The Zero Relaxation Limit for 2×2 Hyperbolic Systems

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Abstract

We study the zero relaxation limit for a class of 2×2 strictly hyperbolic systems of balance laws. In particular we show the strong convergence toward the solution of the formal limit of the system and the validity of an infinite number of Kružkov-type entropy inequalities. Moreover, we give a uniqueness result for this solution.

Keywords: Hyperbolic systems, conservation laws, relaxation problems.

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1 Introduction

In this paper we are concerned with the study of the relaxation of the following system of balance laws

$$\begin{cases} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x = \frac{1}{\varepsilon}(\Phi(u) - v), \end{cases} \tag{1.1}$$

with $u_0(x)$, $v_0(x) \in L^\infty$ and f, g and Φ smooth functions (for instance, f, g and $\Phi \in C^5$). We assume that the system is strictly hyperbolic, namely

$$\Lambda_-(u,v) < \Lambda_+(u,v),$$

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where $\Lambda_-(u,v)$, $\Lambda_+(u,v)$ are the two characteristic speeds of (1.1). We want to prove the convergence of the weak solutions of (1.1) toward the solutions of the scalar conservation law

$$u_t + f(u, \Phi(u))_x = 0. \tag{1.2}$$

This kind of problems has a relevant importance, since they arise in many mathematical models of physical phenomena, such as the kinetic theories of gases, the chemically reacting flows and the theory of linear and nonlinear waves. They have been extensively studied in the engineering literature and, more recently, mathematical problems of this type have been addressed by Whitham in his book [15]. Later there have been several rigorous mathematical results on this subject, in particular we mention Liu [9] and [1] (together with Chen and Levermore). In [1] it has been proved the convergence of the solution of the full system toward a weak solution of the relaxed conservation law. A partial result of convergence and some connections between the stability condition and the existence of invariant regions have been established in [10]. Natalini in [12] considered the semilinear case and prove the validity of the Kružkov entropy conditions for the limit function. With a different approach, this problem was investigated also in [4], where applications to models arising in chromatography can be found. The authors in [8] prove a similar result in the case of the Whitham hydrodynamic model of the traffic flow.

In the second section of this paper we investigate the relations between the subcharacteristic condition and the existence of invariant regions for the full relaxing system (1.1).

In the section 3 we prove the basic energy estimates which allow us to control the rate of convergence of the solutions toward the equilibrium as $\varepsilon \downarrow 0$, estimates which are crucial to get the convergence result. This result is obtained by a careful analysis of the linear strictly hyperbolic entropy equation of the system under consideration. Thanks to the energy estimates for this equation, it is then possible to find proper extensions of strictly convex functions which allow us to put into evidence the dissipative nature of the relaxation limit.

In the next section we deal with the study of the convergence of the solutions of (1.1) toward the solutions of (1.2). The arguments used in this section follows the ideas of [1] and they are very close to those one we used in [8].

In the section 5 we deal with the prove of an infinite number of Kružkov-type entropy inequalities. The main difficulty consists in finding some "good" extensions of Kružkov-type entropies $g^k(u)$ of (1.2), namely, for any k, strictly convex entropies $\eta^k(u,v)$, verifying $\eta^k_{vv}(u,v) > 0$, in a domain containing the equilibrium curve. To get this properties, we make use of the results given in the section 3 concerning the linear analysis of the entropy equation.

The last section is devoted to prove the uniqueness of our limit solution. It is well known (see, for instance, [7, 6, 14]) that the Kružkov entropy inequalities are not enough to show the uniqueness of weak solutions to the Cauchy problem for a scalar conservation law; indeed we need to know some kind of L^1 continuity of the solution at time $t = 0$. In our situation, the presence of an initial layer generates an extra positive entropy production at $t = 0$, which makes nontrivial the proof of this property. As in [8], we will prove the uniqueness result when the initial data are in equilibrium, namely when no initial layer occurs.

2 A priori bounds

In this section we give some L^∞ estimates for the weak entropy solutions of the full relaxing system (1.1) by showing the existence of an uniform invariant region for this system, under a certain condition, namely the stability condition (see, for instance, [1])

$$\textbf{(SC)'}\quad \Lambda_-(u,\Phi(u)) \le \frac{d}{du} f(u,\Phi(u)) \le \Lambda_+(u,\Phi(u)).$$

Since this paper is concerned with the relaxation limit of (1.1), we are not interested in the problem of the existence of solutions for this system when the relaxation parameter ε is a fixed, positive quantity. However, there are some cases where this type of results are straightforward, as in the GNL case, by following the ideas of DiPerna [5] and Serre [13] for the homogeneous systems. The aim of this section is to construct an invariant region independent on ε, in order to use the uniform L^∞ estimates, which follows from this fact, to study the relaxation limit. Indeed, if the homogeneous system

$$\begin{cases} u_t + f(u,v)_x = 0 \\ v_t + g(u,v)_x = 0 \end{cases}$$

has a family of invariant regions Σ_i (see [3]), then it is possible to find a region Σ_{i_0} which is invariant for the full system (1.1). From the geometric point of view, the condition **(SC)'** yields the equilibrium curve $\{v = \Phi(u)\}$ to stay in between the level curves of the Riemann invariants (see figure 1).

