Kinetic invariant domains and relaxation limit from a BGK model to isentropic gas dynamics

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Abstract

We obtain rigorously the correspondence between kinetic and macroscopic invariant domains for a vector kinetic BGK system considered in a previous paper of the authors. We establish the convergence, as the relaxation parameter tends to 0, to weak entropy solutions of isentropic gas dynamics equations.

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1 Introduction and main results

In this paper we consider the vector kinetic BGK model already studied in [3], which leads in the relaxation limit to the one-dimensional system of isentropic gas dynamics

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0, \end{cases} \tag{1.1}$$

with $\rho(t,x) \geq 0,\ u(t,x) \in \mathbb{R}$ and $\kappa > 0,\ 1 < \gamma < 3$. The aim of this paper is to obtain rigorously the correspondence between kinetic invariant domains of the BGK model and the macroscopic ones of (1.1), to establish a maximum principle and to prove the convergence as the relaxation parameter tends to 0. The kinetic model has been introduced in [4], and can be written as

$$\partial_t f + \xi \partial_x f = \frac{M[f] - f}{\varepsilon} \quad \text{in }]0,\infty[\times\mathbb{R}\times\mathbb{R}, \tag{1.2}$$

where $f = f(t,x,\xi) \in \mathbb{R}^2$, $t > 0$, $x \in \mathbb{R}$, $\xi \in \mathbb{R}$,

$$f(t,x,\xi) \in D, \tag{1.3}$$

$$D = \{(f_0, f_1) \in \mathbb{R}^2,\ f_0 > 0 \text{ or } f_1 = f_0 = 0\}, \tag{1.4}$$

$$M[f](t,x,\xi) = M\left(\rho(t,x), u(t,x), \xi\right), \tag{1.5}$$

$$\rho(t,x) = \int_{\mathbb{R}} f_0(t,x,\xi)\, d\xi,\ \rho(t,x)u(t,x) = \int_{\mathbb{R}} f_1(t,x,\xi)\, d\xi, \tag{1.6}$$

and

$$M(\rho,u,\xi) = \Big(\chi(\rho,\xi-u), ((1-\theta)u + \theta\xi)\chi(\rho,\xi-u)\Big), \tag{1.7}$$

$$\chi(\rho,\xi) = c_{\gamma,\kappa}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\rho^{\gamma-1} - \xi^2\right)_+^\lambda, \tag{1.8}$$

$$\theta = \frac{\gamma-1}{2},\ \lambda = \frac{1}{\gamma-1} - \frac{1}{2},\ c_{\gamma,\kappa} = \frac{1}{J_\lambda}\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\right)^{-1/(\gamma-1)}, \tag{1.9}$$

$$J_\lambda = \int_{-1}^{1} (1-z^2)^\lambda \, dz = \sqrt{\pi}\Gamma(\lambda+1)/\Gamma(\lambda+3/2). \tag{1.10}$$

The kinetic equilibrium χ in (1.8) has been introduced in [8] as a generating function for entropies, and has also been used in the stability analysis of [9]. This function is also involved in [10] for the so called kinetic formulation, which is slightly different from the BGK equation considered here. The aim of the present paper is to extend to the above case of isentropic gas dynamics the results obtained for scalar equations in [12]. We refer also to [6], [16], [17], [15], [13] for related works.

We consider the system (1.2) with initial data

$$f(0,x,\xi) = f^0(x,\xi) \tag{1.11}$$

satisfying energy bounds. We recall that the energy for (1.1) is given by

$$\eta(\rho,u) = \rho u^2/2 + \frac{\kappa}{\gamma-1}\rho^\gamma, \tag{1.12}$$

and is a mathematical entropy for (1.1). The corresponding kinetic entropy for (1.2) is given, for $f=(f_0,f_1)\in D$, by

$$H(f,\xi) = \frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0 + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1+1/\lambda}}{1+1/\lambda} + \frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0} - \frac{\theta}{1-\theta}\xi f_1, \tag{1.13}$$

$$H(0,\xi)=0. \tag{1.14}$$

We recall that $H\geq 0$. In [3], we proved the existence of a weak solution satisfying a kinetic entropy inequality for any initial data with finite energy. Let us recall the main results of [3].

