Ditopology - Why and How.

Lisbeth Fajstrup

Department of mathematical sciences
Aalborg University

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  - Tracespaces in a precubical complex (M. Raussen)
  - A new category convenient for lifting problems (L. Fajstrup, J. Rosicky)
  - Universal dicoverings as representations of the category $\vec{\pi}_1$. L.F.
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Motivation - directed spaces

- Vectorfields
- Relativity theory - light cones
- Models in control theory
- Concurrency theory - the main focus here.

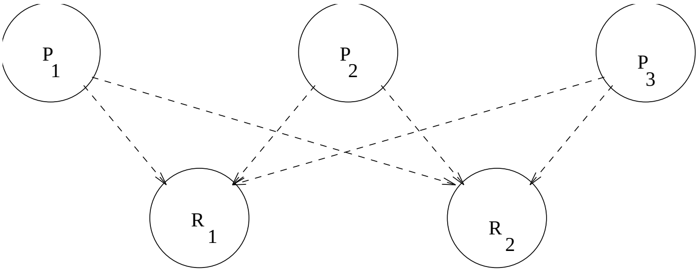
Motivation - concurrency

Modern computers and programs

- Distribute tasks to several processors or computers
- Interact via shared media: Memory, printers, databases,...

Hence they have =coordination problems.

n processes P_i compete for m resources R_j.



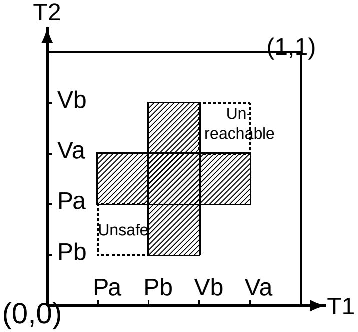
Only k_i processes can be served by R_i at any given time.
Semaphores!
Semantics: A processor has to lock a resource (PR_i) and relinquish (VR_i) the resource later on!
Description/abstraction $P_i : \ldots PR_j \ldots VR_j \ldots$ (Dijkstra)

Practical Problems

- Verification: Will the program behave
  - If there is a “bad” state, is it reachable?
  - Will all possible executions give a “true” result?
- State space explosion: Number of states grows exponentially in the number of processors - so does the number of executions.
- Classification of programs - hierarchy of programs “Anything you can do, I can do (better)”
- What is a good model?
- What if one processor fails - can the program finish?

Geometry of *PV*-programs

The Swiss flag example



PV-diagram from

$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded).

Dipaths that are dihomotopic correspond to equivalent executions.

Deadlocks, unsafe and unreachable regions may occur.

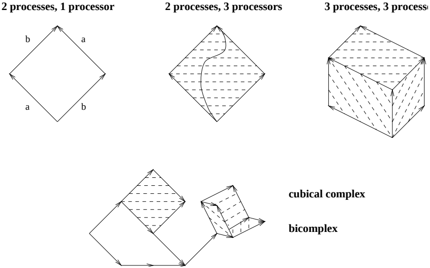
Practical Problems are now geometric

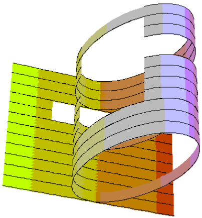
- Verification: Will the program behave
  - If there is a “bad” state, is it reachable? Is there a directed path to it from the initial state(s)
  - Will all possible executions give a “true” result? Study directed paths up to equivalence.
- State space explosion.Now we have infinitely many states - want components!
- Classification of programs - hierarchy of programs. Directed coverings

A model: Higher dimensional automata, HDA

seen as (geometric realizations of) cubical sets

Vaughan Pratt, Rob van Glabbeek, Eric Goubault...





Squares/cubes/hypercubes are filled in if actions on the edges are independent - no interaction/conflict.
Higher dimensional automata are (pre)-cubical sets:

- like simplicial sets, but modelled on (hyper)cubes instead of simplices; gluing by face maps - no degeneracies.
- additionally: preferred directions – not all paths allowed.

The category **d-Top**,(M. Grandis)

Definition

Objects of **d-Top** are d-Spaces: $(X, \vec{P}(X))$ where $X \in$ **Top**, $\vec{P}(X) \subseteq X^I$, the dipaths. $\vec{P}(X)$ is

- closed under concatenation,
- contains the constant paths
- closed under subpath, i.e., composition with $f : I \to I$ increasing but not necessarily surjective.