Proposition 2.1 *Let $w = w(u,v)$ and $z = z(u,v)$ be the Riemann invariants of the system (1.1). For any u_1, $u_2 \in \mathbb{R}$, $u_1 < u_2$, let $w = w_i$, $z = z_i$ be respectively the level curves of w and z through the point $(u_i, \Phi(u_i))$, $i = 1, 2$. If the subcharacteristic condition **(SC)'** holds and if $f_v(u,v) \neq 0$, then for any $u \in [u_1, u_2]$, the equilibrium curve $\{v = \Phi(u)\}$ lies in the domain enclosed by the level curves $w(u,v) = w_i$ and $z(u,v) = z_i$.*

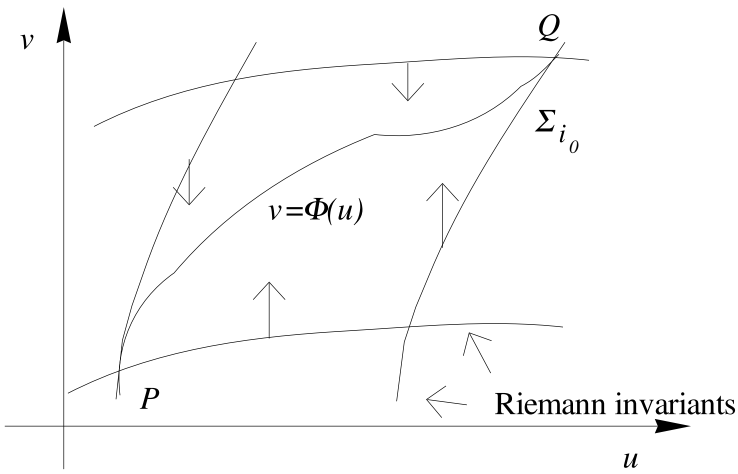

Figure 1: Σ_{i_0} invariant region

Proof. We observe that the tangent vectors of the curves $w = w_i$ are parallel to the vector $\left(1, \dfrac{\Lambda_- - f_u}{f_v}\right)$ and the tangent vectors of the curves $z = z_i$ are parallel to the vector $\left(1, \dfrac{\Lambda_+ - f_u}{f_v}\right)$. The subcharacteristic condition **(SC)'** at any point $(u, \Phi(u))$ provides

$$\frac{\Lambda_-(u, \Phi(u)) - f_u(u, \Phi(u))}{f_v(u, \Phi(u))} \leq \Phi'(u) \leq \frac{\Lambda_+(u, \Phi(u)) - f_u(u, \Phi(u))}{f_v(u, \Phi(u))}. \tag{2.1}$$

Hence, if P and Q are the corner points given in figure 1, we notice that the slope of the tangent vectors to the curve $v = \Phi(u)$ in these two points is between that one of the tangent vectors of the boundary curves. Then, because of the inequality (2.1), the curve can not cross the boundary of the domain Σ_{i_0}. Indeed, even in the extremal situation of a tangential crossing of the boundary, as before, the inequality (2.1) is immediately violated as soon as the equilibrium curve moves outside. ∎

Once we have proved this result concerning the existence of invariant regions for (1.1), we can move to the main problem of our paper, namely the study of the relaxation limit. For this purpose, we will consider the following framework:

(H1) there exists a sequence $(u^\varepsilon, v^\varepsilon)$ of weak entropy solutions of (1.1) bounded in L^∞, uniformly with respect to ε.

3 Energy-type estimates

The aim of this section is to prove an energy estimate which controls the rate of convergence of the solution $(u^\varepsilon, v^\varepsilon)$ of (1.1) toward the equilibrium solution given by the pair $(u, \Phi(u))$, where $u^\varepsilon \longrightarrow u$ in a certain topology. As in [8], we will get this result from the entropy inequality for the full system (1.1). To do this, we study the entropy equation of this system, namely the following strictly hyperbolic equation

$$f_v(u,v)\eta_{uu} + (g_v(u,v) - f_u(u,v))\eta_{uv} - g_u(u,v)\eta_{vv} = 0. \tag{3.1}$$

With the next result we are able to construct a neighborhood of the equilibrium curve $\{(u,v) : v = \Phi(u)\}$, where any C_0^∞ function, which is strictly convex in a given interval, can be extended to be a strictly convex and dissipative entropy for the relaxing system (1.1). We will apply this result in the following sections to the case of C_0^∞ initial data of the Kružkov form.

Proposition 3.1 *Let $g^0(u)$ be a C_0^∞ function which is strictly convex on $[-2M, 2M]$, for a given $M > 0$. Suppose that f, g and Φ are sufficiently regular (for instance, f, g, $\Phi \in C^5$) and suppose the following (stronger) subcharacteristic condition holds:*

(SC) $\Lambda_-(u,\Phi(u)) < \dfrac{d}{du} f(u,\Phi(u)) < \Lambda_+(u,\Phi(u)).$

Then, in the strip $\{(u,v) : u \in [-2M, 2M]\}$, there exists a neighborhood of the equilibrium curve $v = \Phi(u)$ where the solution of the Cauchy problem

$$\begin{cases} f_v\eta^0_{uu} + (g_v - f_u)\eta^0_{uv} - g_u\eta^0_{vv} = 0 \\ \eta^0(u,\Phi(u)) = g^0(u) \\ \eta^0_v(u,\Phi(u)) = 0 \end{cases} \tag{3.2}$$

is strictly convex and verifies $\eta^0_{vv} \geq c > 0$.