Theorem 1.1 *Assume that $f^0 \in L^1(\mathbb{R}_x \times \mathbb{R}_\xi)$ satisfies*

$$f^0(x,\xi) \in D \text{ a.e. in } \mathbb{R}\times\mathbb{R} \tag{1.15}$$

and

$$\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f^0(x,\xi),\xi)\, dx d\xi = C_H^0 < \infty. \tag{1.16}$$

Then there exists a solution f to (1.2)-(1.11) satisfying

$$f \in C([0,\infty[, L^1(\mathbb{R}\times\mathbb{R})), \tag{1.17}$$

$$f \in L^\infty([0,\infty[, L^1(\mathbb{R}\times\mathbb{R})), \tag{1.18}$$

$$\forall t\geq 0,\ f(t,x,\xi)\in D \text{ a.e. in } \mathbb{R}\times\mathbb{R}, \tag{1.19}$$

$$\partial_t\Big(\int_{\mathbb{R}} f\, d\xi\Big) + \partial_x\Big(\int_{\mathbb{R}} \xi f\, d\xi\Big) = 0, \tag{1.20}$$

$$\forall t\geq 0,\ \iint\limits_{\mathbb{R}\times\mathbb{R}} H(f(t,x,\xi),\xi)\, dx d\xi \leq C_H^0. \tag{1.21}$$

Theorem 1.2 *The solution f obtained in Theorem 1.1 satisfies*

$$\begin{aligned}&\left[\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f(t,x,\xi),\xi)\,dxd\xi\right]_0^T \\ &= \frac{1}{\varepsilon}\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}} \left(H'(f,\xi)-\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\right)\right)\cdot(M[f]-f)\,dtdxd\xi,\end{aligned} \tag{1.22}$$

where the derivative $H'(f,\xi)$ of $H(f,\xi)$ with respect to f is given by

$$H'(f,\xi)=\left(\frac{\theta}{1-\theta}\frac{\xi^2}{2}+\frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}f_0^{1/\lambda}-\frac{1}{1-\theta}\frac{1}{2}\frac{f_1^2}{f_0^2},-\frac{\theta}{1-\theta}\xi+\frac{1}{1-\theta}\frac{f_1}{f_0}\right) \tag{1.23}$$

for $f\neq 0$, and $H'(0,\xi)=(\frac{\theta}{1-\theta}\frac{\xi^2}{2},-\frac{\theta}{1-\theta}\xi)$ by convention. Moreover,

$$\left(H'(f,\xi)-\left(\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1}-\frac{u^2}{2},u\right)\right)\cdot(M[f]-f)\leq 0 \quad a.e. \tag{1.24}$$

Indeed, the definition of $H'(0,\xi)$ does not really matter in (1.22), since by applying the result of [5] to each component of f, we have

$$M[f]-f=0 \quad \text{a.e. } t,x,\xi \text{ where } f=0. \tag{1.25}$$

Following [14], let us now introduce the notion of kinetic invariant domains, which is the heart of this paper.

Definition 1.3 *A family $\tilde{D}_\xi$ of subsets of $\mathbb{R}^2$ is a family of kinetic invariant domains if, denoting by f the solution obtained in Theorem 1.1,*

$$f^0(x,\xi)\in\tilde{D}_\xi \text{ a.e. } x,\xi \Rightarrow \forall t\ f(t,x,\xi)\in\tilde{D}_\xi \text{ a.e. } x,\xi. \tag{1.26}$$

We have the following result.

Theorem 1.4 *For any $\omega_{min}<\omega_{max}$, the system (1.2) has the property that*

$$\tilde{D}_\xi=\{f\in D;\ f=0 \text{ or } \omega_{min}\leq\omega_1(f,\xi)\leq\omega_2(f,\xi)\leq\omega_{max}\} \tag{1.27}$$

is a family of convex kinetic invariant domains, where

$$\omega_1(f,\xi)=u(f,\xi)-\frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho(f,\xi)^{\frac{\gamma-1}{2}},\ \omega_2(f,\xi)=u(f,\xi)+\frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho(f,\xi)^{\frac{\gamma-1}{2}}, \tag{1.28}$$

with

$$u(f,\xi)=\frac{f_1/f_0-\theta\xi}{1-\theta}, \tag{1.29}$$

and

$$\rho(f,\xi)=\left(\frac{4\gamma\kappa}{(\gamma-1)^2}\right)^{-1/(\gamma-1)}\left(\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2+(f_0/c_{\gamma,\kappa})^{1/\lambda}\right)^{1/(\gamma-1)}. \tag{1.30}$$

