VISCOUS SPLITTING APPROXIMATION OF MIXED HYPERBOLIC-PARABOLIC CONVECTION-DIFFUSION EQUATIONS

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Abstract. We first analyse a semi-discrete operator splitting method for nonlinear, possibly strongly degenerate, convection-diffusion equations. Due to strong degeneracy, solutions can be discontinuous and are in general not uniquely determined by their data. Hence weak solutions satisfying an entropy condition are sought. We then propose and analyse a fully discrete splitting method which employs a front tracking method for the convection step and a finite difference scheme for the diffusion step. Numerical examples are presented which demonstrate that our method can be used to compute physically correct solutions to mixed hyperbolic-parabolic convection-diffusion equations.

0. Introduction.

In this paper we consider viscous splitting methods for nonlinear, possibly strongly degenerate, convection-diffusion equations of the form

$$(1)\qquad \begin{cases} \partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 A(u), & (x,t) \in Q_T = \mathbb{R} \times \langle 0, T \rangle, \qquad A'(u) \geq 0, \\ u(x,0) = u_0(x), \end{cases}$$

where $u(x,t)$ denotes the (scalar) unknown, $u_0(x)$ is a given function of bounded variation, $f(u)$ and $A(u)$ are given locally smooth bounded functions, and $\varepsilon > 0$ is a (small) scaling parameter.

Convection-diffusion equations arise in a variety of applications, among others turbulence, traffic flow, financial modelling and front propagation. Such equations also constitute an important part of a system of equations describing two phase flow in oil reservoirs [6] as well as a system of equations describing sedimentation processes used for solid-liquid separation in industrial applications [5,4]. When (1) is convection dominated, which is often the case in reservoir simulation, it is well known that conventional numerical methods exhibit non-physical oscillations and/or excessive numerical diffusion in the vicinity of moving shock fronts.

An underlying design principle for many successful numerical methods for equations such as (1), is viscous operator splitting. That is, one splits the time evolution into two partial steps in order to separate the effects of convection and diffusion (viscosity). Variations on the viscous splitting approach have indeed been taken in various contexts by many authors [1,2,25,12,14,10,11,4,5]. Hvistendahl Karlsen and Risebro [17] introduced and analysed a fully discrete splitting method for (1) in the linear diffusion case $A(u) \equiv u$. Their discrete method was based on a front tracking scheme for the convection step and a finite element scheme for the diffusion step. A variant of this method has also been implemented in a prototype two-dimensional black oil reservoir simulator [19]. The idea is to treat multi-dimensional problems by means of dimensional splitting. The resulting method allowed for very long time steps due to the use of front tracking and a novel procedure for reducing potential viscous splitting error in areas containing parabolic shock fronts, see [19] for details.

The assumption of linear diffusion is restrictive from the point of view of applications. For example, the equation arising in models of two phase flow in reservoirs is of the form [6]

$$(2)\qquad \partial_t u + \partial_x f(u) = \varepsilon \partial_x (a(u) \partial_x u) = \varepsilon \partial_x^2 A(u), \qquad A'(u) = a(u),$$

where $a(u)$ is a nonlinear function that vanishes for some values of u. Furthermore, in many applications this equation is so strongly dominated by convection that it is reasonable to set $a(u)$ equal to zero on intervals of positive length. This will imply that the shock fronts, i.e., the transition zones from water to oil, are hyperbolic and thus discontinuous. A variant of (2) also appears in certain sedimentation models and there the diffusion term $a(u)$ is generically zero on an interval of positive length, see [5,4] for details.

The main purpose of our paper is to propose and analyse a working splitting method for the notably more difficult case of degenerate equations, which calls for an appropriate notion of a generalised solution.

When (1) is non-degenerate, i.e., $A'(u) > 0$, it is well known that (1) admits a unique classical solution [24]. This contrasts with the case where (1) is allowed to degenerate at certain points, i.e., $A'(u)$ may vanish for some

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values of u. Solutions are then not necessarily smooth and weak solutions must be sought. A function $u(x,t)$ is here called a weak solution if, for all suitable test functions ϕ, it satisfies

$$\int_{\mathbb{R}}\int_0^T \left(u\partial_t\phi + f(u)\partial_x\phi + \varepsilon A(u)\partial_x^2\phi\right)\,dt\,dx + \int_{\mathbb{R}} u_0(x)\phi(x,0) = 0. \tag{3}$$

The simplest examples of degenerate equations are perhaps provided by the porous medium equation $\partial_t u = \varepsilon\partial_x^2\left(u^m\right),\ m>1$ and the convective porous medium equation $\partial_t u + \partial_x\left(u^n\right) = \varepsilon\partial_x^2\left(u^m\right),\ n,m>1$, which both degenerate at $u=0$. A striking manifestation of this one-point degeneracy is finite speed of propagation. The reader who is interested in an overview of the very extensive literature that exists on degenerate parabolic problems is referred to the papers [7,20,15] and the references therein.

We will from now on refer to (1) as *degenerate* when $A(u)$ is strictly increasing and *strongly degenerate* or *mixed hyperbolic-parabolic* when $A(u)$ is merely non-decreasing.

In the strongly degenerate case there exist two numbers α and β such that $A'(u)=0$ on the interval $[\alpha,\beta]$. A simple example of a strongly degenerate equation is a hyperbolic conservation law $\partial_t u + \partial_x f(u) = 0$. Thus (1) must in general possess discontinuous solutions. Furthermore, discontinuous solutions defined by an integral equality (3) is not uniquely determined by their data. In fact, an additional condition – an entropy condition – is needed to single out the physically relevant weak solution:

We here seek weak solutions that are realizable as the L^1 limit of smooth solutions u_μ of non-degenerate parabolic equations,

$$\partial_t u_\mu + \partial_x f(u_\mu) = \varepsilon\partial_x^2 A(u_\mu) + \mu\partial_x^2 u_\mu, \qquad \mu>0,$$

as the regularization parameter μ tends to zero, yielding the following notion of an entropy solution. We call a bounded measurable function $u(x,t)$ an *entropy weak solution* of the initial value problem (1) if, for all real numbers k and suitable test functions $\phi\geq 0$, it satisfies [27]

$$\begin{aligned}\mathcal{L}(u) = \int_{\mathbb{R}}\int_0^T &\left(|u-k|\partial_t\phi + \operatorname{sign}(u-k)\left(f(u)-f(k)\right)\partial_x\phi + \varepsilon|A(u)-A(k)|\partial_x^2\phi\right)\,dt\,dx \\ &+ \int_{\mathbb{R}}|u_0-k|\phi(x,0)\,dx \geq 0.\end{aligned} \tag{4}$$

Letting $k\to\pm\infty$, we see that (1) holds in the weak sense (3). Furthermore, in the context of hyperbolic equations ($A'\equiv 0$), the entropy condition (4) coincides with the celebrated condition due to Volpert [26] (see also Kruzkov [21]). Using a standard approximation argument it follows from (4) that the entropy inequality

$$\partial_t U(u) + \partial_x P(u) - \varepsilon\partial_x^2 Q(u) \leq 0$$

holds in the weak sense for any convex Lipshitz entropy $U:\mathbb{R}\to\mathbb{R}$ with corresponding entropy fluxes P and Q given by $P' = U'f'$ and $Q' = U'A'$.

Existence of an entropy weak solution to (1) was established by Volpert and Hudjaev [27]. Although the uniqueness of entropy weak solutions in the context of conservation laws is by now a classical result [21,26], a similar (general) uniqueness result for (1) is still not available, thus reflecting the fact that these problems are significantly less well understood than conservation laws. On the other hand, the uniqueness of weak solutions for the purely parabolic case (no convection term) in the class of bounded integrable functions has been proved by Brezis and Crandall [3]. A uniqueness result for the mixed type equation (2) in the BV class with $\partial_x A(u)$ locally integrable has recently been proved by Wu and Yin [28]. We shall briefly review this result together with the jump conditions used in the proof. The jump conditions will also be used to justify the behaviour of our fully discrete method when computing discontinuous solutions.

Having introduced the appropriate notion of a solution, the splitting technique studied in this paper can be summarized as follows: Let $v(x,t) = \mathcal{S}_t v_0(x)$ be the entropy weak solution to the conservation law

$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = v_0(x),$$

and let $w(x,t) = \mathcal{H}_t w_0(x)$ be the entropy weak solution to the degenerate parabolic problem

$$\partial_t w = \varepsilon\partial_x^2 A(w), \qquad w(x,0) = w_0(x). \tag{5}$$

The viscous splitting method is then based on the following approximation

$$u(x,n\Delta t) \approx \left[\mathcal{H}_{\Delta t}\circ\mathcal{S}_{\Delta t}\right]^n u_0. \tag{6}$$

A L^1 type convergence analysis of the splitting (6) was presented in [17] in the context of linear diffusion. In this paper we prove similar results in the case where $A = A(u)$ is non-linear and A' possibly zero on intervals of positive length. Furthermore, we are here able to explicitly quantify the entropy discrepancy (4) associated with the operator splitting. Thus, even in the case of linear diffusion the results presented here are more refined than the results in [17].

With an eye to applications, we propose a fully discrete splitting method where the exact solution operators $\mathcal{S}_t$ and $\mathcal{H}_t$ are replaced by numerical methods. We use front tracking [9,16] as approximate solution operator for the convection operator $\mathcal{S}_t$ and a suitable finite difference scheme as approximate solution operator for the diffusion operator $\mathcal{H}_t$. A detailed convergence analysis is also presented for the fully discrete method.

We mention that Bürger et al. [5,4] employ a finite difference based operator splitting scheme for equations of the type (2) with a time dependent flux function $f = f(t, u)$. This scheme is similar to our front tracking based splitting method, but they have not yet provided mathematical justification of their scheme, see [4].

The remaining part of this paper is organized as follows. In §1 we present a convergence analysis of the semi-discrete splitting formula (6). In §2 we propose and analyse a fully discrete version of (6). We present and discuss some numerical results in §3. Finally, in §4 we make some concluding remarks.