Proof. Let us consider the following change of variables:

$$z = v - \Phi(u).$$

In the new set of variables (u,z), the Cauchy problem (3.2) becomes

$$\begin{cases} \alpha(u,z)\eta^0_{uu} + \beta(u,z)\eta^0_{uz} + \gamma(u,z)\eta^0_{zz} = \Phi''(u)\eta^0_z \\ \eta^0(u,0) = g^0(u) \\ \eta^0_z(u,0) = 0, \end{cases} \tag{3.3}$$

where $\alpha(u,z) = f_v(u, z+\Phi(u))$, $\beta(u,z) = g_v(u,z+\Phi(u)) - f_u(u,z+\Phi(u)) - 2f_v(u,z+\Phi(u))\Phi'(u)$ and $\gamma(u,z) = (\Phi'(u))^2 f_v(u,z+\Phi(u)) - \Phi'(u)(g_v(u,z+\Phi(u)) - f_u(u,z+\Phi(u))) - g_u(u,z+\Phi(u))$. The entropy equation is strictly hyperbolic also in this new formulation; moreover since the stability condition **(SC)** implies $\gamma(u,0) \neq 0$ for any u, then there exists a positive constant $\bar{c}$ such that

$$\gamma(u,z) \neq 0, \text{ for } u \in [-2M, 2M], \ |z| \leq \bar{c}.$$

Therefore the entropy equation can be rewritten in an equivalent form as a strictly hyperbolic system, for any $u \in [-2M, 2M]$, $|z| \leq \bar{c}$:

$$\mathcal{H}_z + A(u,z)\mathcal{H}_u = B(u,z)\mathcal{H}, \tag{3.4}$$

where

$$A(u,z) = \begin{pmatrix} 0 & -1 \\ \frac{\alpha(u,z)}{\gamma(u,z)} & \frac{\beta(u,z)}{\gamma(u,z)} \end{pmatrix}; \quad B(u,z) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\Phi''(u)}{\gamma(u,z)} \end{pmatrix}; \quad \mathcal{H} = \begin{pmatrix} \eta_u^0 \\ \eta_z^0 \end{pmatrix}.$$

Due to the hyperbolicity of the entropy equation in the initial problem (3.2) (or, equivalently, of the system (3.4)), the symmetric matrix

$$T(u,z) = \begin{pmatrix} \frac{\beta^2(u,z)}{\gamma^2(u,z)} - 2\frac{\alpha(u,z)}{\gamma(u,z)} & \frac{\beta(u,z)}{\gamma(u,z)} \\ \frac{\beta(u,z)}{\gamma(u,z)} & 2 \end{pmatrix}$$

is positive definite and moreover the matrix

$$S(u,z) = T(u,z)A(u,z) = \begin{pmatrix} \frac{\beta(u,z)\alpha(u,z)}{\gamma^2(u,z)} & 2\frac{\alpha(u,z)}{\gamma(u,z)} \\ 2\frac{\alpha(u,z)}{\gamma(u,z)} & \frac{\beta(u,z)}{\gamma(u,z)} \end{pmatrix}$$

is symmetric. Therefore the system (3.4) can be rewritten as follows

$$T(u,z)\mathcal{H}_z + S(u,z)\mathcal{H}_u = \widetilde{B}(u,z)\mathcal{H}, \tag{3.5}$$

where

$$\widetilde{B}(u,z) = T(u,z)B(u,z) = \begin{pmatrix} 0 & \frac{\Phi''(u)\beta(u,z)}{\gamma^2(u,z)} \\ 0 & 2\frac{\Phi''(u)}{\gamma(u,z)} \end{pmatrix}.$$

Hence, multiplying (3.5) by $\mathcal{H}$, one has

$$\begin{aligned} D_z\langle \mathcal{H}, T(u,z)\mathcal{H}\rangle + D_u\langle \mathcal{H}, S(u,z)\mathcal{H}\rangle \quad &= \quad \langle \mathcal{H}, (2\widetilde{B}(u,z) + T_z(u,z) \\ &\qquad + S_u(u,z))\mathcal{H}\rangle. \end{aligned}$$

Thus, by integrating in u this relation, we get

$$\frac{d}{dz}\|K(u,z)\mathcal{H}\|^2_{L^2(\mathbb{R})} = \int \langle \mathcal{H}, (2\widetilde{B}(u,z) + T_z(u,z) + S_u(u,z))\mathcal{H}\rangle du, \tag{3.6}$$

where $K(u,z)$ is the positive, symmetric matrix such that $T(u,z) = K^2(u,z)$. Therefore, for any $z \in [0,Z]$, since g^0 has compact support, $\mathcal{H} = 0$ for any u away from a compact set $D \subset \mathbb{R}$. So, in a straightforward way, from (3.6) one deduces that there exists a constant $C > 0$, depending on Z and on the C^2 norm of f, g and Φ in $D \times [0,Z]$, such that

$$\|\mathcal{H}\|_{L^2([0,Z]\times\mathbb{R})} \leq C\|\mathcal{H}(0)\|_{L^2(\mathbb{R})},$$

that is, the H^1 bound for η^0, depending from the H^1 norm of g^0. Differentiating the system (3.5) with respect to u and taking into account the original equation (3.3), one can prove that there exists a constant $C > 0$, depending on Z and on the C^5 norm of f, g and Φ in $D \times [0,Z]$, such that

$$\|\eta^0\|_{H^5([0,Z]\times\mathbb{R})} \leq C\|g^0\|_{H^5(\mathbb{R})}. \tag{3.7}$$

