.-O., *The homological theory of maximal Cohen-Macaulay approximations*, Colloque en l'honneur de Pierre Samuel (Orsay, 1987). Mém. Soc. Math. France (N.S.) No. 38 (1989), 5–37.

[5] Auslander, M., Reiten, I., *Applications of contravariantly finite subcategories*, Adv. Math. 86 (1991), no. 1, 111–152.

[6] Auslander, M., Reiten, I., *Homologically finite subcategories*, Representations of algebras and related topics (Kyoto, 1990), 1–42, London Math. Soc. Lecture Note Ser., 168, Cambridge Univ. Press, Cambridge, 1992.

[7] Auslander, M., Reiten, I., *On a generalized version of the Nakayama conjecture*, Proc. Amer. Math. Soc. 52 (1975), 69–74.

[8] Auslander, M., Smalø, S. O., *Preprojective modules over artin algebras*, J. Algebra 66 (1980), no. 1, 61–122

[9] Bass, H., *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc. 95, 1960, 466–488.

[10] Bass, H., *Injective dimension in Noetherian rings*, Trans. Amer. Math. Soc. 102, 1962, 18–29.

[11] Buan, A. B., Solberg, Ø., *Relative cotilting theory and almost complete cotilting modules*, CMS Conference Proc., vol. 24 (1998) 77–92.

[12] Cartan, H., Eilenberg, S., *Homological algebra*, Princeton University Press, 1956.

[13] Dräxler, P., Happel, D., *A proof of the generalized Nakayama conjecture for algebras with $J^{2l+1} = 0$ and A/J^l representation finite*, J. Pure Appl. Algebra 78 (1992), no. 2, 161–164.

[14] Feustel, C. D., Green, E. L., Kirkman, E., Kuzmanovich, J., *Constructing projective resolutions*, Comm. Algebra 21 (1993), no. 6, 1869–1887.

[15] Green, E. L., Happel, D., Zacharia, D., *Projective resolutions over Artin algebras with zero relations*, Illinois J. Math. 29 (1985), no. 1, 180–190.

[16] Green, E. L., Kirkman, E., Kuzmanovich, J., *Finitistic dimensions of finite-dimensional monomial algebras*, J. Algebra 136 (1991), no. 1, 37–50.

[17] Green, E. L., Zimmermann-Huisgen, B., *Finitistic dimension of Artinian rings with vanishing radical cube*, Math. Z. 206 (1991), no. 4, 505–526.

[18] Gruson, L, Raynaud, *Critères de platitude et de projectivité*, Invent. Math. 13 (1971) 1–89.

[19] Happel, D., Reiten, I., Smalø, S. O., *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 120 (1996), no. 575, viii+ 88 pp.

[20] Happel. D., Unger, L., *Complements and the generalized Nakayama conjecture*, CMS Conf. Proc., vol. 24 (1998) 293–310.

[21] Huisgen-Zimmermann, B., Smalø, S. O., *A homological brigde between finite and infinite dimensional representations of algebras*, Algebr. Represent. Theory 1 (1998), no. 2, 169-188.

[22] Igusa, K., Zacharia, D., *Syzygy pairs in a monomial algebra*, Proc. AMS, 108 (1990), no. 3, 601–604.

[23] Jans, J. P., *Duality in Noetherian rings*, Proc. Amer. Math. Soc. 12, 1961, 829–835.

[24] Jans, J. P., *Some generalizations of finite projective dimension*, Illinois J. Math. 5 1961, 334–344.

[25] Jensen, C. U., Lenzing, H., *Model-theoretic algebra with particular emphasis on fields, rings, modules*, Algebra, Logic and Applications, 2. Gordon and Breach Science Publishers, New York, 1989. xiv+443 pp.

[26] Krause, H., *Finitistic dimension and Ziegler spectrum*, Proc. Amer. Math. Soc. 126 (1998), no. 4, 983–987.

[27] H. Krause, Ø. Solberg, *Filtering modules of finite projective dimension*, preprint.

[28] Membrillo-Hernandez, F.H., Salmeron, L., *A geometric approach towards the finitistic dimension conjecture*, Arch. Math. 67 (1996) 448–456.

[29] Müller, B. J., *The classification of algebras by dominant dimension*, Canad. J. Math. 20, 1968, 398–409.

[30] Nakayama, T., *On algebras with complete homology*, Abh. Math. Sem. Univ. Hamburg 22, 1958, 300–307.

[31] Nunke, R. J., *Modules of extensions over Dedekind rings*, Illinois J. Math. 3, 1959, 222–241.

[32] Schofield, A. H, *Bounding the global dimension in terms of the dimension*, Bull. London Math. Soc. 17 (1985), no. 4, 393–394.

[33] Smalø, S. O., *The supremum of the difference between the big and little finitistic dimensions is infinite*, Proc. Amer. Math. Soc. 126 (1998), no. 9, 2619-2622.

[34] Tachikawa, H., *Quasi-Frobenius rings and generalizations,* $\mathrm{QF}-3$ *and* $\mathrm{QF}-1$ *rings*, Notes by Claus Michael Ringel. Lecture Notes in Mathematics, Vol. 351. Springer-Verlag, Berlin-New York, 1973. xi+172 pp.

[35] Zimmermann-Huisgen, B., *Homological domino effects and the first finitistic dimension conjecture*, Invent. Math. 108 (1992) 369–383.