A d-map $f : X \to Y$ is a continuous map satisfying $\gamma \in \vec{P}(X) \Rightarrow f \circ \gamma \in \vec{P}(Y)$

Example: $\vec{I}$=(I, increasing paths)

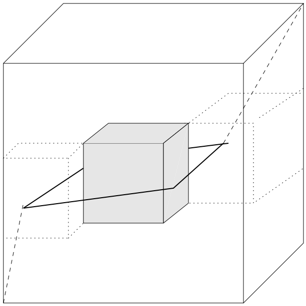
d-Top contains the geometric models of HDA and PV-programs. And a lot more.

Dihomotopy

$H : X \times I \to Y$ is a Dihomotopy if all $H(-, t)$ are d-maps.

- $\vec{\pi}_1(X, x, y) = \vec{P}(X, x, y)/\sim$, where $\sim$ is dihomotopy with fixed endpoints.
- There may be few or no directed loops.
- Concatenation: $\vec{\pi}_1(X, x, y) \times \vec{\pi}_1(X, y, z) \to \vec{\pi}_1(X, x, z)$
- Not a group structure - dipaths are not reversible.
- $\vec{\pi}_1(X)$ is a category. Objects: Points. Morphisms: dihomotopy classes of dipaths.
- Van Kampen: $X = int(X_1) \cup int(X_2)$, then $\vec{\pi}_1(X)$ is a pushout in Cat of the obvious diagram. (M. Grandis)

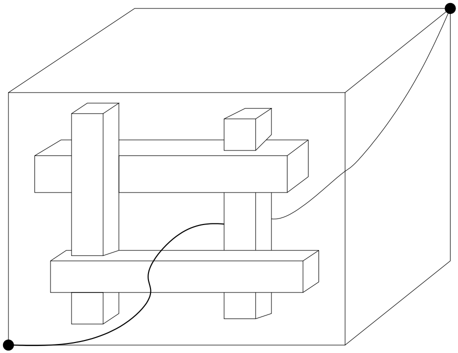
No cancellation rule



"Birth and death" of
d-homotopy classes

Dihomotopy is finer than homotopy with fixed endpoints

Example: Two wedges in the forbidden region



All dipaths from minimum to maximum are homotopic.
A dipath through the “hole” is not dihomotopic to a dipath on the boundary.

Dicomponents - without loops

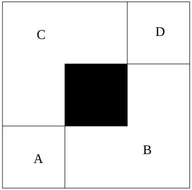
$[\sigma] : x \to y$ in $\vec{\pi}_1(X)$ induces maps (concatenation)

$$[\sigma]^* : \vec{\pi}_1(X, y, z) \to \vec{\pi}_1(X, x, z)$$

$$[\sigma]_* : \vec{\pi}_1(X, w, x) \to \vec{\pi}_1(X, w, y)$$

σ is weakly invertible if all such maps are bijections.

- $W \subset Mor(\vec{\pi}_1)$ is the set of weakly invertibles.
- Σ is a subset of W satisfying left and right extension properties as required for a category of fractions. (Plus technical conditions)
- x and y are in the same component, wrt. Σ if there is a zig zag path $\sigma_1 \sigma_2^{-1}$ from x to y



Components

- Calculated - program!- for PV-programs without loops - E. Goubault, E. Haucourt. $\longrightarrow$ Classification of executions.$\longrightarrow$ Verification.
- With loops: Components are subsets of $X \times X$, (M.Raussen.)
- Calculations - Use (di)coverings to unfold the loops.(?)

X a (saturated)[1] d-space.
$\varphi, \psi \in \vec{P}(X)(x, y)$ are called reparametrization equivalent if there are $\alpha, \beta \in \vec{P}(\vec{I}, 0, 1)$ such that $\varphi \circ \alpha = \psi \circ \beta$ (“same oriented trace”).
(Fahrenberg-Raussen, 07): Reparametrization equivalence is an equivalence relation (transitivity).
$\vec{T}(X)(x, y) = \vec{P}(X)(x, y)/_{\simeq}$, the Trace Space
$\vec{\pi}_1(X, x, y) = \pi_0(\vec{T}(X)(x, y))$
Higher di-invariants from homology, homotopy,... of Trace spaces.

