WEAKLY COUPLED SYSTEMS OF QUASILINEAR HYPERBOLIC EQUATIONS

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Abstract. We obtain some general comparison results for entropy solutions of weakly coupled systems of conservation laws with source. For some particular couplings, by using these comparison methods, we show the global existence of the solutions to the Cauchy problem.

1. Introduction

We consider the Cauchy problem for weakly coupled first order quasilinear hyperbolic equations:

$$\partial_t u_i + \partial_x f_i(u_i) = g_i(U), \quad i = 1, \dots, N \ , \tag{1.1}$$

$$U(x,0) = U^0(x) \ , \tag{1.2}$$

where $U = (u_1, \dots, u_N)$ is the unknown function and $F(U) = (f_1(u_1), \dots, f_N(u_N))$, $G(U) = (g_1(U), \dots, g_N(U))$ are given locally Lipschitz continuous functions; U^0 is a given bounded measurable N–valued function.

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The system (1.1) is said to be weakly coupled because the differential part is in diagonal form. Then the coupling — or the source — G does not depend on the derivatives of the unknown U.

In general these systems arise as simplified or asymptotic models in many applications. For example in combustion theory the popular Majda's model for detonation [11] was intensively studied in [20], [21] and [10] (see Section 4 *b)*). Another interesting example is given by a 2×2 quasilinear system arising in chromatography [17], [18]; a computational study of this model was given in [16], while well–posedness was established in [22]. This system belongs to the class of quasimonotone systems considered in Section 4 *a)*.

The aim of this paper is to start a general theory of these weakly coupled systems. First we obtain a general local existence and continuation result: there is $T = T(\|U^0\|_\infty) > 0$ such that there exists a unique bounded entropy solution of (1.1)–(1.2) in $\Sigma_T = \mathbb{R} \times (0, T)$.

Actually the L^∞–norm of the local entropy solution does not need to be bounded: either the entropy solution is global or there exists $T^* < +\infty$ such that, for each $T < T^*$, U is defined and bounded in Σ_T and $\|U\|_{L^\infty(\Sigma_T)} \to +\infty$ as $T \to T^*_-$.

The scalar case was investigated in [12], [14], [15] by using comparison methods in the framework first introduced in [8]. In particular global existence, blow–up and qualitative behaviour results were established.

In the vector case we first develop some new comparison tools, well-known in the parabolic case (see [4] and references therein). Therefore we apply these comparison methods to some special couplings.

In particular, in Section 4, we consider a class of 2×2 quasilinear systems with quasimonotone interactions, the Majda's model for detonation and some related couplings. Next we discuss the class of the conservative interactions. In the semilinear case these couplings were investigated in [5], [1] and [2] by using energy methods. Here we extend these results to the quasilinear case, by using only the comparison tools.

Finally let us observe that these well-posedness and comparison results can be applied in a number of situations. Some new results have been recently obtained for the Chapman–Jouguet limit of the Majda's model [7] and for the convergence to equilibrium of conservation laws with relaxation [13].

2. Uniqueness and comparison

This section is devoted to some general results concerning stability and comparison properties of entropy solutions of the Cauchy problem (1.1)–(1.2). Let Σ_T denote the strip $\mathbb{R} \times (0,T)$ $(T>0)$, $\Sigma = \Sigma_\infty$. First we consider the scalar problem

$$\partial_t u + \partial_x f(u) = g \ , \quad \text{in } \ \Sigma_T \ , \tag{2.1}$$

$$u(\cdot,0) = u^0 \ , \quad \text{in } \ \mathbb{R} \ , \tag{2.2}$$

where f is a locally Lipschitz continuous function and $g \in L^\infty_{\text{loc}}(\Sigma_T)$; we consider only bounded initial data u^0. Let us recall some standard definitions [8]. Denote $[r]_\pm = \frac{1}{2}(|r| \pm r)$ and $H^\pm(r) = \frac{1}{2}(\operatorname{sgn} r \pm 1)$. $H^+ = H$ is just the Heaviside function.

Definition 2.1. *A function $u \in L^\infty(\Sigma_T)$ is said to be an entropy subsolution of (2.1)–(2.2) in Σ_T, if for any $\varphi \in C_0^\infty(\Sigma_T)$, $\varphi \geq 0$, and $k \in \mathbb{R}$, we have*

$$\iint \{[u-k]_+ \partial_t \varphi + H^+(u-k)((f(u)-f(k))\partial_x \varphi + g\varphi)\} \, \mathrm{d}x \, \mathrm{d}t \geq 0 \tag{2.3}$$

and, for any interval I,

$$\operatorname*{esslim}_{t \to 0+} \int_I [u(x,t) - u^0(x)]_+ \, \mathrm{d}x = 0 \ . \tag{2.4}$$

An entropy supersolution of (2.1)–(2.2) is defined replacing in (2.3)–(2.4) $[\quad]_+$, H^+ by $[\quad]_-$, H^-. An entropy solution is defined as being both entropy subsolution and supersolution.

Next we consider the vectorial case (1.1)–(1.2).

Definition 2.2. *A vector–valued function $U \in L^\infty(\Sigma_T)^N$ is said to be an entropy subsolution of the Cauchy problem (1.1)–(1.2) if each component u_i is an entropy subsolution of*

$$\partial_t u_i + \partial_x f_i(u_i) = g_i(U) \ , \quad \text{in} \quad \Sigma_T \ , \tag{1.1$_i$}$$

$$u_i(\cdot,0) = u_i^0 \ , \quad \text{in} \quad \mathbb{R} \ . \tag{1.2$_i$}$$

Entropy supersolutions and solutions are similarly defined.