1. The semi-discrete method.

Fix $T > 0$ and $\Delta t > 0$, and let N be such that $N\Delta t = T$. Let now u^n denote the approximate solution to (1) at a fixed time $t_n = n\Delta t$ $(n = 0, \dots, N-1)$, $u^0 = u_0$. Next, we explain how to construct u^{n+1} from u^n.

Let $v(x,t)$ be the solution to the hyperbolic conservation law

$$\partial_t v + \partial_x f(v) = 0, \qquad v(x,0) = u^n(x), \qquad (x,t) \in \mathbb{R} \times \langle 0, \Delta t\rangle\,. \tag{7}$$

Note that there exists a unique entropy weak solution to (7) provided u^n has bounded total variation and f is locally C^2, see [26]. Letting $\mathcal{S}_t$ denote the solution operator of (7) at time t, we define $u^{n+1/2} = \mathcal{S}_{\Delta t} u^n$.

Let $A_\mu(u) = A(u) + \frac{\mu}{\varepsilon} u$, for $\mu > 0$ and let $w_\mu(x,t)$ be the solution of the parabolic problem

$$\partial_t w_\mu = \varepsilon \partial_x^2 A_\mu(w_\mu), \qquad w_\mu(x,0) = u^{n+1/2}(x), \qquad (x,t) \in \mathbb{R} \times \langle 0, \Delta t\rangle\,. \tag{8}$$

Note that there exists a unique classical solution to (8) provided $u^{n+1/2}$ has bounded total variation and A is locally C^2. Let $\mathcal{H}_t^\mu$ denote the solution operator associated with (8) at time t.

Although our finite difference scheme for solving (5) (see §2) is not based on adding an artificial diffusion term to the equation, this is indeed the case for many numerical schemes. Thus, using (8) instead of (5) when defining the semi-discrete splitting is reasonable from a numerical point of view. However, another (technical) reason is that we avoid appealing to approximation arguments when working with the derivatives due to the smoothness of the solution to (8). On the other hand, Theorem 1.5 below does not assume any relation between Δt and μ so that μ can go to zero independently of Δt. The case $\mu \equiv 0$ is therefore covered by our analysis.

With the current notation in hand, we can define the approximation u^{n+1} by the product formula

$$u^{n+1} = \left[\mathcal{H}_{\Delta t}^\mu \circ \mathcal{S}_{\Delta t}\right] u^n. \tag{9}$$

It will be more convenient to work with functions defined in the whole upper plane, and not merely on the time strips $t = t_n$. Consider therefore the function $u_\eta(x,t)$, $\eta = (\Delta t, \mu)$, where

$$u_\eta(x,t) = u^n(x), \qquad (x,t) \in \mathbb{R} \times \langle t_{n-1}, t_n\rangle\,, \qquad n = 1, \dots, N. \tag{10}$$

We next establish convergence of the splitting approximations to an entropy weak solution. To this end, recall that the space $BV(\mathbb{R})$ consists of all $L^1_{\mathrm{loc}}(\mathbb{R})$ functions $z(x)$ whose first order derivative $\frac{dz}{dx}$ is represented by a (locally) finite Borel measure. The total variation of z is by definition the total mass of this measure, i.e.,

$$|z|_{BV} = \int_{\mathbb{R}} \left|\frac{dz}{dx}\right|.$$

Lemma 1.1. *We have the following maximum principle*

$$\|u_\eta(\cdot,t)\|_\infty \le \|u_0\|_\infty, \qquad \forall t > 0. \tag{11}$$

Proof. This claim follows from the well-known fact that the solution operators $\mathcal{S}_t$ and $\mathcal{H}_t^\mu$ do not introduce new minima or maxima. □

Lemma 1.2. *We have the following bound on the total variation*

$$|u_\eta(\cdot,t)|_{BV} \le |u_0|_{BV}, \qquad \forall t > 0. \tag{12}$$

Proof. The lemma follows since the solution operators $\mathcal{S}_t$ and $\mathcal{H}_t^\mu$ do not increase the total variation of their initial data, see [26,27] and also §2. □

Lemma 1.3. *There is a finite constant $C > 0$, independent of η, such that*

$$\|u_\eta(\cdot, t_2) - u_\eta(\cdot, t_1)\|_1 \le C\sqrt{|t_2 - t_1|}, \qquad \forall t_1, t_2 > 0. \tag{13}$$

Proof. Without loss of generality, we assume that $t_1 = t_n$ and $t_2 = t_{n+1}$ for some fixed n. We first establish weak Lipschitz continuity in the time variable of the splitting solution u_η.

To this end, integrating the equation (8) against a test function $\phi(x)$ over $\mathbb{R} \times \langle \tau_1, \tau_2\rangle$ yields the result

$$\begin{aligned}\left|\int_{\mathbb{R}} (w_\mu(x, \tau_2) - w_\mu(x, \tau_1))\, \phi(x)\, dx\right| &= \left|\int_{\tau_1}^{\tau_2}\int_{\mathbb{R}} \varepsilon \partial_x A_\mu(w_\mu)\phi'(x)\, dx\, dt\right| \\ &\le (\tau_2 - \tau_1)\|\phi'\|_\infty \max_u |A'_\mu(u)||w_\mu(\cdot, t)|_{BV}.\end{aligned}$$

Letting $\tau_1 = 0,\ \tau_2 = \Delta t,\ w_0 = u^{n+1/2}$ and using Lemma 1.2, we get

$$\left|\int_{\mathbb{R}} \left(\mathcal{H}^\mu_{\Delta t} u^{n+1/2} - u^{n+1/2}\right) \phi\, dx\right| = \mathcal{O}(1)\|\phi'\|_\infty \Delta t.$$

Since solutions of hyperbolic conservation laws have finite speed of propagation, we get

$$\|\mathcal{S}_{\Delta t} u^n - u^n\|_1 = \mathcal{O}(1)\Delta t.$$

Consequently, the following weak continuity result holds

$$\left|\int_{\mathbb{R}} (u_\eta(x, t_2) - u_\eta(x, t_1))\, \phi(x)\, dx\right| = \mathcal{O}(1)\left(\|\phi\|_\infty + \|\phi'\|_\infty\right)(t_2 - t_1). \tag{14}$$

Next, let $\omega_\rho(x)$ be a standard C_0^∞-mollifier with smoothing radius ρ. Let $d(x) = u_\eta(x, t_2) - u_\eta(x, t_1)$, and define $\beta(x) = \operatorname{sign}(d(x))$ for $|x| \le r - \rho$ and $\beta(x) = 0$ for $|x| > r - \rho$, where $r > \rho$. Moreover, define $\beta^\rho = \omega_\rho * \beta$, and note that $\beta^\rho \in C^\infty$ with support in $[-r, r]$. By choosing $\phi = \beta^\rho$ in (14) it follows that

$$\begin{aligned}&\left|\int_{-r}^{r} |u_\eta(x, t_2) - u_\eta(x, t_1)|\, dx\right| \\ &\quad\le \left|\int_{-r}^{r} ||d(x)| - \beta^\rho(x)d(x)|\, dx\right| + \left|\int_{-r}^{r} |\beta^\rho(x)d(x)|\, dx\right| \\ &\quad\le C_1\rho + C_2(t_2 - t_1)/\rho.\end{aligned}$$

Choosing $\rho = \sqrt{t_2 - t_1}$ and letting $r \to \infty$, we obtain (13). □

In view of Lemmas 1.1 and 1.2, a classical application of Helly's theorem yields the existence of a subsequence $\{u_{\eta_j}(\cdot, t)\}$ converging in $L^1_{\text{loc}}(\mathbb{R})$ to a function $u(\cdot, t)$ in $L^\infty(\mathbb{R}) \cap BV(\mathbb{R})$ for each fixed t. By a diagonalization argument we obtain the existence of a further subsequence, still denoted by $\{u_{\eta_j}(\cdot, t)\}$, which converges for all t in some dense countable subset of $\langle 0, T\rangle$. By appealing to Lemma 1.3, we obtain convergence for all t in $\langle 0, T\rangle$.

Consequently, we have the following lemma.

Lemma 1.4. *Let $\{\eta = (\Delta t, \mu)\}$ be a sequence of discretization parameters tending to zero. Then there exists a subsequence $\{\eta_j\}$ and a bounded measurable function u such that $u_{\eta_j} \to u$ in $L^1_{\text{loc}}(Q_T)$ as $j \to \infty$.*

We now justify the term 'approximate solution' by proving the following theorem.

Theorem 1.5. *Suppose that u_0 is of bounded total variation, and that $f(u)$ and $A(u)$ are locally twice continuously differentiable. Let $\{u_\eta\}$ be the semi-discrete splitting sequence given by (10). Then the entropy discrepancy $\mathcal{L}(u_\eta)$ is of order $\sqrt{\Delta t}$. Furthermore, there exists a subsequence $\{u_{\eta_j}\}$ converging to an entropy weak solution of the mixed hyperbolic-parabolic convection-diffusion problem*

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 A(u), \qquad u(x, 0) = u_0(x), \qquad (x, t) \in Q_T, \qquad A'(u) \ge 0.$$

Proof. We first estimate the entropy discrepancy (4) associated with the auxiliary sequence $\{\hat{u}_\eta\}$,

$$\hat{u}_\eta(x, t) = \begin{cases} \mathcal{S}_{2(t - t_n)} u^n, & (x, t) \in \mathbb{R} \times \langle t_n, t_{n+1/2}\rangle, \\ \left[\mathcal{H}^\mu_{2(t - t_{n+1/2})} \circ \mathcal{S}_{\Delta t}\right] u^n, & (x, t) \in \mathbb{R} \times \langle t_{n+1/2}, t_{n+1}\rangle. \end{cases} \tag{15}$$

This way of extending (9) to a function defined for all t was first used by Crandall and Madja [8], see also [17]. Observe that $\|u_\eta(\cdot,t)-\hat{u}_\eta(\cdot,t)\|_1=\mathcal{O}(\sqrt{\Delta t})$. This implies that also $\hat{u}_{\eta_j}\to u$ in $L^1_{\mathrm{loc}}(Q_T)$, where $\{\eta_j\}$ refers to the subsequence in Lemma 1.4 and u the limit function.