[1] $\varphi \circ \alpha \in \vec{P}(X, x, y)$ and $\alpha \in \vec{P}(\vec{I}, 0.1)$ implies $\varphi \in \vec{P}(X, x, y)$

Definition

A subset $L \subseteq X$ of a d-space X is called

achronal if all $p \in \vec{P}(L) \subset \vec{P}(X)$ are constant.

order convex if $p^{-1}(L)$ is either an interval or empty for all $p \in \vec{P}(X)$;

unavoidable from $B \subset X$ to $C \subset X$ if $\vec{P}(X \setminus L)(B, C) = \emptyset$.

Theorem

Let X denote a d-space, $x_0, x_1 \in X$ and $L \subset X$ a subspace that is achronal,order convex and unavoidable from x_0 to x_1.
Then the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$
is a homeomorphism.

Theorem

Let X be (the geometric realization of) a pre-cubical complex. Let $x_0, x_1 \in X$, $L \subset X$ a subcomplex that is order convex and unavoidable from x_0 to x_1.
Then[2] the concatenation map
$c_L : \vec{T}(X)(x_0, L) \times_L \vec{T}(X)(L, x_1) \to \vec{T}(X)(x_0, x_1),\ (p_0, p_1) \mapsto p_0 * p_1$
is a homotopy equivalence.

Corollary

If $\vec{T}(X)(x_0, l)$ and $\vec{T}(X)(l, x_1)$ are contractible and locally contractible for every $l \in L \cap [x_0, x_1]$, then
$\vec{T}(X)(x_0, x_1)$ is homotopy equivalent to $L \cap [x_0, x_1]$.

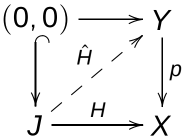
Pieces are trivial, so the topology resides in the gluing. "Huge" trace space identified with "small" space $L \cap [x_0, x_1] \subset X$

[2] add an extra technical condition

Let $J = I \times \vec{I}/(s,0) \sim (0,0)$

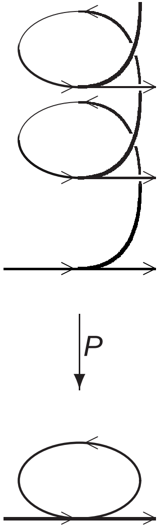
Definition

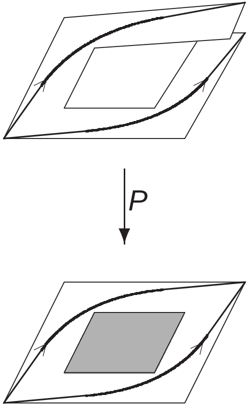
A d-map $p : Y \to X$ is a dicovering, if for all $H : J \to X$ there is a unique lift $\hat{H}$ for any commutative square as below



This means all dipaths and all dihomotopies with fixed initial point lift uniquely given a lift of the initial point.

Two dicoverings-a process with a loop and concurrent processes with a shared object





dTop is too big

- We want universal dicoverings - unfoldings of all directed loops
- Need a category which contains cubical sets and provides universal dicoverings.

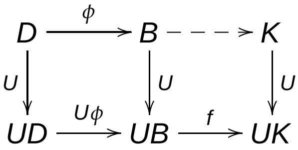
A category generated by a subcategory

Definition

Let **D** be a full subcategory of **C**.

$\mathbf{C_D}$ is the full subcategory defined by

- $B \in \mathbf{C}$ is in $\mathbf{C_D}$ if
  for all $K \in \mathbf{C}$, $f : UB \to UK$ lifts to a **C** morphism if and only if $f \circ U\phi : UD \to UK$ lifts for all $D \in \mathbf{D}$ and all $\phi : D \to B$.
  (The cocone of all $\phi : D \to K$ is U-final.)
- Here U is the forgetful functor to Set.



The full subcategory $\mathcal{B}$

Definition

$\mathcal{B}$ is the full subcategory of **d-Top** with

- objects $I_1 \times I_2 \times \ldots \times I_n$ where I_k is either I with the discrete order (i.e., equality) or $\vec{I}$.
- The order on $I_1 \times I_2 \times \ldots \times I_n$ is the product relation.
- The dipaths are the increasing paths with respect to this relation.