Then, by (3.7) and by the Sobolev embedding theorem, it follows

$$\|\eta^0\|_{C^3([0,Z]\times\mathbb{R})} \leq C\|g^0\|_{H^5(\mathbb{R})}, \tag{3.8}$$

for a positive constant C. Let us denote with H the hessian matrix of $\eta^0(u,v)$, rewritten in the (u,z) variables. A straightforward calculation gives, for any $u \in [-2M, 2M]$

$$\det H(u,0) = -\frac{\alpha(u,0)}{\gamma(u,0)}(g^0(u)'')^2 \geq \delta_1 > 0$$

and

$$\operatorname{tr} H(u,0) = \left(1 - (1 + (\Phi'(u))^2)\frac{\alpha(u,0)}{\gamma(u,0)}\right) g^0(u)'' \geq \delta_2 > 0,$$

since $g^0(u)'' \geq \delta > 0$ for $u \in [-2M, 2M]$ and $\alpha(u,0)\gamma(u,0) > 0$ for the subcharacteristic condition **(SC)**. Now, let us denote $G = \|g^0\|_{H^5(\mathbb{R})}$. From the energy estimate (3.8), it follows

$$\begin{aligned} \det H(u,z) &= \det H(u,0) + D_z[\det H(u,\xi)]z \\ &\geq \delta_1 - CG^2|z| \\ &> 0, \end{aligned}$$

for any $u \in [-2M, 2M]$ and $|z| < min\left\{\bar{c}, \dfrac{\delta_1}{CG^2}\right\}$ and

$$\begin{aligned} \operatorname{tr} H(u,z) &= \operatorname{tr} H(u,0) + D_z[\operatorname{tr} H(u,\xi)]z \\ &\geq \delta_2 - CG|z| \\ &> 0, \end{aligned}$$

for any $u \in [-2M, 2M]$ and $|z| < min\left\{\bar{c}, \frac{\delta_2}{CG}\right\}$. Therefore, $\eta^0(u,v)$ is strictly convex for any $u \in [-2M, 2M]$ and

$$|v - \Phi(u)| < min\left\{\bar{c}, \frac{\delta_1}{CG^2}, \frac{\delta_2}{CG}\right\}.$$

Moreover, denote by $F(u,z)$ the function η^0_{vv} in the variables (u,z), then one has

$$F(u,0) = -\frac{\alpha(u,0)}{\gamma(u,0)} g^0(u)'' \geq \delta_3 > 0$$

and hence

$$\begin{aligned} F(u,z) &= F(u,0) + F_z(u,\xi)z \\ &\geq \delta_3 - CG|z| \\ &> c, \end{aligned}$$

for any $u \in [-2M, 2M]$ and $|z| < min\left\{\bar{c}, \frac{\delta_3 - c}{CG}\right\}$. Thus, $\eta^0(u,v)$ is a strictly convex and strictly dissipative (i.e. $\eta^0_{vv}(u,v) > c > 0$) function, for any $u \in [-2M, 2M]$ and

$$|v - \Phi(u)| < min\left\{\bar{c}, \frac{\delta_1}{CG^2}, \frac{\delta_2}{CG}, \frac{\delta_3 - c}{CG}\right\}.$$

∎

The Proposition 3.1 implies in particular the existence of a compact set $\mathcal{K}$ defined by $\mathcal{K} = \{(u,v) : u \in [-M, M],\ |v - \Phi(u)| \leq c_0,\ c_0 > 0\}$ where η^0 is strictly convex and verifies $\eta^0_{vv} > c > 0$.

Now we are interested to apply this result to the case of the weak solutions of (1.1), with $M = \|u^\varepsilon\|_{L^\infty}$. The following theorem yields an estimate which allows to control the relaxation process.

Theorem 3.2 *Assume the hypotheses **(H1)** and*

(H2) $(u^\varepsilon, v^\varepsilon) \in \mathcal{K}$

*hold and assume the subcharacteristic condition **(SC)** given in proposition 3.1. Finally, assume f, g and $\Phi \in C^5$ and assume that (u_0, v_0) verify*

(H3) $\displaystyle\int_{-\infty}^{+\infty} \eta^0(u_0(x), v_0(x))dx < +\infty.$

Then

$$\|v - \Phi(u)\|_{L^2([0,T]\times\mathbb{R})} = O(1)\sqrt{\varepsilon}. \tag{3.9}$$