Observe that, by definition, our solutions will be always bounded in space.

Theorem 2.1 (Uniqueness). *Let $U^0 \in L^\infty(\mathbb{R})^N$ and $T > 0$. There is at the most one entropy solution to the Cauchy problem (1.1)–(1.2) in Σ_T.*

For any given vectors $A, B \in \mathbb{R}^N$, we set $A \le B$ if $a_i \le b_i$, $i = 1, \ldots, N$; if $A \le B$, the corresponding interval will be denoted by $[A, B]$. Fix an interval $[A, B]$ (with non empty interior). Define

$$L = L(A,B,F) = \max_{1\le i\le N} L_i = \max_{1\le i\le N} \left\| \frac{\partial f_i}{\partial u} \right\|_{L^\infty(a_i,b_i)} ,$$

$$M = M(A,B,G) = \max_{1\le i,j\le N} \left\| \frac{\partial g_i}{\partial u_j} \right\|_{L^\infty([A,B])} .$$

For each real interval $I = (\alpha, \beta)$, set $\tau = \tau(I, L) = \frac{\beta-\alpha}{2L}$. Theorem 2.1 follows immediately from the more precise local argument:

Proposition 2.1. *Let U, V be entropy solutions of the Cauchy problem (1.1)–(1.2) in Σ_T, with $U, V \in [A, B]$. Then, for each interval (α, β) and for a.e. $t \in (0, \min(\tau, T))$, we have*

$$\sum_{i=1}^{N} \int_{\alpha+tL}^{\beta-tL} |u_i(x,t) - v_i(x,t)| \, \mathrm{d}x \le e^{tMN} \sum_{i=1}^{N} \int_{\alpha}^{\beta} |u_i^0(x) - v_i^0(x)| \, \mathrm{d}x \ . \tag{2.5}$$

The following lemma, essentially due to Kružkov [8] (see also [12] for the sub–/supersolutions version), will be used in the proof of Proposition 2.1 and of the comparison theorem. The proof is omitted.

Lemma 2.1. *Let u (v) be an entropy subsolution (supersolution) of problem (2.1)–(2.2) in Σ_T for the right–hand side g (h) and the initial data u^0 (v^0). Fix $a, b \in \mathbb{R}$, $a < b$, such that $u, v \in [a, b]$ in Σ_T. Then for each interval (α, β) we have*

$$\begin{aligned} \int_{\alpha+tK}^{\beta-tK} [u(x,t) - v(x,t)]_+ \, \mathrm{d}x &\le \int_{\alpha}^{\beta} [u^0(x) - v^0(x)]_+ \, \mathrm{d}x \\ &+ \int_0^t \mathrm{d}s \int_{\alpha+sK}^{\beta-sK} H(u(x,s) - v(x,s))(g(x,s) - h(x,s)) \, \mathrm{d}x \ , \end{aligned} \tag{2.6}$$

where $K \geq L = L(a, b, f)$, $0 < t < \min(\tau, T)$.

Proof of Proposition 2.1. Let $i \in \{1, \ldots, N\}$. From the assumptions, u_i, v_i are subsolutions and supersolutions of $(1.1)_i$–$(1.2)_i$ for right–hand sides $g_i(U(x,t))$ and $g_i(V(x,t))$. Hence, from (2.6), we have

$$\int_{\alpha+tL}^{\beta-tL} |u_i(x,t) - v_i(x,t)| \, \mathrm{d}x \leq \int_{\alpha}^{\beta} |u_i^0(x) - v_i^0(x)| \, \mathrm{d}x + \int_0^t \mathrm{d}s \int_{\alpha+sL}^{\beta-sL} |g_i(U(x,s)) - g_i(V(x,s))| \, \mathrm{d}x \ ,$$

since $L \geq L_i$. Set

$$\mu(t) = \sum_{i=1}^{N} \int_{\alpha+tL}^{\beta-tL} |u_i(x,t) - v_i(x,t)| \, \mathrm{d}x \ .$$

We have

$$\sum_{i=1}^{N} |g_i(U) - g_i(V)| \leq MN \sum_{i=1}^{N} |u_i - v_i| \ .$$

So

$$\mu(t) \leq \mu(0) + MN \int_0^t \mu(s) \, \mathrm{d}s \ .$$

The conclusion follows by Gronwall's inequality.□

In the vectorial case, comparison properties need to introduce quasimonotone interactions.

Definition 2.3. *Let Ω be an open, convex set in $\mathbb{R}^N$. A function $G : \Omega \to \mathbb{R}^N$ is said to be quasimonotone (nondecreasing) if each component g_i of G is nondecreasing in u_j for $j \neq i$.*

In the particular case of autonomous systems of ordinary differential equations, $\dot{U} = G(U)$, the quasimonotonicity of G is a necessary and sufficient condition for the flow to preserve the ordering in $\mathbb{R}^N$ (see [19]. This quasimonotonicity condition is also known as Kamke condition). This ordering property is also well–known for reaction–diffusion flows (see for instance [4]).

Next, let us turn to the hyperbolic case.