Let $v_n(t)=\mathcal{S}_t u^n$, $t\in\langle 0,\Delta t\rangle$, and define a new test function φ by $\varphi(x,t)=\phi(x,t/2)$. Then the following inequality holds

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{t_n}^{t_{n+1/2}}\left(\frac{1}{2}|\hat{u}_\eta-k|\partial_t\phi+\operatorname{sign}(\hat{u}_\eta-k)\left(f(\hat{u}_\eta)-f(k)\right)\partial_x\phi\right)dt\,dx\\
&\quad=\frac{1}{2}\int_{\mathbb{R}}\int_0^{\Delta t}\left(|v_n-k|\partial_\tau\varphi(x,\tau+2t_n)+\operatorname{sign}(v_n-k)\left(f(v_n)-f(k)\right)\partial_x\varphi(x,\tau+2t_n)\right)d\tau\,dx\\
&\quad\geq\frac{1}{2}\int_{\mathbb{R}}|\hat{u}_\eta(x,t_{n+1/2})-k|\phi(x,t_{n+1/2})\,dx-\frac{1}{2}\int_{\mathbb{R}}|\hat{u}_\eta(x,t_n)-k|\phi(x,t_n)\,dx.
\end{aligned}
\tag{16}
$$

where we have used the substitution $\tau=2(t-t_n)$. Similarly, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_{t_{n+1/2}}^{t_{n+1}}\left(\frac{1}{2}|\hat{u}_\eta-k|\partial_t\phi+\varepsilon|A_\mu(\hat{u}_\eta)-A_\mu(k)|\partial_x^2\phi\right)dt\,dx\\
&\quad\geq\frac{1}{2}\int_{\mathbb{R}}|\hat{u}_\eta(x,t_{n+1})-k|\phi(x,t_{n+1})\,dx-\frac{1}{2}\int_{\mathbb{R}}|\hat{u}_\eta(x,t_{n+1/2})-k|\phi(x,t_{n+1/2})\,dx.
\end{aligned}
\tag{17}
$$

Introduce the entropy fluxes $P_k(u)=\operatorname{sign}(u-k)\left(f(u)-f(k)\right)$ and $Q_{k,\mu}(u)=|A_\mu(u)-A_\mu(k)|$. Then adding (16) and (17), multiplying by 2 and summing over all $n=0,\dots,N-1$, yields the global inequality

$$
\mathcal{L}(\hat{u}_\eta)\geq E_s^1+E_s^2,
\tag{18}
$$

where

$$
\begin{aligned}
E_s^1&=\sum_n\int_{\mathbb{R}}\left(\int_{t_n}^{t_{n+1}}P_k(\hat{u}_\eta)\partial_x\phi\,dt-2\int_{t_n}^{t_{n+1/2}}P_k(\hat{u}_\eta)\partial_x\phi\,dt\right)dx,\\
E_s^2&=\varepsilon\sum_n\int_{\mathbb{R}}\left(\int_{t_n}^{t_{n+1}}Q_{k,\mu}(\hat{u}_\eta)\partial_x^2\phi\,dt-2\int_{t_{n+1/2}}^{t_{n+1}}Q_{k,\mu}(\hat{u}_\eta)\partial_x^2\phi\,dt\right)dx.
\end{aligned}
\tag{19}
$$

We begin by estimating the second term E_s^2. Writing

$$
Q_{k,\mu}(\hat{u}_\eta(t))=Q_{k,\mu}(\hat{u}_\eta(t_n))+\left[Q_{k,\mu}(\hat{u}_\eta(t))-Q_{k,\mu}(\hat{u}_\eta(t_n))\right],
$$

we can rewrite E_s^2 as $E_s^2=E_s^{2,1}+E_s^{2,2}$, where

$$
\begin{aligned}
E_s^{2,1}&=\varepsilon\sum_n\int_{\mathbb{R}}\left(\int_{t_n}^{t_{n+1}}\partial_x^2\phi\,dt-2\int_{t_{n+1/2}}^{t_{n+1}}\partial_x^2\phi\,dt\right)Q_{k,\mu}(\hat{u}_\eta(t_n))\,dx\\
E_s^{2,2}&=\varepsilon\sum_n\int_{\mathbb{R}}\left(\int_{t_n}^{t_{n+1}}\left[Q_{k,\mu}(\hat{u}_\eta(t))-Q_{k,\mu}(\hat{u}_\eta(t_n))\right]\partial_x^2\phi\,dt\right.\\
&\qquad\left.-2\int_{t_{n+1/2}}^{t_{n+1}}\left[Q_{k,\mu}(\hat{u}_\eta(t))-Q_{k,\mu}(\hat{u}_\eta(t_n))\right]\partial_x^2\phi\,dt\right)dx.
\end{aligned}
$$

Since ϕ is smooth, we may write $\partial_x^2\phi(x,t)=\partial_x^2\phi(x,t_n)+\mathcal{O}(t-t_n)$ for $t\geq t_n$. With the aid of this and Lemma 1.1, it is easy to see that

$$
\left|E_s^{2,1}\right|=\mathcal{O}(1)\Delta t\int_{\operatorname{supp}(\phi)}|Q_{k,\mu}(\hat{u}_\eta(t_n))|\,dx=\mathcal{O}(\Delta t).
$$

Using the reversed triangle inequality, the Lipshitz continuity of $Q_{k,\mu}$ and (13), we get

$$
\left|E_s^{2,2}\right|=\mathcal{O}(1)\sum_n\int_{t_n}^{t_{n+1}}\int_{\mathbb{R}}|\hat{u}_\eta(t)-\hat{u}_\eta(t_n)|\,dx\,dt=\mathcal{O}(\sqrt{\Delta t}).
$$

Consequently, $|E_s^2|=\mathcal{O}(\sqrt{\Delta t})$. Similarly one can deduce that $|E_s^1|=\mathcal{O}(\sqrt{\Delta t})$. Hence we obtain the entropy estimate

$$
\mathcal{L}(\hat{u}_\eta)\geq-\hat{C}\sqrt{\Delta t},
$$

where $\hat{C}$ is a positive constant. Furthermore, this implies that the entropy discrepancy associated with $\{u_\eta\}$ is

$$\mathcal{L}(u_\eta) \geq -C\sqrt{\Delta t}. \tag{20}$$

where C is a finite positive constant. Finally, in view of (20), we get that the limit (see Lemma 1.4)

$$u = \lim_{j\to\infty} u_{\eta_j}$$

is an entropy weak solution to (1). This concludes the proof of the theorem. $\square$

Let us now consider (2) with an initial condition u_0 in $W^{1,\infty}\cap BV\cap L^1$ ($W^{1,\infty}$ denotes the space of Lipschitz functions on $\mathbb{R}$). As mentioned before, a general uniqueness result for entropy weak solutions is still not available. However, a general result was proved recently by Wu and Yin [28] for the so-called BV entropy weak solutions. This notion of a solution depends heavily on the theory of BV functions of several variables and geometric measures. We therefore recall some relevant definitions and results from this theory before returning to [28].

Let Ω be an open subset of $\mathbb{R}^d$ ($d > 1$). The space $BV(\Omega)$ of function of bounded variation consists of all L^1_{loc} functions u whose first order partial derivatives $\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_d}$ are represented by finite Borel measures. The total variation of u is by definition the sum of the total masses of these Borel measures. A useful concept proposed in [26] is the notion of averaged superposition. Let u be in $L^\infty(\Omega)\cap BV(\Omega)$ and let p be locally C^1. Then the averaged superposition of the function u by the function p is defined for H_{d-1} - almost every y in Ω by

$$\hat{p}(u)(y) = \begin{cases} p(u(y)), & \text{if } y \text{ is a approximate continuity point of } u,\\ \int_0^1 p\left(\xi u^+(y) + (1-\xi)u^-(y)\right)\, d\xi, & \text{if } y \text{ is a approximate jump point of } u,\end{cases}$$

where H_{d-1} denotes the $(d-1)$ dimensional Hausdorff measure. Furthermore, $u^-(y)$ and $u^+(y)$ are the left and right approximate limits of $u(y)$ with respect to a unit normal $\nu(y)\in\mathbb{R}^d$ (see below). The function $\hat{p}(u)$ is measurable and integrable with respect to the Borel measure $\frac{\partial u}{\partial y_i}$, so that the non-conservative product $\hat{p}(u)\frac{\partial u}{\partial y_i}$ makes sense as a finite Borel measure. Moreover, the chain rule

$$\frac{p(u)}{\partial y_i} = \frac{\partial \hat{p}(u)}{\partial u}\frac{\partial u}{\partial y_i} \tag{21}$$

holds in the sense of measures. We are now in position to introduce the notion of a BV entropy weak solution. The essential idea behind this solution concept is to impose more regularity on the solution so that the derivative of the diffusion term exists in some sense. This is realized by the following definition.

A function $u(x,t)$ in $BV(Q_T)$ is called a *BV entropy weak solution* if the following two conditions hold (here k is a real number and $\phi \geq 0$ a suitable test function)

$$\hat{a}(u)\partial_x u \in L^1_{\text{loc}}(Q_T), \tag{22}$$

$$\int_{\mathbb{R}}\int_0^T \left(|u-k|\partial_t\phi + \text{sign}(u-k)\left(f(u)-f(k)-\varepsilon\hat{a}(u)\partial_x u\right)\partial_x\phi\right)dt\,dx + \int_{\mathbb{R}}|u_0-k|\phi(x,0)\,dx \geq 0. \tag{23}$$

Let us recall some basic consequences of (22) and (23), as proved in [27]. The condition (22) implies that the measure $\hat{a}(u)\partial_x u$ is absolutely continuous, which means that $\hat{a}(u)\partial_x u$ can be expressed by the integral of a function in $L^1(Q_T)$ (the Radon-Nikodym theorem). We shall always identify $\hat{a}(u)\partial_x u$ with this L^1 function. It is easy to see that (23) implies that the equation holds in the weak sense (3). Consequently, the generalized derivative $\partial_x(\hat{a}(u)\partial_x u)$ is a locally finite measure since $\partial_t u$ and $\partial_x f(u)$ are locally finite measures, and the equation (2) holds in the sense of equality of measures.