Proof. The entropy inequality for η^0 gives:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \eta^0(u(x,T), v(x,T))dx - \int_{-\infty}^{+\infty} \eta^0(u_0(x), v_0(x))dx \\
&\leq \int_0^T \int_{-\infty}^{+\infty} \eta^0_v(u(x,t), v(x,t)) \frac{\Phi(u) - v}{\varepsilon} dxdt \\
&= -\int_0^T \int_{-\infty}^{+\infty} \eta^0_{vv}(u(x,t), \bar{v}) \frac{(\Phi(u) - v)^2}{\varepsilon} dxdt \\
&\leq -\frac{c}{\varepsilon} \int_0^T \int_{-\infty}^{+\infty} (\Phi(u) - v)^2 \leq 0,
\end{aligned}$$

where $\bar{v} \in [0, v]$. Therefore, since η^0 is nonnegative,

$$\frac{1}{\varepsilon} \int_0^T \int_{-\infty}^{+\infty} (\Phi(u) - v)^2 \leq \frac{1}{c} \int_{-\infty}^{+\infty} \eta^0(u_0(x), v_0(x))dx \leq C.$$

∎

4 Convergence

In this section, we study the convergence of the solution $(u^\varepsilon, v^\varepsilon)$ of the full system (1.1) toward the formal limit of this system, namely the scalar conservation law

$$u_t + f(u, \Phi(u))_x = 0. \tag{4.1}$$

In particular, as in [1, 8], we will prove the strong convergence of u^ε to a weak solution of (4.1) in L^p_{loc}, for any $p < +\infty$, for those solutions verifying the hypothesis **(H2)** of theorem 3.2, namely, when the basic energy estimate (3.9) holds.

Theorem 4.1 *Assume the hypothesis **(H1)**, the hypotheses **(H2)**, **(H3)** of theorem 3.2 and assume f, g and $\Phi \in C^5$. Finally, assume the sub-characteristic condition **(SC)** of proposition 3.1 holds. Then, extracting if necessary a subsequence, $u^\varepsilon \longrightarrow u$ as $\varepsilon \downarrow 0$ strongly in L^p_{loc}, for any $p < +\infty$. Moreover, $u(x,t)$ is a weak solution of the Cauchy problem*

$$\begin{cases} u_t + f(u, \Phi(u))_x = 0 \\ u(x, 0) = u_0(x). \end{cases}$$

Proof. As in [8], thanks to the compensated compactness results (see, for instance, [2]), it suffices to prove

$$u_t + f(u, \Phi(u))_x \in \text{comp}\, H^{-1}_{loc}$$

and

$$f(u,\Phi(u))_t + \left(\int^u \left(\frac{d}{ds} f(s,\Phi(s))\right)^2 ds\right)_x \in \operatorname{comp} H^{-1}_{loc}.$$

From the estimate (3.9), it is clear that

$$u_t + f(u,\Phi(u))_x = (f(u,\Phi(u)) - f(u,v))_x \in \operatorname{comp} H^{-1}_{loc}.$$

Moreover, if we denote with $\psi^0(u)$ the entropy-flux associated to the entropy $g^0(u)$ of (4.1), we have:

$$\begin{aligned} g^0(u)_t + \psi^0(u)_x &\leq (\eta^0(u,\Phi(u)) - \eta^0(u,v))_t \\ &\quad + (q^0(u,\Phi(u)) - q^0(u,v))_x \\ &\quad + \frac{1}{\varepsilon}(\eta^0_v(u,v) - \eta^0_v(u,\Phi(u)))(\Phi(u) - v) \\ &= I_1^\varepsilon + I_2^\varepsilon + I_3^\varepsilon, \end{aligned}$$

where $q^0(u,v)$ is the entropy-flux associated with the entropy $\eta^0(u,v)$ of the full system (1.1). Now, in a standard way, (see [1, 8]), the L^∞ bounds and the energy estimate (3.9) imply that I_1^ε and I_2^ε tend to zero in H^{-1}_{loc}, as $\varepsilon \downarrow 0$ and I_3^ε is bounded in L^1, uniformly with respect to ε. Thus, the Murat lemma [11] yields

$$g^0(u)_t + \psi^0(u)_x \in \operatorname{comp} H^{-1}_{loc}. \tag{4.2}$$

Let us denote with $\widetilde{\eta}(u,v)$ the entropy of the system (1.1) which verifies

$$\begin{cases} \widetilde{\eta}(u,\Phi(u)) = f(u,\Phi(u)) \\ \widetilde{\eta}_v(u,\Phi(u)) = 0 \end{cases}$$

and let $\widehat{\eta}(u,v) = \eta^0(u,v) + \mu\widetilde{\eta}(u,v)$ for a positive constant μ. From the condition **(H2)**, it follows that for a sufficiently small constant $\mu \neq 0$, $\widehat{\eta}(u,v)$ is strictly convex. Moreover, $\widehat{\eta}_v(u,\Phi(u)) = 0$. Thus, proceeding as before, one has:

$$(g^0(u) + \mu\Phi(u))_t + \left(\psi^0(u) + \mu \int^u \left(\frac{d}{ds} f(s,\Phi(s))\right)^2 ds\right)_x \in \operatorname{comp} H^{-1}_{loc}. \tag{4.3}$$

Therefore, from (4.2) and (4.3), it follows

$$f(u,\Phi(u))_t + \left(\int^u \left(\frac{d}{ds} f(s,\Phi(s))\right)^2 ds\right)_x \in \operatorname{comp} H^{-1}_{loc}.$$

Then the convergence result is straightforward (see [2]). ■

5 Entropy conditions

In this section we want to prove the validity of an infinite number of Kružkov-type entropy inequalities for the solution $u(x,t)$ of the relaxed conservation law (4.1), given by the theorem 4.1. The main difficulty is to find appropriate extensions of a family $g^k(u)$ of Kružkov-type entropies of (4.1). In particular, we need to show that, in an suitable domain containing the equilibrium curve, for any k, the extensions $\eta^k(u,v)$ are strictly convex and strictly dissipative, namely $\eta^k_{vv}(u,v) > 0$. We state this property in the following definition.