Theorem 2.2 (Comparison). *Let* $\underline{U}$ $(\overline{U})$ *be an entropy subsolution (supersolution) of* (1.1)–(1.2) *in* Σ_T *with initial data* $\underline{U}^0$ $(\overline{U}^0)$. *Assume that there are* $A, B \in \mathbb{R}^N$, $A \le B$, *such that:*

$$G \text{ is quasimonotone in } [A,B] \ , \tag{2.7}$$

$$\underline{U}, \overline{U} \in [A,B] \text{ a.e. in } \Sigma_T \ . \tag{2.8}$$

If $\underline{U}^0 \le \overline{U}^0$ a.e. *in* $\mathbb{R}$, *then* $\underline{U} \le \overline{U}$ a.e. *in* Σ_T.

Set

$$M^+ = M^+(A,B,G) = \max_{\substack{i \ne j \\ 1 \le i,j \le N}} \left\| \frac{\partial g_i}{\partial u_j} \right\|_{L^\infty([A,B])} .$$

Proposition 2.2. *In the assumptions of* Theorem 2.2, *for each interval* (α, β) *and for* a.e. $t \in (0, \min(\tau, T))$, *we have:*

$$\begin{aligned} &\sum_{i=1}^N \int_{\alpha+tL}^{\beta-tL} [\underline{u}_i(x,t) - \overline{u}_i(x,t)]_+ \,\mathrm{d}x \\ &\qquad \le e^{t(M+(N-1)M^+)} \sum_{i=1}^N \int_\alpha^\beta [\underline{u}_i^0(x) - \overline{u}_i^0(x)]_+ \,\mathrm{d}x \ . \end{aligned} \tag{2.9}$$

Proof. As in the proof of Proposition 2.2, we have, applying again (2.6),

$$\begin{aligned} \int_{\alpha+tL}^{\beta-tL} [\underline{u}_i(x,t) - \overline{u}_i(x,t)]_+ \,\mathrm{d}x &\le \int_\alpha^\beta [\underline{u}_i^0(x) - \overline{u}_i^0(x)]_+ \,\mathrm{d}x \\ &+ \int_0^t \mathrm{d}s \int_{\alpha+sL}^{\alpha-sL} H(\underline{u}_i(x,s) - \overline{u}_i(x,s))(g_i(\underline{U}(x,s)) - g_i(\overline{U}(x,s)))\,\mathrm{d}x \ . \end{aligned}$$

Set

$$\mu^+(t) = \sum_{i=1}^N \int_{\alpha+tL}^{\beta-tL} [\underline{u}_i(x,t) - \overline{u}_i(x,t)]_+ \,\mathrm{d}x \ .$$

By using (2.7) we obtain

$$\begin{aligned} &\sum_{i=1}^N H(\underline{u}_i - \overline{u}_i)(g_i(\underline{U}) - g_i(\overline{U})) \\ &\qquad \le (M + (N-1)M^+) \sum_{i=1}^N [\underline{u}_i - \overline{u}_i]_+ \,\mathrm{d}x \end{aligned}$$

and then

$$\mu^+(t) \leq \mu^+(0) + (M + (N-1)M^+) \int_0^t \mu^+(s)\,\mathrm{d}s \;.\square$$

The estimate (2.9) show that, if $\underline{U}^0 \leq \overline{U}^0$ a.e. in some interval, then this inequality is established in a domain of determinacy. So these uniqueness and comparison properties take in account for the hyperbolic nature of the problem, namely for the finite speed of propagation.

Monotonicity methods can be used also in some applications with non quasimonotone coupling. In particular the following result could be useful.

Proposition 2.3. *Let $\underline{U}$ $(U, \overline{U})$ be a subsolution for $\underline{G}$ (solution for G, supersolution for $\overline{G}$) of (1.1)–(1.2) in Σ_T with initial data $\underline{U}^0$ $(U^0, \overline{U}^0)$. Assume that there are $A, B \in \mathbb{R}^N$, $A \leq B$, such that:*

$$\underline{G} \leq G \leq \overline{G} \quad \text{in} \quad [A,B]\;, \tag{2.10}$$

$$\underline{G}, \overline{G} \text{ are quasimonotone in} \quad [A,B]\;, \tag{2.11}$$

$$\underline{U}, U, \overline{U} \in [A,B] \quad \text{in} \quad \Sigma_T\;. \tag{2.12}$$

If $\underline{U}^0 \leq U^0 \leq \overline{U}^0$, then $\underline{U} \leq U \leq \overline{U}$ in Σ_T.

Proof. It is enough to observe that U is a supersolution for $\underline{G}$ and a subsolution for $\overline{G}$. Then we apply Theorem 2.2.$\square$

Remark. Let G be a Lipschitz continuous coupling, defined in $[A,B]$. For $\underline{G}$ we can consider the inferior quasimonotone envelope of G in $[A,B]$, i.e.: the supremum of the quasimonotone functions smaller than G in $[A,B]$. For $\overline{G}$ we can consider the superior quasimonotone envelope.

3. Local existence and continuation

Concerning the existence of an entropy solution to the Cauchy problem (1.1)–(1.2), the following result holds true.