The proof of existence of a BV entropy weak solution dates back to the work [27]. The L^1 stability and thus uniqueness, however, was established recently in [28]. The proof makes significant use of the following (correct) jump conditions, which also were derived in [28].

Let Γ_u be the set of jumps of $u(x,t)$, $\nu = (\nu_t,\nu_x)$ the unit normal to Γ_u, $u^-(x_0,t_0)$ and $u^+(x_0,t_0)$ the approximate limits of u at $(x_0,t_0)\in\Gamma_u$ from the sides of the half-planes $(t-t_0)\nu_t + (x-x_0)\nu_x < 0$ and $(t-t_0)\nu_t + (x-x_0)\nu_x > 0$ respectively, $u^l(x,t)$ and $u^r(x,t)$ denote the left and right approximate limits of $u(\cdot,t)$ respectively as a function of x. Then H_1 - almost everywhere on Γ_u,

$$\begin{aligned} &(u^+-u^-)\nu_t + (f(u^+)-f(u^-))\nu_x - \varepsilon\left((\hat{a}(u)\partial_x u)^r - (\hat{a}(u)\partial_x u)^l\right)|\nu_x| = 0,\\ &a(u) = 0, \qquad \forall u \in I(u^-,u^+), \end{aligned} \tag{24}$$

where $I(a,b)$ denotes the interval bounded by a and b. Furthermore,

$$
\begin{aligned}
&|u^+ - k|\nu_t + \operatorname{sign}(u^+ - k)\big[f(u^+) - f(k) - \varepsilon\left((\hat{a}(u)\partial_x u)^r \operatorname{sign}^+(\nu_x) - (\hat{a}(u)\partial_x u)^r \operatorname{sign}^-(\nu_x)\right)\big]\nu_x \\
&\le |u^- - k|\nu_t + \operatorname{sign}(u^- - k)\big[f(u^-) - f(k) - \varepsilon\left((\hat{a}(u)\partial_x u)^l \operatorname{sign}^+(\nu_x) - (\hat{a}(u)\partial_x u)^l \operatorname{sign}^-(\nu_x)\right)\big]\nu_x,
\end{aligned} \tag{25}
$$

where $\operatorname{sign}^+ = \operatorname{sign}$ and $\operatorname{sign}^- = \operatorname{sign}^+ - 1$. Observe that these jump conditions *do not* coincide with the jump conditions satisfied by entropy solutions of hyperbolic conservation laws. Later we will use the Rankine-Hugoniot type condition (24) to explain the qualitative behaviour of our fully discrete splitting method.

We should mention that the jump conditions proposed in [27] are in general not correct and thus the uniqueness proof presented there is incomplete, see [28] for more details.

We now consider the semi-discrete splitting method (10), where $\mathcal{H}_t^\mu$ is now the exact solution operator associated with the non-degenerate parabolic equation with the diffusion term on non-conservative form

$$
\partial_t w = \varepsilon \partial_x^2 \left(a_\mu(w)\partial_x w\right), \qquad a_\mu(w) = a(w) + \frac{\mu}{\varepsilon}, \qquad \mu > 0.
$$

According to Theorem 1.5 a subsequence $\{u_{\eta_j}\}$ converges to an entropy weak solution u of (2) as $j \to \infty$. Without loss of generality, let us suppose that the whole sequence $\{u_\eta\}$ converges to u.

Assuming that u is an element in $BV(Q_T)$, we want to show that u satisfies the regularity condition (22). To this end, introduce the two sequences $\{\tilde{u}_\eta\}$ and $\{g_\eta\}$,

$$
\tilde{u}_\eta(x,t) = \left[\mathcal{H}_{t-t_n} \circ \mathcal{S}_{\Delta t}\right] u^n, \qquad g_\eta(x,t) = \sqrt{\varepsilon a\left(\tilde{u}_\eta\right)}\partial_x \tilde{u}_\eta, \qquad (x,t) \in \mathbb{R} \times \langle t_n, t_{n+1}\rangle .
$$

Since $\|u_\eta - \tilde{u}_\eta\|_1 = \mathcal{O}(\sqrt{\Delta t})$, $\tilde{u}_\eta \to u$. Multiplying the differential equation for $\tilde{u}_\eta(x,t)$ on $\mathbb{R} \times \langle t_n, t_{n+1}\rangle$ by $\tilde{u}_\eta(x,t)$, integrating over Q_T and then doing integration by parts in space, we get

$$
\begin{aligned}
\|g_\eta\|_2 &\le \int_{\mathbb{R}} \int_0^T \varepsilon a_\mu(\tilde{u}_\eta)\left(\partial_x \tilde{u}_\eta\right)^2 \, dt\, dx \\
&= -\int_{\mathbb{R}} \int_0^T \varepsilon \partial_x \left(a_\mu\left(\tilde{u}_\eta\right)\partial_x \tilde{u}_\eta\right)\tilde{u}_\eta \, dt\, dx = -\sum_n \int_{\mathbb{R}} \int_{t_n}^{t_{n+1}} \frac{1}{2}\partial_t \left(\tilde{u}_\eta\right)^2 \, dt\, dx \\
&= -\frac{1}{2}\sum_n \int_{\mathbb{R}} \left(\left(\tilde{u}_\eta\big|_{t=t_{n+1}}\right)^2 - \left(\tilde{u}_\eta\big|_{t=t_n}\right)^2\right) dx \\
&= -\frac{1}{2}\sum_n \int_{\mathbb{R}} \left(\left[\left(u^{n+1}\right)^2 - \left(u^n\right)^2\right] + \left[\left(u^n\right)^2 - \left(u^{n+1/2}\right)^2\right]\right) dx \\
&= -\frac{1}{2}\int_{\mathbb{R}} \left[\left(u^N\right)^2 - \left(u^0\right)^2\right] dx + \frac{1}{2}\sum_n \int_{\mathbb{R}} \left[\left(u^{n+1/2}\right)^2 - \left(u^n\right)^2\right] dx,
\end{aligned}
$$

where we have, without loss of generality, assumed that $\left(a_\mu\left(\tilde{u}_\eta\right)\partial_x \tilde{u}_\eta\right)\tilde{u}_\eta \to 0$ as $|x| \to \infty$.

Since the operators $\mathcal{S}_t$ and $\mathcal{H}_t$ both are L^1 contractive, we know that $\|\tilde{u}_\eta(\cdot,t)\|_1 \le \|\tilde{u}_\eta(\cdot,0)\|_1$. Thus since $\tilde{u}_\eta$ is uniformly bounded and the initial function is integrable, the first term is clearly bounded independent of η,

$$
\left|\frac{1}{2}\int_{\mathbb{R}} \left[\left(u^N\right)^2 - \left(u^0\right)^2\right] dx\right| \le \left\|u^0\right\|_\infty \left\|u^0\right\|_1 = \mathcal{O}(1). \tag{26}
$$

Exploiting the L^1 Lipschitz continuity of $\mathcal{S}_t$ and again that $\tilde{u}_\eta$ is bounded, we obtain for the second term that

$$
\left|\frac{1}{2}\sum_n \int_{\mathbb{R}} \left[\left(u^{n+1/2}\right)^2 - \left(u^n\right)^2\right] dx\right| \le \|u_0\|_\infty \sum_n \|\mathcal{S}_{\Delta t}u^n - u^n\|_1 = \mathcal{O}(1)T. \tag{27}
$$

From (26) and (27) we conclude that the following L^2 bound is valid

$$
\|g_\eta\|_{L^2(Q_T)} \le M(T), \tag{28}
$$

where $M(T)$ is a finite constant independent of η. By virtue of (28) we conclude that $\{g_\eta\}$ is weakly compact in $L^2(Q_T)$. Without loss of generality, we may assume that the entire sequence $\{g_\eta\}$ converges weakly in $L^2(Q_T)$

to a function g. Let G be defined such that $\partial G(u)/\partial u = \sqrt{\varepsilon a(u)}$ and let ϕ be a test function. We can then calculate

$$\begin{aligned}\int_{\mathbb{R}}\int_0^T g(x,t)\phi(x,t)\,dt\,dx &= \lim_{\eta\to 0}\int_{\mathbb{R}}\int_0^T \partial_x G(\tilde{u}_\eta)\phi\,dt\,dx = \lim_{\eta\to 0}\int_{\mathbb{R}}\int_0^T \left(-G(\tilde{u}_\eta)\partial_x\phi\right)\,dt\,dx\\ &= \int_{\mathbb{R}}\int_0^T \left(-G(u)\partial_x\phi\right)\,dt\,dx = \int_{\mathbb{R}}\int_0^T \frac{\partial\hat{G}(u)}{\partial u}\partial_x u\phi\,dt\,dx = \int_{\mathbb{R}}\int_0^T \hat{r}(u)\partial_x u\phi\,dt\,dx, \qquad r(u) = \sqrt{a(u)},\end{aligned}$$

where we have used the chain rule (21). Consequently, we have shown that $\hat{r}(u)\partial_x u$ exists in the sense of distributions in $L^2_{\mathrm{loc}}(Q_T)$. Finally, from the facts $u \in L^\infty(Q_T)$, $\hat{r}(u)\partial_x u \in L^2_{\mathrm{loc}}(Q_T)$ and the second part of (24), we get that $\hat{a}(u)\partial_x u \in L^2_{\mathrm{loc}}(Q_T)$ and hence $\hat{a}(u)\partial_x u \in L^1_{\mathrm{loc}}(Q_T)$, see [27] for details.