Definition 5.1 Let φ^k, $k \in [a,b]$ be a sequence of C_0^∞ functions and let η^k the entropy of the system (1.1) which verifies

$$\begin{cases} \eta^k(u,\Phi(u)) = \varphi^k(u) \\ \eta^k_v(u,\Phi(u)) = 0. \end{cases}$$

We say that a sequence (u,v) of exact (or approximate) solutions of (1.1) is φ^k*-stable* if $(u,v) \in \Omega$, where Ω is a closed neighborhood (with nonempty interior) of the equilibrium curve $\{(u,v) : v = \Phi(u)\}$ where $\eta^k(u,v)$ is strictly convex and verifies $\eta^k_{vv}(u,v) > 0$ for any $k \in [a,b]$.

The next remark implies that the definition above is well-posed, when the functions φ^k are chosen to be C_0^∞ functions of the Kružkov type.

Remark 5.2 Let $g^k(u)$ be a C_0^∞ function of the form $|u-k|$ and let $\eta^k(u,v)$ be the corresponding "extended" entropy given by the following Cauchy problem:

$$\begin{cases} f_v\eta^k_{uu} + (g_v - f_u)\eta^k_{uv} - g_u\eta^k_{vv} = 0 \\ \eta^k(u,\Phi(u)) = g^k(u) \\ \eta^0_v(u,\Phi(u)) = 0. \end{cases}$$

If we choose the function $g^0(u)$ of proposition 3.1 to be a C_0^∞ function of the form $|u|$, then $g^k(u)$ is a translation of $g^0(u)$. Therefore, all the entropies $\eta^k(u,v)$, $k \in [-M,M]$, are strictly convex and strictly dissipative in the compact neighborhood $\mathcal{K}$ of the equilibrium curve $\{v = \Phi(u)\}$, constructed in the proposition 3.1. Thus, the condition **(H2)** of theorem 3.2 allows to get the entropy inequality for $g^k(u)$ from the entropy inequality of $\eta^k(u,v)$, for any $k \in [-M,M]$. ∎

Theorem 5.3 *Let $g^k(u)$ be a C_0^∞ approximation of $|u-k|$ and let $(u^\varepsilon, v^\varepsilon)$ be a g^k-stable sequence of solutions of (1.1). Assume f, g and $\Phi \in C^5$ and assume the hypothesis **(H1)** and the hypothesis **(H3)** of theorem 3.2* *hold. Finally, assume the subcharacteristic condition* ***(SC)*** *of proposition 3.1 holds. Then, for any $k \in [-M, M]$, the limit function $u(x,t)$ verifies*

$$g^k(u)_t + \psi^k(u)_x \leq 0 \quad \text{in } \mathcal{D}', \tag{5.1}$$

where $\psi^k(u) = \int^u \frac{d}{ds} f(s, \Phi(s)) \frac{d}{ds} g^k(s) ds.$

Proof. As in the proof of theorem 4.1, one has

$$\begin{aligned} g^k(u)_t + \psi^k(u)_x &\leq (\eta^k(u, \Phi(u)) - \eta^k(u,v))_t \\ &\quad + (q^k(u, \Phi(u)) - q^k(u,v))_x \\ &\quad + \frac{1}{\varepsilon}(\eta^k_v(u,v) - \eta^k_v(u, \Phi(u)))(\Phi(u) - v) \\ &= I^\varepsilon_{1,k} + I^\varepsilon_{2,k} + I^\varepsilon_{3,k}, \end{aligned}$$

where $q^k(u,v)$ is the entropy-flux associated with the entropy $\eta^k(u,v)$ of (1.1). Now, as before, $I^\varepsilon_{1,k}$ and $I^\varepsilon_{2,k}$ tend to zero in H^{-1}_{loc}, as $\varepsilon \downarrow 0$ and

$$I_{3,k} = -\frac{1}{\varepsilon}\eta^k_{vv}(u, \bar{v})(\Phi(u) - v)^2 \leq 0,$$

where $\bar{v} \in [0, v]$, which concludes the proof. ∎

6 Uniqueness

In this section we deal with the study of the uniqueness of the limit function $u(x,t)$. Hence, in order to apply the Kružkov result [7], we need the following hypothesis:

(H4) the limit function $u(x,t)$ verifies

$$|u - k|_t + F^k(u)_x \leq 0 \quad \text{in } \mathcal{D}',$$

for $k \in [-M, M]$ and $F^k(u) = \int^u \operatorname{sign}(s-k) \frac{d}{ds} f(s, \Phi(s)) ds.$

Moreover, as it was clearly pointed out in [7, 6, 14], these relations are not enough to conclude the uniqueness result: one needs also a generalized L^1_{loc} continuity of the solution at $t = 0$. The crucial part in the proof of this relation is to show a contraction property in a suitable metric space. The main difficulty in our case is the presence of an initial layer, which is created since in general the initial data are not in equilibrium, namely $v_0 \neq \Phi(u_0)$. In particular, an extra entropy production is generated at $t = 0$ (see also [8]):

$$E_0 = -\int_{-\infty}^{+\infty} \Psi(x)(\eta(u_0(x), \Phi(u_0(x))) - \eta(u_0(x), v_0(x)))dx,$$

which is nonnegative for any test function $\Psi \geq 0$ and for any convex entropy η, such that $\eta_v(u\Phi(u)) = 0$. As in [8], we prove the uniqueness result under the above hypothesis **(H4)** and under the compatibility condition $v_0 = \Phi(u_0)$.