Theorem 3.1 (Local existence). *For any $U^0 \in L^\infty(\mathbb{R})^N$, there is $T > 0$ (depending only on $\|U_0\|_\infty$) such that in Σ_T there exists a (unique) entropy solution U of* (1.1)–(1.2) *and $U \in C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R})^N)$.*

By following the proof given in [8] for the scalar case, we approximate the problem (1.1)–(1.2) by its parabolic regularization:

$$\partial_t U^\varepsilon + \partial_x F^\varepsilon(U^\varepsilon) = G^\varepsilon(U^\varepsilon) + \varepsilon \partial_x^2 U^\varepsilon \ , \quad \text{in} \quad \Sigma_T \ , \tag{3.1}$$

$$U^\varepsilon(\cdot, 0) = U^{0\varepsilon} \ , \quad \text{in} \quad \mathbb{R} \ . \tag{3.2}$$

Here F^ε, G^ε and $U^{0\varepsilon}$ are suitable regularizations of the function F, G and U^0 (see for instance [6]).

First we establish the following L^∞–estimate.

Proposition 3.1. *There are $T, R > 0$, which depend only on $\|U^0\|_\infty$, such that for every $\varepsilon > 0$ there exists a unique classical solution U^ε to* (3.1)–(3.2) *in Σ_T and*

$$\|U^\varepsilon\|_{L^\infty(\Sigma_T)^N} \leq R \ . \tag{3.3}$$

Proof. Let us prove (3.3). Pick $R_0 > 0$ such that $U^0 \in B_{R_0} = \{U : |U| \leq R_0\}$ and fix $R > \sqrt{N} R_0$. Let $\chi \in C^\infty([0,\infty))$, $0 \leq \chi \leq 1$, $\chi \equiv 1$ in $[0,1]$, $\operatorname{supp} \chi \subset [0,2]$. Let $\varphi_R(U) = \chi(R^{-1}|U|)$ and set

$$G_R(U) = \varphi_R(U) G(U) \ .$$

If $K_R = \max_{B^{2R}} |G(U)|$, we have $|G_R(U)| \leq K_R$ and so $|(G^\varepsilon)_R(U)| \leq K_{R+1}$, for $\varepsilon < 1$.

Let $\tilde{U}^\varepsilon$ be the global solution to problem (3.1)–(3.2) with G^ε replaced by G^ε_R. By the maximum principle, we obtain

$$|\tilde{u}^\varepsilon_i(x,t)| \leq R_0 + t K_{R+1} \ ,$$

for $i = 1, \ldots, N$. Therefore, setting $T = \frac{R - \sqrt{N} R_0}{\sqrt{N} K_{R+1}}$, we have $\tilde{U}^\varepsilon \in B_R$, for $0 \leq t < T$. Since $G^\varepsilon_R \equiv G^\varepsilon$ for $|U| \leq R$, by uniqueness, we obtain $\tilde{U}^\varepsilon \equiv U^\varepsilon$ in Σ_T and the conclusion follows. $\Box$

Proof of Theorem 3.1. We establish a compactness framework in L^1_{loc} for the bounded sequence $\{U^\varepsilon\}$ in Σ_T. Let us set

$$W = W^{\varepsilon,h}(x,t) = U^\varepsilon(x+h,t) - U^\varepsilon(x,t) ,$$

for $|h| < h_0$. To obtain the L^1–estimate of the modulus of continuity of U^ε in the space variable, we notice that the function W satisfies the linear problem

$$\partial_t W + \partial_x(BW) = \varepsilon \partial_x^2 W + CW , \tag{3.4}$$

where $B = \operatorname{diag}(b_i)$, $C = \operatorname{diag}(c_i)$,

$$b_i(x,t) = \int_0^1 f_i'(\alpha u_i(x+h,t) + (1-\alpha)u_i(x,t))\,\mathrm{d}\alpha$$

and

$$c_i(x,t) = \int_0^1 \frac{\partial g_i}{\partial u_i}(u_1, \dots, \alpha u_i(x+h,t) + (1-\alpha)u_i(x,t), \dots, u_N)\,\mathrm{d}\alpha .$$

Let $\varphi_i \in C^\infty(\Sigma_T)$, $\varphi_i \geq 0$. Multiplying $(3.4)_i$ by $(\operatorname{sgn} w_i)\varphi_i$ gives

$$\begin{aligned} &\partial_t(|w_i|\varphi_i) + \partial_x(b_i|w_i|\varphi_i) - \varepsilon\partial_x^2(|w_i|\varphi_i) \\ &\quad \leq (\partial_t\varphi_i + b_i\partial_x\varphi_i + c_i\varphi + \varepsilon\partial_x^2\varphi_i)|w_i| - 2\varepsilon\partial_x\ (|w_i|\partial_x\varphi_i) . \end{aligned} \tag{3.5}$$

Let us choose φ_i as the solution of the following backward parabolic problem, for $0 < \tau \leq T$;

$$\begin{cases} \partial_t\varphi_i + b_i\partial_x\varphi_i + c_i\varphi_i + \varepsilon\partial_x^2\varphi_i = 0 , & \text{in} \quad \Sigma_\tau , \\ \varphi_i(x,\tau) = \psi_i(x) , \end{cases}$$

with $\operatorname{supp}\psi_i \subseteq [-a,a]$, $\|\psi_i\|_\infty \leq 1$.