Since the limit $u(x,t)$ is assumed to be in $BV(Q_T)$ and $\hat{a}(u)\partial_x u$ is an absolutely continuous measure, the following equality

$$\partial_x\left(\mathrm{sign}(u-k)(A(u)-A(k))\right] = \mathrm{sign}(u-k)\partial_x\left(A(u)-A(k)\right) = \mathrm{sign}(u-k)\hat{a}(u)\partial_x u$$

holds in the sense of measures. This implies that $u(x,t)$ satisfies the regularity condition (23), see [26,27].

Note that the assumption $u(x,t)\in BV(Q_T)$ cannot be avoided for the operator splitting method. The reason is that the estimate given in Lemma 1.3, which is optimal since the hyperbolic operator $\mathcal{S}_t$ in general maps its data into $BV(\mathbb{R})$, is not sufficient to ensure that $\{u_\eta\}$ is in $BV(Q_T)$. In fact, we need L^1 Lipschitz continuity in time to conclude that $\{u_\eta\}$ is in $BV(Q_T)$. We may sum up as follows: A subsequence of the splitting sequence $\{u_\eta\}$ given by (10) converges to an entropy weak solution of (2) thanks to Theorem 1.5. Assuming that the limit u is in $BV(Q_T)$, we have shown that (22) and (23) hold. Consequently, the entire sequence $\{u_\eta\}$ must converge to be the unique BV entropy weak solution of (2).

Finally, the notion of BV entropy weak solutions can also be extended to the initial-value problem [29]. The arguments presented in this paper carry over to this case as well, we refer to the technical report [13] for details.

2. A fully discrete method.

In this section we present a numerical method based on the splitting technique described earlier. We shall use a front tracking method for the convection step (7) and a finite difference scheme for the diffusion step (5).

We start by describing the finite difference scheme. We emphasise that our difference scheme will be based on 'differencing' (5) and not (8). Furthermore, if the equation under consideration is in non-conservative form (2) to begin with, we insist on writing the equation in conservative form, i.e., writing $\partial_x(a(u)\partial_x u) = \partial_x^2 A(u)$ for some A such that $A' = a$. Selecting a mesh size $h > 0$ and a time step $\tau > 0$, the value of our difference approximation at $(x_j, t_m) = (jh, m\tau)$ will be denoted by W_j^m. The difference scheme takes the following form

$$W_j^{m+1} = W_j^m + \mu\left(A\left(W_{j-1}^m\right) - 2A\left(W_j^m\right) + A\left(W_{j+1}^m\right)\right), \qquad m = 0,\dots,N_\tau - 1, \tag{29}$$

where $N_\tau\tau = \Delta t$, $\mu = \varepsilon\frac{\tau}{h^2}$, and the discretization parameters h,τ are chosen so that

$$2\mu\max_u |A'(u)| \le 1. \tag{30}$$

Furthermore, we assume (for simplicity) that h and Δt are related as $\Delta t = ch$ for some constant $c > 0$. The iteration (29) is initiated by setting

$$W_j^0 = \frac{1}{h}\int_{x_j}^{x_{j+1}} w_0(x)\,dx. \tag{31}$$

Let $\mathcal{H}_t^h$ denote the operator which takes an initial function $w_0(x)$ to the approximate solution of (8) obtained by the finite difference scheme (29), i.e.,

$$\mathcal{H}_t^h w_0(x) = \sum_{m,j} W_j^m \chi_j^m(x,t),$$

where χ_j^m denotes the characteristic function of $\langle x_j, x_{j+1}\rangle \times \langle t_m, t_{m+1}\rangle$. Observe that (29) can be written

$$W_j^{m+1} = \mu A'\left(\theta_j^m\right)W_{j-1}^m + \left[1 - \mu A\left(\theta_j^m\right) - \mu A\left(\theta_{j+1}^m\right)\right]W_j^m + \mu A'\left(\theta_{j+1}^m\right)W_{j+1}^m,$$

where θ_j^m is a number between W_{j-1}^m and W_j^m. In view of (30), we get that $\max_j\left|W_j^{m+1}\right| \le \max_j\left|W_j^m\right|$. It follows from this that the difference approximation obeys the maximum principle

$$\|\mathcal{H}_{\Delta t}^h w_0\|_\infty \le \|w_0\|_\infty. \tag{32}$$

Let $Z_j^m = W_{j+1}^m - W_j^m$. Similarly, from the difference equation for Z_j^{m+1} we get that $\sum_j \left|Z_j^{m+1}\right| \leq \sum_j \left|Z_j^m\right|$. Consequently, the approximation is BV stable in the sense that

$$\left|\mathcal{H}_{\Delta t}^h w_0\right|_{BV} \leq |w_0|_{BV}\,. \tag{33}$$

Let $\phi(x)$ be a test function and $\phi^h(x) = \sum_j \phi_j \chi_{\langle x_j, x_{j+1}\rangle}(x)$, where $\phi_j = \phi(x_j)$. Multiplying the difference equation (29) by ϕ_j and subsequently applying summation by parts, we get

$$\begin{aligned}\left|h\sum_j \phi_j\left(W_j^{N_\tau} - W_j^0\right)\right| &= \mu h\left|\sum_m\sum_j\left(\phi_{j+1}-\phi_j\right)\left(A\left(W_{j+1}^m\right) - A\left(W_j^m\right)\right)\right| \\ &\leq N_\tau(\mu h^2)\max_u |A'(u)|\|\phi'\|_\infty \sup_m \sum_j \left|W_{j+1}^m - W_j^m\right| = \mathcal{O}(1)\Delta t\|\phi'\|_\infty.\end{aligned}$$

Using this estimate and that $h = c\Delta t$, we get the following discrete analog of (14)

$$\begin{aligned}\left|\int_{\mathbb{R}}\phi\left(\mathcal{H}_{\Delta t}^h w_0 - w_0\right)\,dx\right| &\leq \left|\int_{\mathbb{R}}\phi^h\left(\mathcal{H}_{\Delta t}^h w_0 - w_0\right)\,dx\right| + \left|\int_{\mathbb{R}}\left(\phi-\phi^h\right)\left(\mathcal{H}_{\Delta t}^h w_0 - w_0\right)\,dx\right| \\ &= \mathcal{O}(1)\Delta t\left(\|\phi\|_\infty + \|\phi'\|_\infty\right).\end{aligned} \tag{34}$$

Next, we want to show that the difference approximation satisfies the following discrete entropy inequality

$$U_k\left(W_j^{m+1}\right) - U_k\left(W_j^m\right) - \mu\left(Q_k\left(W_{j-1}^m\right) - 2Q_k\left(W_j^m\right) + Q_k\left(W_{j+1}^m\right)\right) \leq 0, \tag{35}$$

where $U_k(u) = |u-k|$ and $Q_k(u) = |A(u) - A(k)|$. To this end, we use (29) to rewrite (35) as

$$\begin{aligned}&U_k\left(W_j^m\right) + \mu\left(Q_k\left(W_{j-1}^m\right) - 2Q_k\left(W_j^m\right) + Q_k\left(W_{j+1}^m\right)\right) \\ &\qquad \geq \left|W_j^m + \mu\left(A\left(W_{j-1}^m\right) - 2A\left(W_j^m\right) + A\left(W_{j+1}^m\right)\right) - k\right|.\end{aligned} \tag{36}$$

To show (36), define the function

$$H(u,v,w) = v + \mu\left(A(u) - 2A(v) + A(w)\right),$$

and observe that H is a non-decreasing function of all its arguments due to (30). Introduce the standard notations $a \vee b = \max(a,b)$ and $a \wedge b = \min(a,b)$. A straightforward calculation now yields

$$\begin{aligned}&U_k\left(W_j^m\right) + \mu\left(Q_k\left(W_{j-1}^m\right) - 2Q_k\left(W_j^m\right) + Q_k\left(W_{j+1}^m\right)\right) \\ &\quad = W_j^m \vee k - W_j^m \wedge k + \mu\big[A\left(W_{j-1}^m \vee k\right) - A\left(W_{j-1}^m \wedge k\right) \\ &\qquad\quad - 2A\left(W_j^m \vee k\right) + 2A\left(W_j^m \wedge k\right) + A\left(W_{j+1}^m \vee k\right) - A\left(W_{j+1}^m \wedge k\right)\big] \\ &\quad = H\left(W_{j-1}^m \vee k, W_j^m \vee k, W_{j+1}^m \vee k\right) - H\left(W_{j-1}^m \wedge k, W_j^m \wedge k, W_{j+1}^m \wedge k\right) \\ &\quad \geq H\left(W_{j-1}^m, W_j^m, W_{j+1}^m\right) \vee H(k,k,k) - H\left(W_{j-1}^m, W_j^m, W_{j+1}^m\right) \wedge H(k,k,k) \\ &\quad = \left|H\left(W_{j-1}^m, W_j^m, W_{j+1}^m\right) - k\right|\end{aligned}$$

since $H(k,k,k) = k$. This proves (36). Let $\phi(x,t)$ be a suitable test function and let $\phi_j^m = \phi(x_j, t_m)$. Multiplying (35) by $\phi_j^m h$, summing over all j,m and then applying summation by parts in space and time, we get

$$\begin{aligned}&h\tau\sum_j\sum_m \left|W_j^m - k\right|\left[\frac{\phi_j^m - \phi_j^{m-1}}{\tau}\right] + h\tau\sum_j\sum_m \varepsilon\left|A\left(W_j^m\right) - A(k)\right|\left[\frac{\phi_{j-1}^m - 2\phi_j^m + \phi_{j+1}^m}{h^2}\right] \\ &\quad + h\sum_j\left|W_j^0 - k\right|\phi_j^0 - h\sum_j\left|W_j^{N_\tau} - k\right|\phi_j^{N_\tau} \geq 0.\end{aligned}$$

Letting $w_h(x,t) = \mathcal{H}_t^h w_0(x)$ we can replace this inequality by

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^{\Delta t}\left(|w_h - k|\partial_t\phi + \varepsilon\left|A(w_h) - A(k)\right|\partial_x^2\phi\right)\,dt\,dx \\ &\qquad \int_{\mathbb{R}}|w_h(x,0) - k|\phi(x,0)\,dx - \int_{\mathbb{R}}|w_h(x,\Delta t) - k|\phi(x,\Delta t)\,dx + \mathcal{O}(\Delta t^2) \geq 0.\end{aligned} \tag{37}$$

The solution operator for the hyperbolic part of the equation is replaced by a solution generated by front tracking, which is based on solving Riemann problems and tracking shock collisions. Let $\mathcal{S}_t^\delta$ denote the approximate solution operator associated with (7) at time t, i.e., $\mathcal{S}_t^\delta v_0$ is the result of using front tracking on the piecewise constant function v_0 up to time t. In view of the convergence analysis, recall that the front tracking approximation is an exact solution to a conservation law with piecewise linear flux function f^δ and piecewise constant initial data v_0, which implies that approximate solution operator $\mathcal{S}_t^\delta$ inherits the properties of the exact solution operator $\mathcal{S}_t$. For a detailed description and analysis of this method we refer to [9,16,22].