Lemma 6.1 *Let $\varphi(u)$ be any C_0^∞ function equal to $|u|^2$ for $u \in [-M, M]$ and let f, g and $\Phi \in C^5$. Let us assume the hypothesis **(H1)**,the hypothesis **(H3)** of theorem 3.2 and assume the subcharacteristic condition **(SC)** of proposition 3.1 holds. Moreover, assume that the solution (u, v) of the system (1.1) is the limit, as $\nu \downarrow 0$, of the solution of the following parabolic approximation of (1.1):*

$$\begin{cases} u_t + f(u,v)_x = \nu u_{xx} \\ v_t + g(u,v)_x = \frac{1}{\varepsilon}(\Phi(u) - v) + \nu v_{xx}. \end{cases} \tag{6.1}$$

Finally, assume the sequence $(u^{\varepsilon,\nu}, v^{\varepsilon,\nu})$ is φ-stable (namely, we take $\varphi^k \equiv \varphi$ in the definition 5.1). Then, for any nonnegative test function $\Psi(x)$, one has:

$$\int_0^T \int_{-\infty}^{+\infty} \Psi(x)|u(x,t)|^2 dx dt \leq \int_0^T \int_{-\infty}^{+\infty} \Psi(x)|u_0(x)|^2 dx dt + O(1)T^2, \tag{6.2}$$

uniformly in ν, ε.

Proof. It suffices to prove (6.2) with $\varphi(u)$ instead of $|u|^2$. Let $\Psi(x)$ be a nonnegative test function and let $(\eta(u,v), q(u,v))$ be the entropy-entropy flux pair satisfying

$$\begin{cases} \eta(u, \Phi(u)) = \varphi(u) \\ \eta_v(u, \Phi(u)) = 0. \end{cases}$$

Then, in a standard way, one has:

$$\begin{aligned} &\int_{-\infty}^{+\infty} \Psi(x)\eta(u(x,t), v(x,t))dx - \int_{-\infty}^{+\infty} \Psi(x)\eta(u_0(x), v_0(x))dx \\ &\quad = \frac{1}{\varepsilon}\int_0^t \int_{-\infty}^{+\infty} \Psi(x)\eta_v(u(x,s), v(x,s))(\Phi(u(x,s)) - v(x,s))dxds \\ &\qquad -\nu \int_0^t \int_{-\infty}^{+\infty} \Psi(x) D^2\eta[u_x, v_x] dxds \\ &\qquad + \int_0^t \int_{-\infty}^{+\infty} \Psi'(x) q(u(x,s), v(x,s))dxds \\ &\qquad +\nu \int_0^t \int_{-\infty}^{+\infty} \Psi''(x)\eta(u(x,s), v(x,s))dxds \end{aligned}$$

$$\leq \int_0^t \int_{-\infty}^{+\infty} \Psi'(x) q(u(x,s), v(x,s)) dx ds$$
$$+\nu \int_0^t \int_{-\infty}^{+\infty} \Psi''(x) \eta(u(x,s), v(x,s)) dx ds, \tag{6.3}$$

where with $D^2\eta[u_x, v_x]$ we denote the quadratic form in (u_x, v_x) associated with the hessian matrix of η. Thus, passing into the limit $\nu \downarrow 0$ in (6.3) and integrating over $[0, T]$, one has

$$\int_0^T \int_{-\infty}^{+\infty} \Psi(x) \eta(u(x,t), v(x,t)) dx dt$$
$$- \int_0^T \int_{-\infty}^{+\infty} \Psi(x) \eta(u_0(x), v_0(x)) dx dt \leq C \|q\|_{L^\infty} T^2.$$

Finally, as $\varepsilon \downarrow 0$, from (3.9) and from the compatibility condition $v_0 = \Phi(u_0)$, it follows

$$\int_0^T \int_{-\infty}^{+\infty} \Psi(x) \varphi(u(x,t)) dx dt - \int_0^T \int_{-\infty}^{+\infty} \Psi(x) \varphi(u_0(x)) dx dt$$
$$\leq C \|q\|_{L^\infty} T^2.$$

■

Remark 6.2 With a simple argument, it is possible to show that (6.2) yields:

$$\int_0^T \int_K |u(x,t)|^2 dx dt \leq \int_0^T \int_{-\infty}^{+\infty} |u_0(x)|^2 dx dt + O(1) T^2, \tag{6.4}$$

for any compact $K \subset \mathbb{R}$ (see [8]). ■

Now we are ready to prove the main result of this section.