As in [8], we have two suitable constants C_1, $C_2 > 0$, such that:

$$|\varphi_i(x,t)| \leq C_1 \min\{1, e^{-\frac{1}{\varepsilon}[|x|-a-C_2(\tau-t)]}\} .$$

Integrating (3.5) and choosing a sequence ψ_i which converges to the characteristic function of the interval $[-a,a]$, we get

$$\begin{aligned} &\int_{-a}^a |u_i^\varepsilon(x+h,\tau) - u_i^\varepsilon(x,\tau)|\,\mathrm{d}x \\ &\qquad\qquad \leq \int |u_i^{0\varepsilon}(x+h) - u_i^{0\varepsilon}(x)|\varphi_i^\varepsilon(x,0)\,\mathrm{d}x \leq w^{T,a}(|h|) . \end{aligned}$$

Here the modulus of continuity $w^{T,a}(|h|)$ does not depend on ε.

The L^1–estimate of the modulus of continuity in time follows from ([8], Lemma 5, p. 233). Hence we have, for $\tau > 0$,

$$\int_{-a}^{a} |u_i^\varepsilon(x,t+\tau) - u_i^\varepsilon(x,t)|\,\mathrm{d}x \le \tilde{w}^{T,a}(\tau)\ ,$$

with $\tilde{w}^{T,a}(\tau) = C_a \min\limits_{|h|\le h_0} \left[|h| + w^{T,a}(|h|) + \frac{\tau}{h^2}\right]$.

The convergence of the sequence U^ε to a (unique) entropy solution follows easily, by using the usual diagonal compactness arguments in $L^1_{\mathrm{loc}}(\Sigma_T)$ and $C([0,T];\, L^1_{\mathrm{loc}}(\mathbb{R}))$.□

Theorem 3.2 (Continuation). *Let U be the local entropy solution of (1.1)–(1.2). Two possibilities occur: either $U \in L^\infty(\Sigma_T)^N$ for any $T > 0$ or there exists $T^* < +\infty$ such that, for any $T < T^*$, U is defined on Σ_T and $\lim\limits_{T\to T^*_-} \|U\|_{L^\infty(\Sigma_T)^N} = +\infty$.*

In the first case the entropy solution U is said to be global, in the second case the solution blows up in finite time and T^* is the blow–up time.

Proof. Let τ be such that for any $T \in (0,\tau)$, the solution U is defined in Σ_T and

$$L = \lim_{T\to\tau_-} \|U\|_{L^\infty(\Sigma_T)^N} < +\infty\ .$$

Since $U \in C([0,\tau);\, L^1_{\mathrm{loc}}(\mathbb{R})^N)$, so for any $t \in (0,\tau)$, $\|U(\cdot,t)\|_{L^\infty(\mathbb{R})^N} \le L$. The conclusion follows from Theorems 2.1, 3.1, by using standard arguments.□

4. Applications

a) Quasimonotone couplings

Let us consider the following 2×2 systems with quasimonotone coupling:

$$\begin{cases} \partial_t u_1 + \partial_x f_1(u_1) = (a_2(u_2) - a_1(u_1))\ , \\ \partial_t u_2 + \partial_x f_2(u_2) = (a_1(u_1) - a_2(u_2)), \end{cases} \tag{4.1}$$

where the functions a_i are smooth with $a_i'(u) > 0$ for $u > 0$ and $a_i(0) = 0$ $(i = 1, 2)$.

A first particular case is the well-known Carleman's model, given by the semilinear system

$$\begin{cases} \partial_t u_1 + \partial_x u_1 = (u_2^2 - u_1^2) \ , \\ \partial_t u_2 - \partial_x u_2 = (u_1^2 - u_2^2), \end{cases} \tag{4.2}$$

which was studied in [9] as the simplest model for the discrete velocity gas.

Another example is given by the following quasilinear system arising in chromatography [17], [18]

$$\begin{cases} \partial_t u_1 + \partial_x f(u_1) = \frac{1}{\varepsilon}(u_2 - A(u_1)) \ , \\ \partial_t u_2 = -\frac{1}{\varepsilon}(u_2 - A(u_1)), \end{cases} \tag{4.3}$$

where $\varepsilon > 0$ and $A' \geq 0,\ A(0) = 0$. This system was studied in [22], while a computational study was given in [16].

Theorem 4.1. *Let $U^0 \in L^\infty(\mathbb{R})^2$, $U^0 \geq 0$. Then the solution of the Cauchy problem (4.1)–(1.2) is global and there exists a constant $C > 0$, only depending on the L^∞ norm of the initial data, such that:*

$$0 \leq \max(\|u_1\|_\infty, \|u_2\|_\infty) \leq C. \tag{4.4}$$

Moreover the inequality (2.5) holds with $M = 0$.

Proof. Notice that the right–hand side is quasimonotone for $U^0 \geq 0$. Set $R_i = \|u_i\|_\infty$ $(i = 1, 2)$. Therefore Theorem 2.2 applies with $\underline{U} = (0, 0)$ and $\overline{U} = (x_1, x_2)$, where (x_1, x_2) solves the problem

$$\begin{cases} \dot{x}_1 = (a_2(x_2) - a_1(x_1)) \ , \\ \dot{x}_2 = (a_1(x_1) - a_2(x_2)), \end{cases}$$

and $(x_1(0), x_2(0)) = (R_1, R_2)$. It is easy to check that $\overline{U}$ is bounded by a constant which depends only on (R_1, R_2). The inequality (2.5), with $M = 0$, follows from (2.6) by a direct checking.□

Observe that inequality (2.5), with $M = 0$, implies a uniform estimates of the modulus of continuity in L^1 for the solutions of problems (4.2) and (4.3); in particular for (4.3) the estimate does not depend on $\varepsilon > 0$, as well as inequality (4.4). This establishes the compactness of the solutions in L^1 as in the proof of Theorem 3.1.

b) Majda's model for combustion

Consider the following 2×2 system

$$\begin{cases} \partial_t u_1 + \partial_x f_1(u_1) = \alpha\varphi(u_1)u_2 \ , \\ \partial_t u_2 + \partial_x f_2(u_2) = -\beta\varphi(u_1)u_2 \ , \end{cases} \tag{4.5}$$

where $\alpha, \beta > 0$ and $\varphi : \mathbb{R} \to [0, 1]$.