A convenient category for lifting problems

A d-Space X is in $\mathbf{d\text{-}Top}_{\mathcal{B}}$ if

- $V \subset X$ is open if and only if $\phi^{-1}(V)$ is open for all d-maps $\phi : b \to X$, where $b \in \mathcal{B}$.

Notice: It suffices to check the discretely ordered cubes, since $I \to \vec{I}$ is a d-map and a homeomorphism.
There are no requirements on $\vec{P}(X)$, since $\vec{I} \in \mathcal{B}$
Analogue: k-spaces, generated in **Top** by compact topological spaces.

Lifting properties

For $f : A \longrightarrow B$, $g : C \longrightarrow D$, morphisms in a category $\mathcal{K}$, we write

$$f \square g \qquad (f \perp g)$$

if, in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

there is a (unique) diagonal $d : B \longrightarrow C$ with $df = u$ and $gd = v$.

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{H}^{\square} = \{g | f \square g \text{ for each } f \in \mathcal{H}\}, (e.g. Serre fibrations)$$
$$^{\square}\mathcal{H} = \{f | f \square g \text{ for each } g \in \mathcal{H}\},$$
$$\mathcal{H}^{\perp} = \{g | f \perp g \text{ for each } f \in \mathcal{H}\}, (e.g. coverings)$$
$$^{\perp}\mathcal{H} = \{f | f \perp g \text{ for each } g \in \mathcal{H}\}.$$

Example: $p : Y \to X$ in **d-Top** is a dicovering if and only if $p \in \{(0,0) \to J\}^{\perp}$.

Factorization systems

A pair $(\mathcal{L}, \mathcal{R})$ of morphisms of $\mathcal{K}$ is called a weak factorization system if

1. $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$

and

(2) any morphism h of $\mathcal{K}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

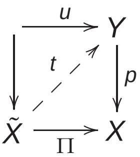
The pair $(\mathcal{L}, \mathcal{R})$ is called a factorization system if condition (1) is replaced by

(1') $\mathcal{R} = \mathcal{L}^{\perp}$, $\mathcal{L} = {}^{\perp}\mathcal{R}$.

For a factorization system, the factorization is unique up to isomorphism.

Factorization and universal morphisms

Suppose $(\mathcal{L}, \mathcal{R})$ is a factorization system in $\mathcal{K}$ and $* \in \mathcal{K}$ is an initial object.
Let $* \to \tilde{X} \to X$ be the factorization of $* \to X$.
Then $\Pi : \tilde{X} \to X$ is universal in $\mathcal{R}$ in this sense:
For any $p : Y \to X \in \mathcal{R}$ there is a unique $t : \tilde{X} \to Y$ s.t.
$p \circ t = \Pi$
Reason: Apply unique right lifting property to



moreover, $t : \tilde{X} \to Y$ is in $\mathcal{R}$

When do we have factorizaton systems

Theorem

with J.Rosicky.
Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\text{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system in $\mathcal{K}$.

Theorem

with J.Rosicky.
Let $(\mathcal{K}, U)$ be a fibre-small topological category and let $\mathcal{I}$ be a full small subcategory of $\mathcal{K}$. Then the category $\mathcal{K}_{\mathcal{I}}$ is locally presentable.

Corollary

For a set $\mathcal{C}$ of morphisms in $\textbf{d-Top}_{\mathcal{B}}$*,* $(\text{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ *is a factorization system.*

Locally presentable categories

Let λ be a regular cardinal.

Definition

A category $\mathcal{K}$ is locally λ presentable if it is cocomplete and there is a set $\mathcal{A}$ of λ presentable objects of $\mathcal{K}$ s.t. for all $K \in \mathcal{K}$, there is a λ-directed poset (each subset of cardinality smaller than λ has an upper bound) and an associated diagram of objects from $\mathcal{A}$ with colimit K. (K is a λ directed colimit of objects in $\mathcal{A}$.)
$\mathcal{K}$ is locally presentable, if it is locally λ presentable for some λ.

Definition

An object A of $\mathcal{K}$ is λ presentable if $hom(A, -)$ preserves λ directed colimits.