The fully discrete splitting approximation is now given by the formula

$$u_\eta(x,t) = \left[\mathcal{H}_{\Delta t}^h \circ \pi \circ \mathcal{S}_{\Delta t}^\delta\right]^n u^0, \qquad (x,t) \in \mathbb{R} \times \langle t_{n-1}, t_n\rangle, \qquad n = 1, \dots, N, \tag{38}$$

where π denotes the projection operator given by (31), $u^0 = \pi u_0$ and $\eta = (\Delta t, h, \delta)$. Recall that the operator π does not introduce new extrema nor does it increase the total variation, and it satisfies the error estimate $\|z - \pi z\|_1 = \mathcal{O}(h)$ for any function $z \in BV(\mathbb{R})$. Now having Section 1 and the estimates (32) - (34) in mind, it is not difficult to see that there must exist a finite constant $M > 0$, independent of η, such that

$$\|u_\eta(\cdot,t)\|_\infty \le M, \qquad |u_\eta(\cdot,t)|_{BV} \le M, \qquad \|u_\eta(\cdot,t_2) - u_\eta(\cdot,t_1)\|_1 \le M\sqrt{|t_2 - t_1|}.$$

Thus we have the following convergence result.

Lemma 2.1. *Let $\{\eta = (\Delta t, h, \delta)\}$ be a sequence of discretization parameters tending to zero. Then there exists a subsequence $\{\eta_j\}$ and a bounded measurable function u such that $u_{\eta_j} \to u$ in $L^1_{\text{loc}}(Q_T)$ as $j \to \infty$.*

Proceeding as in the proof of Theorem 1.4, we get that the entropy discrepancy associated with $\{\hat u_\eta\}$, where $\hat u_\eta$ is defined analogously to (15), takes the form

$$\mathcal{L}(\hat u_\eta) \ge E_s^1 + E_s^2 + E_p + E_f,$$

where E_s^1 and E_s^2 are given in (19) and the 'new' error terms E_p (due to the projection step in (38)) and E_f (due to the flux approximation used by front tracking) are given by

$$E_p = \sum_n \int_{\mathbb{R}} \Big(U_k\big(u^{n+1/2}\big) - U_k\big(\pi u^{n+1/2}\big)\Big)\, \phi^n \, dx,$$

$$E_f = \int_{\mathbb{R}} \int_0^T \big(P_k(\hat u) - P_k^\delta(\hat u_\eta)\big)\, \partial_x \phi, dt\, dx,$$

where $u^{n+1/2} = \mathcal{S}_{\Delta t}^\delta u^n$, $\phi^n = \phi(\cdot, t_{n+1/2})$, $t_{n+1/2} = (n+1/2)\Delta t$ and $P_k^\delta(u) = \operatorname{sign}(u-k)\big(f^\delta(u) - f^\delta(k)\big)$.

Let us begin by estimating E_p. We can rewrite E_p as follows

$$\begin{aligned} E_p = &\sum_n \sum_j \int_{x_j}^{x_{j+1}} \Big(U_k\big(u^{n+1/2}\big) - U_k\big(\pi u^{n+1/2}\big)\Big)\, \phi^n(x_j)\, dx \\ &+ \sum_n \sum_j \int_{x_j}^{x_{j+1}} \Big(U_k\big(u^{n+1/2}\big) - U_k\big(\pi u^{n+1/2}\big)\Big)\, \big(\phi^n(x) - \phi^n(x_j)\big)\, dx = E_p^1 + E_p^2. \end{aligned}$$

Let $\{U_{k,m}\}$ be a sequence of smooth entropies converging uniformly to U_k for almost every u as $m \to \infty$. Using the averaging nature of the operator π and the convexity of the entropy $U_{k,m}$, we get

$$\begin{aligned} &\sum_n \sum_j \int_{x_j}^{x_{j+1}} \Big(U_{k,m}\big(u^{n+1/2}\big) - U_{k,m}\big(\pi u^{n+1/2}\big)\Big)\, \phi^n(x_j)\, dx \\ &= \sum_n \sum_j \int_{x_j}^{x_{j+1}} \Big(U_{k,m}'\big(\pi u^{n+1/2}\big)\big(\pi u^{n+1/2} - u^{n+1/2}\big) + \frac{1}{2} U_{k,m}''(u^*)\big(u^{n+1/2} - \pi u^{n+1/2}\big)^2\Big)\, \phi^n(x_j)\, dx \ge 0, \end{aligned}$$

where $u^*(x)$ is a 'number' between $u^{n+1/2}(x)$ and $\pi u^{n+1/2}(x)$. Taking the limit $m \to \infty$, we get $E_p \le 0$. This implies that

$$\mathcal{L}(\hat u_\eta) \ge E_s^1 + E_s^2 + E_p^2 + E_f.$$

Using the Lipschitz continuity of U_k and that $Nh = \mathcal{O}(1)$, we can bound E_p^2 as follows

$$\big|E_p^2\big| = \mathcal{O}(1) h \sum_n \int_{\mathbb{R}} |u^n - \pi u^n|\, dx = \mathcal{O}(h).$$

The terms E_s^1 and E_s^2 have already been estimated in the proof of Theorem 1.5. Assuming that f is locally C^2, we can choose the approximation f^δ such that $\big\|f - f^\delta\big\|_\infty = \mathcal{O}(\delta^2)$ and thus $\big\|P_k - P_k^\delta\big\|_\infty = \mathcal{O}(\delta^2)$, which yields $|E_f| = \mathcal{O}(\delta^2)$. Summing up, we have $\mathcal{L}(\hat u_\eta) \ge -\tilde C\big(\sqrt{\Delta t} + h + \delta^2\big)$ for some finite constant $\tilde C > 0$.

Consequently, the entropy discrepancy associated with the sequence $\{u_\eta\}$ is also of the form

$$\mathcal{L}(u_\eta) \ge -C\big(\sqrt{\Delta t} + h + \delta^2\big). \tag{39}$$

We can therefore conclude that the following theorem is valid.

Theorem 2.2. *Suppose that u_0 is of bounded total variation, $f(u)$ is locally piecewise twice continuously differentiable and that the derivative of $A(u)$ exists locally and is bounded. Let $\{u_\eta\}$ be the fully discrete splitting sequence given by (38). Then the entropy discrepancy $\mathcal{L}(u_\eta)$ is of order $\sqrt{\Delta t}+h+\delta^2$. Furthermore, there exists a subsequence $\{u_{\eta_j}\}$ converging to an entropy weak solution of the convection-diffusion problem*

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 A(u), \qquad u(x,0) = u_0(x), \qquad (x,t) \in Q_T, \qquad A'(u) \geq 0.$$

Remark. In applications, one will often use Strang splitting instead of a Godunov splitting, i.e.,

$$u_\eta(x,t) = \left[\mathcal{H}^h_{\Delta t/2} \mathcal{S}^h_{\Delta t} \mathcal{H}^h_{\Delta t/2}\right]^n u^0, \qquad (x,t) \in \mathbb{R} \times \langle t_{n-1}, t_n\rangle, \qquad n = 1, \ldots, N$$

instead of (38). It is clear that Theorem 2.2 holds for this construction as well. Furthermore, it should be noted that our method is unconditionally stable in the sense that the (splitting) time step Δt is not limited by the space discretization h. Finally, Theorem 2.2 is valid under the rather mild condition that A' is locally bounded. In particular, for the equation (2) this means that $a(u)$ only need to be a locally bounded function.

3. Numerical experiments.

This section reports on numerical experiments with the method (38) for the mixed hyperbolic-parabolic equation (2). The experiments are designed to indicate the convergence properties of the proposed method as well as to demonstrate the difference between the jump condition (24) for mixed type equations and the Rankine-Hugoniot jump condition for hyperbolic equations. In both examples we will consider problems with one discontinuity. Let $x = x(t)$ denote the curve of this discontinuity and $s = x'(t)$ the corresponding shock speed. Then (24) can be written as

$$\left[x'(t)u(x,t) - f(u) + \varepsilon \partial_x A(u)\right]\Big|_{x=x(t)-0}^{x=x(t)+0} = 0.$$

Letting u^l and u^r denote the left and right limits of $u(\cdot,t)$ respectively, the expression for the shock speed s takes the form

$$s = \frac{\left(f(u^r) - f(u^l)\right) - \varepsilon\left((\partial_x A(u))^r - (\partial_x A(u))^l\right)}{u^r - u^l}. \tag{40}$$

We will make use of the following two diffusion functions

$$\begin{aligned} a_1(u) &= \begin{cases} 0, & \text{for } u \in [0, 0.5\rangle, \\ 1, & \text{for } u \in [0.5, 1], \end{cases} \qquad A_1(u) = \int_0^u a_1(\xi)\, d\xi, \\ a_2(u) &= \begin{cases} 0, & \text{for } u \in [0, 0.1\rangle, \\ 1, & \text{for } u \in [0.1, 1], \end{cases} \qquad A_2(u) = \int_0^u a_2(\xi)\, d\xi. \end{aligned} \tag{41}$$

In all the experiments we use Riemann initial data of the form

$$u_0(x) = \begin{cases} 1, & \text{for } x \leq 0, \\ 0, & \text{for } x > 0. \end{cases}$$

In addition to the local time step τ used by the difference scheme (29), the splitting solution (38) has the three discretization parameters; Δt, h and δ. The flux parameter δ is kept fixed (very small) throughout this section. Furthermore, given the space step h, the local time step τ is chosen according to (30). The splitting time step Δt is always related to the space step h through a given CFL number. Hence the only input parameters that we specify in our experiments are the space step h and the CFL number. We denote the approximate solutions by u_h. The approximations associated with a_1 and a_2 are often denoted by u_h^1 and u_h^2 respectively. The solutions u_h are by definition piecewise constant, but in smooth regions they can be replaced by piecewise linear interpolants in order to increase the accuracy. To determine the rate of convergence of our method we measure the error E_T (at a fixed time T) for a decreasing sequence of space steps h. The error is measured in the L^1 norm

$$E_T = \|u_h(\cdot,T) - u_{\text{ref}}(\cdot,T)\|_1,$$

where u_{ref} is a reference solution calculated with (38) using a very fine discretization. When trying to determine the convergence rate we assume that the error is of the form $E_T \approx Ch^\alpha$ for some constant C independent of h. We use least-squares fit to obtain the exponent α.