For $f_2 \equiv 0$, $u_2 \geq 0$ and φ monotone increasing, the system (4.5) has been intensively studied as a simplified model for detonation [11]. In particular the existence and the uniqueness of entropy solutions were first established in [20] by using the Lax–Friedrichs scheme. Afterwards the convergence of the viscosity approximation was proved in [21] and [10]. We refer to [23] for a discussion about some recent related works concerning the stability of the viscous profiles and the inviscid elementary waves.

We can consider now the system (4.5) by using comparison methods. Notice that in the following proposition the monotonicity of the function φ is not assumed.

Theorem 4.2. *Assume that $\varphi : \mathbb{R} \to [0, 1]$ is Lipschitz continuous. Then, for any $U^0 \in L^\infty(\mathbb{R})^2$, there is a unique entropy solution U to the Cauchy problem (4.5)–(1.2) and the following estimate holds:*

$$\begin{aligned} |u_1(x,t)| &\leq \|u_1^0\|_\infty + \alpha t \|u_2^0\|_\infty \ , \\ |u_2(x,t)| &\leq \|u_2^0\|_\infty \ . \end{aligned} \tag{4.6}$$

Proof. By assumptions $g_1 = \alpha\varphi(u_1)u_2$ is monotone in u_2, since $\varphi \geq 0$. Also

$$-\beta|u_2| \leq g_2(u_1, u_2)\,\mathrm{sgn} u_2 \leq 0 \ .$$

Let us set

$$\overline{g}_1 = \underline{g}_1 = g_1$$

and

$$\overline{g}_2 = \begin{cases} 0 \ , & u_2 \geq 0 \ , \\ -\beta u_2 \ , & u_2 \leq 0 \ , \end{cases}$$

$$\underline{g}_2 = \begin{cases} -\beta u_2 \ , & u_2 \geq 0 \ , \\ 0 \ , & u_2 \leq 0 \ . \end{cases}$$

So $\underline{g}_2 \leq g_2 \leq \overline{g}_2$, and $\underline{g}_2$, $\overline{g}_2$ are both monotone with respect to u_1. Hence we can apply Proposition 2.3. Take $X^0 = (\|u_1^0\|_\infty, \|u_2^0\|_\infty)$ and let $\overline{X} = (\overline{x}_1, \overline{x}_2)$ be the solution of the problem

$$\dot{X} = \overline{G}(X) \ , \qquad X(0) = X^0 \ .$$

Then, for any $t \geq 0$

$$\begin{aligned} x_1^0 \leq \overline{x}_1(t) &\leq x_1^0 + \alpha t x_2^0 \ , \\ \overline{x}_2(t) &= x_2^0 \end{aligned}$$

and

$$U \leq \overline{X} \ .$$

In the same way let $\underline{X} = (\underline{x}_1, \underline{x}_2)$ be the solution of

$$\dot{X} = \underline{G}(X) \ , \qquad X(0) = -X^0 \ .$$

Then, for any $t \geq 0$

$$\begin{aligned} -x_1^0 \geq \underline{x}_1(t) &\geq -(x_1^0 + \alpha t x_2^0) \ , \\ \underline{x}_2(t) &= -x_2^0 \end{aligned}$$

and

$$U \geq \underline{X}. \square$$

We notice also that no assumptions were made about the convexity of the fluxes or the sign of the state variables. We can also consider other cases in which φ is not bounded. For instance, it is easy to see that if

$$G = (u_1 u_2, -u_1 u_2)$$

and if U_0 is a bounded nonnegative data, we have a global entropy solution U and the following estimates hold:

$$\begin{aligned} 0 \leq u_1(x,t) &\leq e^{t\|u_2^0\|_\infty} \|u_1^0\|_\infty \ , \\ 0 \leq u_2(x,t) &\leq \|u_2^0\|_\infty \ . \end{aligned}$$

In fact the following general statement holds true.

Theorem 4.3. *Consider the problem (1.1)–(1.2) with* $N = 2$. *Let us assume there exist some constants* $c_1, c_2 > 0$ *and some continuous functions* $\alpha_1, \alpha_2 : \mathbb{R} \to (0, \infty)$ *such that:*

$$\begin{aligned} g_1(U)u_1 &\leq \alpha_1(u_2)u_1^2 + \alpha_2(u_2) \ , \\ g_2(U)u_2 &\leq c_1 u_2^2 + c_2 \ , \end{aligned} \tag{4.7}$$

for any $U = (u_1, u_2) \in \mathbb{R}^2$. *Hence the solution of (1.1)–(1.2) is global for any bounded initial data and we have*

$$\begin{aligned} |u_2(x,t)| &\leq e^{c_1 t}\left(\|u_2^0\|_\infty^2 + \frac{c_2}{c_1}\right)^{1/2} = M_2(t) \ , \\ |u_1(x,t)| &\leq e^{\overline{\alpha}_1(t)t}\left(\|u_1^0\|_\infty^2 + \frac{\overline{\alpha}_2(t)}{\overline{\alpha}_1(t)}\right)^{1/2} = M_1(t) \end{aligned} \tag{4.8}$$

where $\overline{\alpha}_1(t) = \max_{|\xi| \leq M_2(t)} \alpha_1(\xi)$ *and* $\overline{\alpha}_2(t) = \max_{|\xi| \leq M_2(t)} \alpha_2(\xi)$.