Universal dicoverings

pd-Top is the category of pointed d-spaces:

- $(X,x) \in \textbf{pd-Top}_{\mathcal{B}}$ if $X \in \textbf{d-Top}_{\mathcal{B}}$, $x \in X$.
- $f : (Y,y) \to (X,x)$, a pointed d-map is a dicovering if the corresponding d-map is a dicovering.
- $(*,*)$ is an initial object in **pd-Top**, so there is a universal dicovering $(\tilde{X},\tilde{x}) \to (X,x)$ of every pd-Space X.

Dicoverings in general could be huge spaces: Let (W,w) be a discretely ordered space. Then $pr_2 : (W \times X, (w,x)) \to (X,x)$ is a dicovering.
In $\textbf{d-Top}_{\mathcal{B}}$, $\emptyset$ is initial and $\emptyset \to X$ is a dicovering, hence universal.

Well pointed d-spaces

The future of a point:
$\uparrow_X x_0 = \{x \in X | \exists \gamma : \vec{I} \to X, \gamma(0) = x_0, \gamma(1) = x\}$

Definition

A pointed d-space (X, x_0) is well pointed, $(X, x_0) \in$ **wpd-Top**, if $X = \uparrow_X x_0$.

- Let $p : (Y, y_0) \to (X, x_0)$ be a dicovering, $(X, x_0) \in$ **wpd-Top**, then $p : (\uparrow_Y y_0, y_0) \to (X, x_0)$ is a dicovering in **wpd-Top**.
- Let $\Pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be the universal dicover in **pd-Top** Then $\Pi : (\uparrow_{\tilde{X}} \tilde{x}_0, \tilde{x}_0) \to (X, x_0)$ is the universal dicover in **wpd-Top**.

The universal dicoverings of a fixed space

Let $X \in \mathbf{d\text{-}Top}_{\mathcal{B}}$, $x \in X$

- $X_x = \uparrow_X x \in \mathbf{wpd\text{-}Top}_{\mathcal{B}}$
- $\Pi_x : \tilde{X}_x \to X_x$ the universal dicover.
- For $y \in X_x$, $i_{yx} : X_y \to X_x$ the inclusion.

Definition

$f : \tilde{X}_y \to \tilde{X}_x$ is a dicovering morphism if it preserves fibers:
$\Pi_x \circ f = i_{yx} \circ \Pi_y$

Dicovering morphisms

For $[\sigma] \in \vec{\pi}_1(X, x, y)$ we define $[\tilde{\sigma}] : \tilde{X}_y \to \tilde{X}_x$:
Let $\tilde{\sigma}$ be the unique lift of σ to $\tilde{X}_x$. Then

$$\Pi_x : \uparrow_{\tilde{X}_x} \tilde{\sigma}(1) \to X_y$$

is a dicovering.
Hence there is a d-map

$$\tilde{X}_y \to \uparrow_{\tilde{X}_x} \tilde{\sigma}(1)$$

compose with the inclusion to get $[\tilde{\sigma}] : \tilde{X}_y \to \tilde{X}_x$

Classification of dicovering morphisms

Another description of $[\tilde{\sigma}]$:
for $[\gamma] \in \tilde{X}_y$,
$[\tilde{\sigma}]([\gamma]) = [\gamma * \sigma]$

Theorem

These are the only dicovering morphisms. Hence we get a representation of $\vec{\pi}_1(X)$.

Other properties of **d-Top**$_{\mathcal{B}}$

- There is a "boxification" functor B : **d-Top** $\rightarrow$ **d-Top**$_{\mathcal{B}}$, right adjoint to the inclusion. $B(X)$ adds more opens to X.
- Boxification preserves dicoverings; dipaths and dihomotopies to X are dipaths and dihomotopies in BX
- The universal dicovering as a set is $\tilde{X}_{x_0} = \{[\gamma] | \gamma : \vec{I} \rightarrow X, \ \gamma(0) = x_0\}$ - projection is the endpoint map.

conclusions

- Ditopology has applications, and is a subject in itself.
- Directions make life easier in some cases - there are not as many dipaths. And harder in some cases - lack of group structure.
- Methods from nondirected algebraic topology usually need a twist to apply.
- Still much to do - calculations!