Theorem 6.3 *Let φ be as in Lemma 6.1 and let f, g and $\Phi \in C^5$. Assume the hypothesis **(H1)**,the hypothesis **(H3)** of theorem 3.2 and assume the subcharacteristic condition **(SC)** of proposition 3.1 holds. Moreover, assume that the solution (u, v) of the system (1.1) is the limit, as $\nu \downarrow 0$, of the solutions $(u^{\varepsilon,\nu}, v^{\varepsilon,\nu})$ of (6.1). Finally, assume the sequence $(u^{\varepsilon,\nu}, v^{\varepsilon,\nu})$ is φ-stable. Then we have:*

$$\lim_{T \longrightarrow 0^+} \frac{1}{T} \int_0^T \int_K |u(x,t) - u_0(x)| dx dt = 0,$$

*for any compact $K \subset \mathbb{R}$. In particular, if the condition **(H4)** holds, $u(x.t)$ is the unique entropy solution of (4.1) with $u_0(x)$ as initial condition.*

Proof. In this proof we proceed as in [14, 8]. Let us denote with $d(u) = |u|^2$ and let $D(u, u_0) = d(u) - d(u_0) - d'(u_0)(u - u_0) = (u - u_0)^2$. For any compact $K \subset \mathbb{R}$, the Schwarz inequality yields

$$\begin{aligned} \int_K |u - u_0| dx &\leq C_K \left(\int_K (u - u_0)^2 dx \right)^{\frac{1}{2}} \\ &= C_K \left(\int_K D(u, u_0) dx \right)^{\frac{1}{2}}. \end{aligned}$$

Therefore, the Jensen inequality implies

$$\frac{1}{T} \int_0^T \int_K |u - u_0| dx dt \leq C_K \left(\frac{1}{T} \int_0^T \int_K D(u, u_0) dx dt \right)^{\frac{1}{2}}. \tag{6.5}$$

Now, let f_n be a sequence of C_0^∞ functions such that

$$\|f_n - d'(u_0)\|_{L^1} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Then, from (6.4) it follows

$$\begin{aligned} \int_0^T \int_K D(u, u_0) dx dt &\leq \int_0^T \int_K f_n (u_0 - u) dx dt \\ &\quad + T(\|u\|_{L^\infty} + \|u_0\|_{L^\infty}) \|d'(u_0) - f_n\|_{L^1} \\ &\quad + \int_0^T \int_K |u|^2 dx dt - \int_0^T \int_K |u_0|^2 dx dt \\ &\leq \int_0^T \int_K f_n (u_0 - u) dx dt \\ &\quad + T(\|u\|_{L^\infty} + \|u_0\|_{L^\infty}) \|d'(u_0) - f_n\|_{L^1} \\ &\quad + \int_0^T \int_{\mathbb{R} - K} |u_0|^2 dx dt + CT^2. \end{aligned}$$

Let $\{K_i\}$ be an increasing sequence of compact set which cover $\mathbb{R}$: $K_i \subset K_{i+1}$ for any i, $K \subset K_1$ and $\displaystyle\bigcup_{i=1}^{\infty} K_i = \mathbb{R}$. Now, repeating the previous argument for all the K_i, since

$$\int_0^T \int_K D(u, u_0) dx dt \leq \int_0^T \int_{K_i} D(u, u_0) dx dt,$$

we get that

$$\begin{aligned} \frac{1}{T} \int_0^T \int_K D(u, u_0) dx dt &\leq \frac{1}{T} \int_0^T \int_{-\infty}^{+\infty} f_n (u_0 - u) dx dt + \\ &\quad + (\|u\|_{L^\infty} + \|u_0\|_{L^\infty}) \|d'(u_0) - f_n\|_{L^1} \\ &\quad + CT. \end{aligned} \tag{6.6}$$

Hence, thanks to (6.5) and (6.6), we have only to prove

$$\lim_{T\longrightarrow 0^+}\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(u_0-u)dxdt=0. \tag{6.7}$$

Let $(u^{\nu,\varepsilon},v^{\nu,\varepsilon})$ be the solution of the system (6.1). Now,

$$\begin{aligned}
\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(x)(u_0-u^{\nu,\varepsilon})dxdt &= -\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty}\int_0^t u_s^{\nu,\varepsilon} f_n(x)dsdxdt\\
&= \frac{1}{T}\int_0^T\int_{-\infty}^{+\infty}\int_0^t (v_x^{\nu,\varepsilon}-\nu u_{xx}^{\nu,\varepsilon})f_n(x)dsdxdt\\
&= -\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty}\int_0^t (v^{\nu,\varepsilon}f_n'(x)+\nu u^{\nu,\varepsilon}f_n''(x))dsdxdt\\
&\le C_nT.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(u_0-u)dxdt &= \lim_{\nu,\varepsilon\longrightarrow 0^+}\frac{1}{T}\int_0^T\int_{-\infty}^{+\infty} f_n(u_0-u^{\nu,\varepsilon})dxdt\\
&\le C_nT,
\end{aligned}$$

which proves (6.7). ■

Remark 6.4 It is possible to prove a uniqueness result even in the case of initial data not in equilibrium, namely in the case of a sequence $(u_0^\varepsilon,v_0^\varepsilon)$, uniformly bounded in L^∞, which converges toward the equilibrium states almost everywhere, provided that the finite energy condition **(H3)** of theorem 3.2 is uniform on ε. ■

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