Proof. Let $T > 0$ be fixed. Set

$$b_2 = -a_2 = e^{c_1 T}\left(\|u_2^0\|_\infty^2 + \frac{c_2}{c_1}\right)^{1/2} = M_2(T)$$

and

$$b_1 = -a_1 = e^{\overline{\alpha}_1(T)T}\left(\|u_1^0\|_\infty^2 + \frac{\overline{\alpha}_2(T)}{\overline{\alpha}_1(T)}\right)^{1/2} = M_1(T) \ .$$

Let $\overline{G}$ ($\underline{G}$) be the superior (inferior) quasimonotone envelope of G in the interval $[A, B]$. Then, the functions $\overline{g}_1$, $\overline{g}_2$ ($\underline{g}_1, \underline{g}_2$) verify (4.7) in $[A, B]$ with $\alpha_i = \overline{\alpha}_i(T)$ =const.. In fact we have the following property, for example, for $\overline{g}_1$: for any $U = (u_1, u_2) \in [A, B]$, there exists u_2^* such that $U^* = (u_1, u_2^*) \in [A, B]$ and $\overline{g}_1(U) = g_1(U^*)$.

For $X^0 = (\|u_1^0\|_\infty, \|u_2^0\|_\infty)$, let $\overline{X}$ be the solution of the problem

$$\dot{X} = \overline{G}(X) \ , \quad X(0) = X^0 \ . \tag{4.9}$$

By multiplying (4.9) by $\overline{X}$, we obtain

$$\frac{d}{dt}\overline{x}_i^2 = 2\overline{g}_i(\overline{X})\overline{x}_i \ , \qquad i = 1, 2 \ ,$$

which gives, for all $t \in (0,T)$,

$$|\overline{x}_i(t)| \leq M_i(t) \ , \qquad i = 1, 2 \ .$$

Moreover, by Proposition 2.3,

$$U(x,t) \leq \overline{X}(t) \ ,$$

for a.e. $(x,t) \in \Sigma_T$. In the same way, let $\underline{X}$ be the solution of

$$\dot{X} = \underline{G}(X) \ , \qquad X(0) = -X^0 \ ,$$

then, for all $t \in (0,T)$,

$$|\underline{x}_i(t)| \leq M_i(t) \ , \qquad i = 1, 2 \ ,$$

and

$$\underline{X}(t) \leq U(x,t) \ ,$$

for a.e. $(x,t) \in \Sigma_T$.$\square$

c) Nonlinear optics

Next we consider the following 2×2 system with quadratic coupling

$$\begin{cases} \partial_t u_1 + \partial_x f_1(u_1) = u_2^2 \ , \\ \partial_t u_2 + \partial_x f_2(u_2) = -u_1 u_2 \ . \end{cases} \tag{4.10}$$

In the semilinear case this system arises in nonlinear optics [3]. It was considered in [5] as a kind of (quadratic) conservative interaction. A quadratic interaction is said to be conservative if there is $\lambda > 0$ such that, for any $U \in \mathbb{R}^2$,

$$u_1 g_1(U) + \lambda u_2 g_2(U) = 0 \ . \tag{4.11}$$

In the conservative semilinear case, the global existence of the solution is obtained by using energy methods and the method of characteristics. In fact it is possible to show the global existence for all bounded initial data if and only if

$$G(U) = (\beta_2 u_2(c_1 u_1 + c_2 u_2), \ -\beta_1 u_1(a_1 u_1 + a_2 u_2)) \ ,$$

for $\beta_1, \ \beta_2 \geq 0$ (see [5], [1] and [2] for more details and references). In the quasilinear case it seems hard to apply the same methods. Nevertheless we can obtain some results by using comparison arguments.

Theorem 4.4. *For any $U^0 \in L^\infty(\mathbb{R})^2$, the solution of* (4.10)–(1.2) *is global and we have the following estimate:*

$$\begin{aligned} &-\|u_1^0\|_\infty \le u_1(x,t) \le \|u_1^0\|_\infty + t\|u_2^0\|_\infty e^{2t\|u_1^0\|_\infty} ; \\ &|u_2(x,t)| \le \|u_2^0\|_\infty e^{t\|u_1^0\|_\infty} . \end{aligned} \tag{4.12}$$

Proof. Set

$$m_i = \|u_i^0\|_\infty , \qquad i = 1, 2 .$$

For any fixed $T > 0$, consider the interval $[A, B]$ with

$$\begin{aligned} &a_1 = -m_1 , \quad b_1 = m_1 + Tm_2 e^{2Tm_1} ; \\ &b_2 = -a_2 = m_2 e^{Tm_1} . \end{aligned}$$