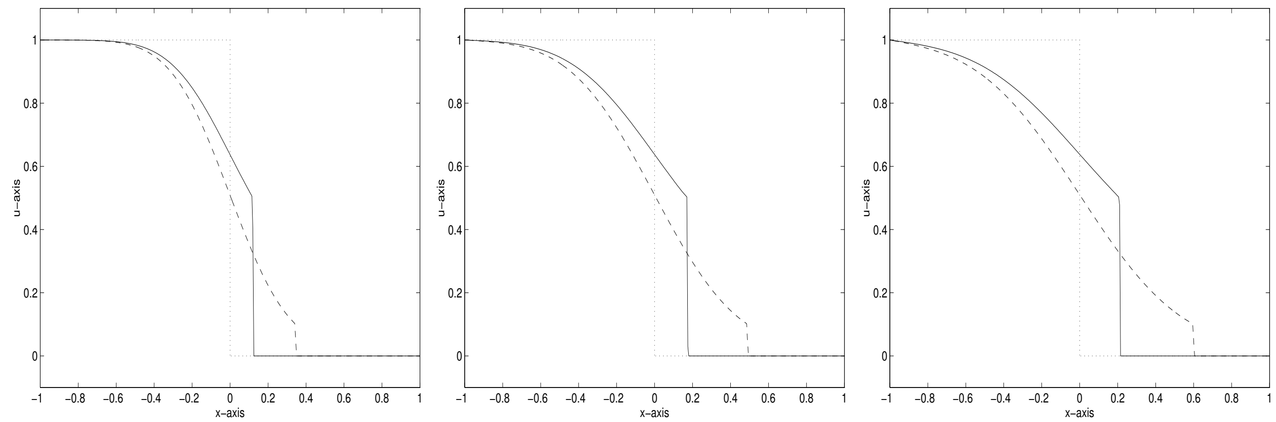

Figure 1 (Example 1). The dotted, solid and dashed lines represent u_0, u_h^1 and u_h^2 respectively at times $T = 0.3$ (left), $T = 0.6$ (middle) and $T = 0.9$ (right) with space step $h = 0.005$ and scaling parameter $\varepsilon = 0.1$.

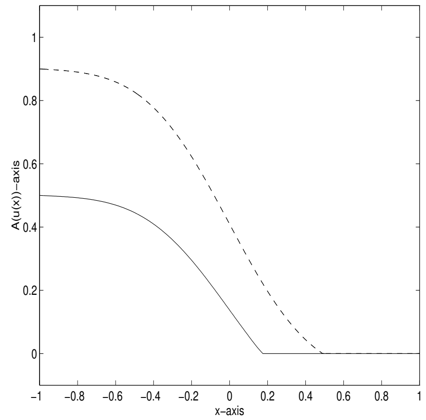

Figure 2 (Example 1). The solid and dashed lines represent the diffusion functions $A_1(u_h^1(x))$ and $A_2(u_h^2(x))$ respectively, where the functions $u_h^1(x)$ and $u_h^2(x)$ are shown in Figure 1 (middle).

Example 1. In this example we focus on the convergence properties of the difference scheme (29) for the purely parabolic equation (5). In particular, we want to verify that the scheme propagates discontinuities according to the jump condition (40). In Figure 1 we have plotted the solutions u_h^1 and u_h^2 at times $T = 0.3, 0.6$ and 0.9 with $\varepsilon = 0.1$. Due to the degeneracy of a_1 in the interval $[0, 0.5]$ the solution u_h^1 is discontinuous in this interval and the discontinuity propagates with a finite speed. Similarly the solution u_h^2 corresponding to a_2 is discontinuous in $[0, 0.1]$. From Figure 1 it is clear that the speed of the discontinuity depends on its size. This can easily be explained from (40) by observing that $u^r = 0$ and $(\partial_x A(u))^r = 0$ (see Figures 1 an 2), which imply that $s = -\varepsilon(\partial_x A(u))^l/u^l$. In Figure 2 we have plotted the diffusion functions $A_1(u_h^1(x))$ and $A_2(u_h^2(x))$ viewed as functions of x. Despite the fact that the solutions u_h are discontinuous, the diffusion functions $A(u_h(x))$ are continuous as functions of x.

In Figure 3 we display the approximate solutions u_h^1 and u_h^2 with $h = 0.05$ and $h = 0.025$ at time $T = 0.3$. In Table 1 we display the L^1 errors for a decreasing sequence of h values at time $T = 0.3$ and the corresponding rates of convergence. These results indicate that the method is first order accurate independent on the size of the discontinuity. Furthermore, we conclude that the approximation is good on a fairly coarse grid.

Table 1 (Example 1)

	u_h^1	u_h^2
$2/h$	L^1 error ($\times 10^2$)	L^1 error ($\times 10^2$)
20	5.39	5.00
40	2.66	2.38
80	1.13	1.12
160	0.50	0.50
L^1 order	1.15	1.11

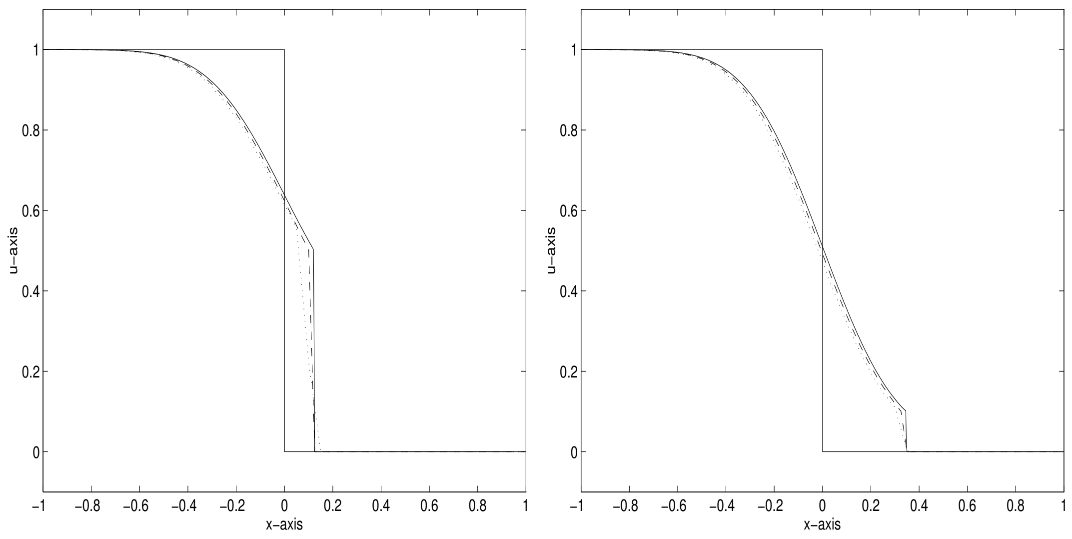

Figure 3 (Example 1). The solid, dashed and dotted lines represent u_{ref}, u_h with $h = 0.025$ (80 nodes) and u_h with $h = 0.05$ (40 nodes) respectively. Left plot: $u_h = u_h^1$. Right plot: $u_h = u_h^2$.

Example 2. We now consider the mixed hyperbolic-parabolic equation (2) with the convective flux $f = u^2 - u$, the diffusive flux $a = a_1$ (see (41)) and scaling parameter $\varepsilon = 0.5$. In Figure 4 (left) we have plotted the solution at times $T = 0.3, 0.6$ and 0.9. We observe that a discontinuity is formed in the interval $[0, 0.5]$ and that it moves with some finite speed to the right. Furthermore, the speed is highest for small times and decreases as time runs. Let us see how this dynamic is reflected in the jump condition (40), which now reads $s = (f(u^l) - \varepsilon(\partial_x A(u))^l)/u^l$. For small times the diffusive transport term $\varepsilon\,(\partial_x A(u))^l$ is the dominating term because u^l is close to 1. But as the discontinuity $(u^l, u^r) = (0.5, 0)$ is formed, the speed is reduced because of the negative term $f(u^l)$.

In Figure 4 (right) we have plotted $A_1(u_h^1(x))$ as a function of x at the same times. As in Example 1 (Figure 2), the function $A_1(u_h^1(x))$ is smooth even though $u_h^1(x)$ is not. Finally, we note that the solution of the corresponding hyperbolic equation (7) coincides with the initial data, thus clearly demonstrating the qualitative difference between a 'hyperbolic' discontinuity and a 'mixed hyperbolic-parabolic' discontinuity.