Since $G = (u_2^2, -u_1 u_2)$ is not quasimonotone, we consider the inferior and superior quasimonotone envelopes of G. We have for $\overline{G}$:

$$\overline{g}_1 = a_2^2 ;$$

$$\overline{g}_2 = \begin{cases} -u_1 u_2 , & u_2 \le 0 , \\ -a_1 u_2 , & u_2 \ge 0 . \end{cases}$$

Also, for $\underline{G}$, we have:

$$\underline{g}_1 = \begin{cases} 0 , & u_2 \le 0 , \\ u_2^2 , & u_2 \ge 0 ; \end{cases}$$

$$\underline{g}_2 = \begin{cases} -u_1 u_2 , & u_2 \le 0 , \\ -b_1 u_2 , & u_2 \ge 0 . \end{cases}$$

First we consider the system of ordinary differential equations

$$\dot{X} = \overline{G}(X) , \qquad X(0) = M = (m_1, m_2) .$$

The solution is given, for $t \in [0, T]$, by:

$$\overline{x}_1(t) = m_1 + t b_2^2 , \qquad \overline{x}_2(t) = m_2 e^{m_1 t}$$

we observe that for $t \in [0, T]$, $\overline{X}(t) \in [A, B]$. In the same way, consider the system

$$\dot{X} = \underline{G}(X) , \qquad X(0) = -M .$$

Now the solution is given by

$$\underline{x}_1(t) = -m_1 \ , \qquad \underline{x}_2(t) = -m_2 e^{m_1 t} \ .$$

Finally, by using Proposition 2.3, we have for a.e. $(x,t) \in \Sigma_T$:

$$\underline{X}(t) \le U(x,t) \le \overline{X}(t)$$

and the conclusion follows.□

To consider the general conservative (quadratic) interactions, which verify (4.11), let us notice that by simple transformations we can reduce our study to the case

$$G(U) = (u_2(u_2 - u_1), \mu u_1(u_1 - u_2)) \ , \quad \mu > 0 \ . \tag{4.13}$$

We have the following partial result.

Theorem 4.5. *Consider the problem* (1.1)–(1.2) *for* $N = 2$ *and* G *given by* (4.13) *with* $\mu = 1$. *Let* $U^0 \in L^\infty(\mathbb{R})^2$, *with* $u_2^0 \ge 0$ *a.e. in* $\mathbb{R}$. *Then the solution is global and we have:*

$$\begin{aligned} -R \le u_1(x,t) \le R + 4tR^2 e^{2tR} \ , \\ 0 \le u_2(x,t) \le 2Re^{Rt} \ , \end{aligned} \tag{4.14}$$

where $R = \max(\|u_1^0\|_\infty, \|u_2^0\|_\infty)$.

Proof. For any fixed $T > 0$, consider the interval $[A, B]$ with

$$\begin{aligned} a_1 &= -R \ , \qquad b_1 = R + 4TR^2 e^{2TR} \ ; \\ a_2 &= 0 \ , \qquad b_2 = 2Re^{RT} \ . \end{aligned}$$

Since the coupling G is no longer quasimonotone in this interval, we take its quasimonotone envelopes.

For $\overline{G}$, we have:

$$\overline{g}_1 = \begin{cases} 0 \ , & u_2 \le u_1 \ , \\ u_2(u_2 - u_1) \ , & u_2 \ge u_1 \ ; \end{cases}$$
$$\overline{g}_2 = \begin{cases} R(u_2 + R) \ , & u_1 \le u_2 + R \ , \\ u_1(u_1 - u_2) \ , & u_1 \ge u_2 + R \ . \end{cases}$$

In the same way we have:

$$\underline{g}_1 = \begin{cases} -\frac{u_1^2}{4} \ , & u_2 \leq \frac{u_1}{2} \ , \\ u_2(u_2 - u_1) \ , & u_2 \geq \frac{u_1}{2} \ ; \end{cases}$$
$$\underline{g}_2 = \begin{cases} -\frac{u_2^2}{4} \ , & u_1 \leq \frac{u_2}{2} \ , \\ u_1(u_1 - u_2) \ , & u_1 \geq \frac{u_2}{2} \ . \end{cases}$$

First we consider the system of ordinary differential equations

$$\dot{X} = \overline{G}(X) \ , \quad X(0) = (R, R) \ .$$

Setting $Q(t) = \int_0^t \overline{x}_2(s)\,\mathrm{d}s$, it is easily seen that we have

$$\begin{aligned} \overline{x}_1(t) &= Re^{-Q(t)} + \int_0^t e^{(Q(s)-Q(t))}\overline{x}_2^2(s)\,\mathrm{d}s \ , \\ \overline{x}_2(t) &= R(2e^{Rt} - 1) \ . \end{aligned}$$

Observe that for $t \in [0, T]$, $\overline{X}(t) \in [A, B]$. Next, we have that $\underline{X}(t) = (-R, 0)$ is a stationary solution of the problem

$$\dot{X} = \underline{G}(X) \ , \qquad X(0) = (-R, 0) \ .$$

Therefore, again from the Proposition 2.3, we have the following inequalities in Σ_T:

$$\underline{X}(t) \leq U(x, t) \leq \overline{X}(t)$$

and the estimate (4.12) follows easily.□

An analogous statement could be made for $U_0 \in L^\infty(\mathbb{R})^2$, with $u_1^0 \geq 0$ a.e. in $\mathbb{R}$.

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