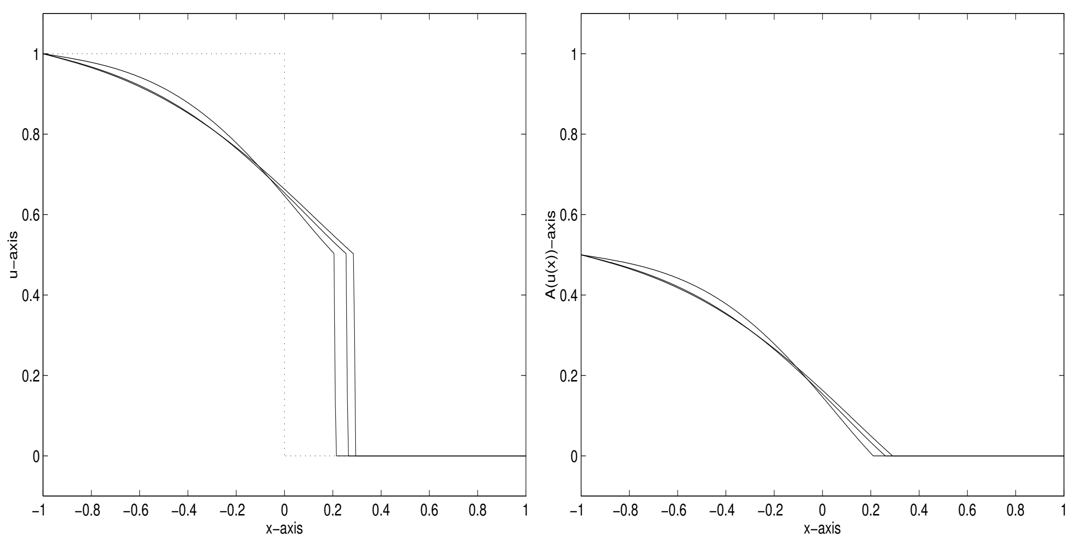

Figure 4 (Example 2). Left plot: The dotted and solid lines represent the initial data u_0 ($\equiv$ solution of hyperbolic part) and the approximation u_h^1 respectively at times $T = 0.3, 0.6, 0.9$ with $h = 0.005$, CFL 3 and $\varepsilon = 0.5$. Right plot: The corresponding diffusion terms $A_1(u_h^1(x))$ plotted as a function of x.

Example 3. In this example we consider (2) with the flux function $f = u^2$ and the scaling parameter $\varepsilon = 0.1$. The approximate solution u_h^1 is displayed in Figure 5 at time $T = 0.5$. In Table 2 we show the L^1 errors and the corresponding convergence rates for three different CFL numbers. In Figure 6 and Table 3 we show the corresponding results for u_h^2. The rate of convergence has been tested for CFL numbers 0.625, 1.25, 2.5, 5 and 10 (only results for CFL numbers 0.625, 2.5 and 10 are shown in Tables 2 and 3). The results indicate that the method is first order accurate (even) for discontinuous solutions, which can be related to the fact that front tracking has a linear convergence rate [22]. We observe that for high CFL numbers the error on a coarse grid is dominated by the temporal splitting error. Compare for instance Figure 3 (left) and Figure 5 (right) with

$h = 0.05$ (40 nodes). The splitting error is reduced by increasing the number of nodes (thus reducing the time step), see Figure 5 (right) with 160 nodes. However, if $\Delta t \gg \varepsilon$ the splitting error will dominate. We also observe that the severity of the splitting error for high CFL numbers depends on the length of the interval of degeneracy. Compare the accuracy of u_h^1 and u_h^2 for CFL 10 (Table 2 and 3). In other words, the splitting error is produced in the parabolic regions. However, this is not a new observation and techniques have recently been developed that substantially reduces this splitting error for non-degenerate equations. We refer to Hvistendahl Karlsen and Risebro [18] for details (see also [19]).

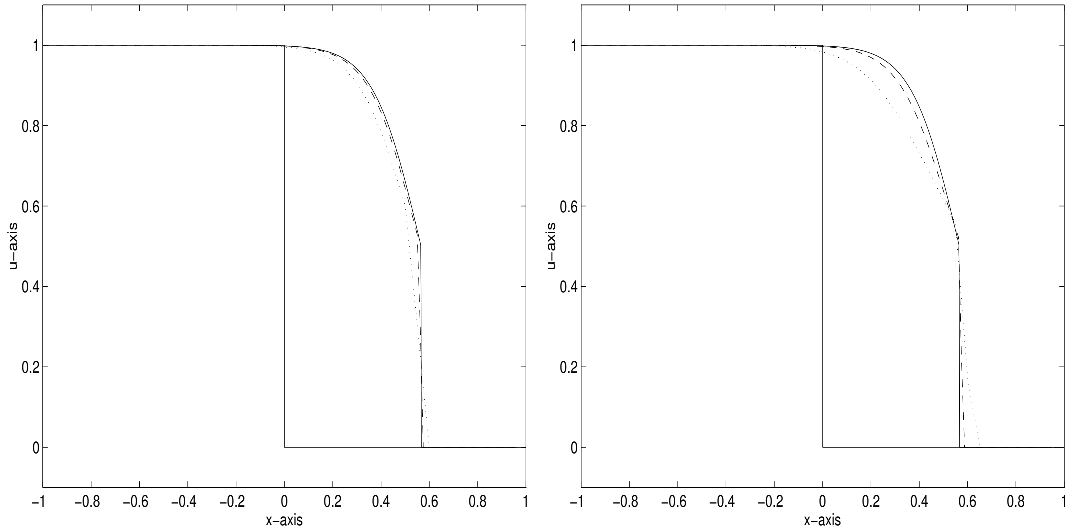

Figure 5 (Example 3). The solid, dashed and dotted lines represent u_{ref}, u_h^1 with $h = 0.0125$ (160 nodes) and u_h^1 with $h = 0.05$ (40 nodes) respectively at time $T = 0.5$ with $\varepsilon = 0.1$. Left plot: calculations done with CFL 1.25. Right plot: calculations done with CFL 10.

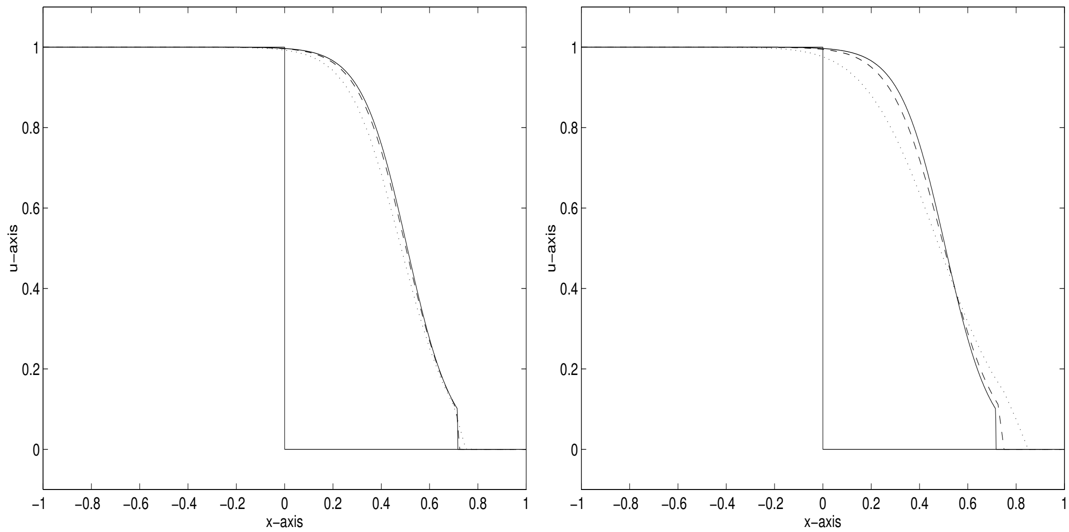

Figure 6 (Example 3). The solid, dashed and dotted lines represent u_{ref}, u_h^2 with $h = 0.0125$ (160 nodes) and u_h^2 with $h = 0.05$ (40 nodes) respectively at time $T = 0.5$ with $\varepsilon = 0.1$. Left plot: calculations done with CFL 1.25. Right plot: calculations done with CFL 10.

Table 2 (Example 3, u_h^1)

CFL	0.625	2.5	10
$2/h$	L^1 error ($\times 10^2$)	L^1 error ($\times 10^2$)	L^1-error ($\times 10^2$)
20	6.37	5.72	10.08
40	3.33	3.15	5.38
80	1.53	1.36	2.91
160	0.77	0.73	1.42
320	0.32	0.27	0.71
L^1 order	1.07	1.09	0.96

Table 3 (Example 3, u_h^2)

CFL	0.625	2.5	10
$2/h$	L^1 error($\times 10^2$)	L^1 error($\times 10^2$)	L^1 error($\times 10^2$)
20	5.70	5.79	12.69
40	2.73	2.82	6.72
80	1.34	1.38	3.32
160	0.62	0.63	1.61
320	0.27	0.28	0.77
L^1 order	1.09	1.09	1.01

4. Concluding remarks.

A numerical method for one-dimensional mixed hyperbolic-parabolic convection-diffusion equations has been formulated and analysed in this work. The primary goal has been to demonstrate both mathematically and numerically that the method can be used to compute the (physically correct) solutions of a very broad class of equations, including hyperbolic conservation laws, parabolic equations as well as any mixed hyperbolic-parabolic equation. Suitable stability results for this method and its convergence to an entropy weak solution have been established. The numerical results have indicated that the method is first order accurate (also with discontinuities present), nonoscillatory, shock-capturing, and unconditionally stable in the sense that splitting time step is not restricted by the spatial discretization parameter. The regularity of $A(u)$ has been established by analysis as well as demonstrated numerically (see Figures 2 and 4). We have shown that $\partial_x A(u)$ is in $L^2(Q_T)$, however, Figures 2 and 4 indicate that a sharper smoothness estimate on $\partial_x A(u)$ should be possible to obtain. This observation also explain why *we* choose to discretize $\partial_x A(u)$ and not $a(u)\partial_x u$. In a forthcoming work we analyse finite difference approximations of mixed type problems of the type (2). In particular, by means of these approximations we prove rigorously that $A(u)$ is smooth. Finally, we plan to extend our method to multi-dimensional equations with space and time dependent coefficients as well as start using implicit finite difference solvers.

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