Moreover, the set $\tilde{D}_\xi$ is associated with the invariant domain of (1.1)

$$\tilde{D} = \{(\rho, u) \in \mathbb{R}^+ \times \mathbb{R};\ \ \rho = 0 \text{ or } \omega_{min} \le \omega_1 \le \omega_2 \le \omega_{max}\}, \tag{1.31}$$

where the Riemann invariants ω_1, ω_2 are given by

$$\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}},\ \ \omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}, \tag{1.32}$$

in the sense that

$$\forall (\rho, u) \in \tilde{D},\ \ M(\rho, u, \xi) \in \tilde{D}_\xi \text{ a.e.}\xi, \tag{1.33}$$

and

$$\begin{gathered}\text{for any } f(\xi) \in L^1(\mathbb{R}_\xi) \text{ such that } f(\xi) \in \tilde{D}_\xi \text{ a.e. } \xi,\\ (\rho, u) \in \tilde{D} \text{ with } (\rho, \rho u) = \int_{\mathbb{R}} f(\xi)\, d\xi.\end{gathered} \tag{1.34}$$

In particular, if the initial data of Theorem 1.1 satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ for a.e. x, ξ, then (ρ, u) defined by (1.6) verify $\forall t \ge 0$, $(\rho(t,x), u(t,x)) \in \tilde{D}$ for a.e. x. Besides $f(t,x,\xi) \in \tilde{D}_\xi$ $\forall t \ge 0$ and consequently f has compact support with respect to ξ, $supp_\xi f \subset [\omega_{min}, \omega_{max}]$.

According to [9], we define entropy solutions to (1.1) as solutions with finite energy, i.e. $\sup_t \int \eta(\rho(t,x), u(t,x))\, dx < \infty$, which satisfy the entropy inequalities

$$\partial_t(\eta_S) + \partial_x(G_S) \le 0 \tag{1.35}$$

where η_S and G_S are suitable entropy and entropy flux for (1.1) parametrized by a convex function S with subquadratic growth that will be defined precisely in the next section. We have the following convergence result.

Theorem 1.5 *Let us denote by f_ε the solution of Theorem 1.1 with the same initial data $f^0(x,\xi) \in L^1(\mathbb{R}\times\mathbb{R})$ that satisfies $f^0(x,\xi) \in \tilde{D}_\xi$ a.e. for some $\omega_{min} < \omega_{max}$, and the energy bound (1.16). Then $(\rho_\varepsilon, u_\varepsilon)$ defined by (1.6) are uniformly bounded in L^∞, and passing if necessary to subsequences, $(\rho_\varepsilon, \rho_\varepsilon u_\varepsilon)$ converge a.e. in $]0,\infty[\times\mathbb{R}$ when $\varepsilon \to 0$ to an entropy solution $(\rho, \rho u)$ to (1.1), with initial data $(\rho^0, \rho^0 u^0) = \int f^0 d\xi$.*

We have to notice that in particular, we can take as initial data

$$f^0(x,\xi) = M(\rho^0(x), u^0(x), \xi), \tag{1.36}$$

with

$$\begin{gathered}(\rho^0(x), u^0(x)) \in \tilde{D} \quad a.e.,\\ \int_{\mathbb{R}} \rho^0(x)\, dx < \infty, \qquad \int_{\mathbb{R}} \eta(\rho^0(x), u^0(x))\, dx < \infty.\end{gathered} \tag{1.37}$$

The analysis relies on the stability results of [9] together with the use of the kinetic entropies $H_S(f,\xi)$ introduced in [4], that are associated to the entropies η_S via the properties

$$\int H_S(M(\rho, u, \xi), \xi)\, d\xi = \eta_S(\rho, u), \tag{1.38}$$

$$\eta_S(\rho,u) \le \int H_S(f(\xi),\xi)\,d\xi, \qquad (\rho,\rho u) = \int f(\xi)d\xi. \tag{1.39}$$

They verify also the important relation

$$H_S'(M(\rho,u,\xi),\xi) = \eta_S'(\rho,u), \tag{1.40}$$

whenever $M(\rho,u,\xi) \neq 0$. In general, a weaker form holds with a subdifferential,

$$\partial H_S(M(\rho,u,\xi),\xi) \ni T_S(\rho,u), \tag{1.41}$$

where $T_S(\rho,u)$ is the Lagrange multiplier corresponding to the minimization problem associated to (1.39), and coincides with $\eta_S'(\rho,u)$ when $\rho > 0$ (see Proposition 4.1).

Let us finally remark that the BGK system (1.2) leads formally, at first order in ε, to the viscous model

$$\begin{aligned} \partial_t\rho + \partial_x(\rho u) &= \varepsilon\mu\partial_x(\rho^{\gamma-1}\partial_x\rho), \\ \partial_t(\rho u) + \partial_x(\rho u^2 + \kappa\rho^\gamma) &= \varepsilon\mu\partial_x(\rho^{\gamma-1}\partial_x(\rho u)), \end{aligned} \tag{1.42}$$

with $\mu = \kappa\gamma(3-\gamma)/(\gamma-1)$.

2 Notations

In order to define the kinetic entropies, we need to introduce some notations. We set

$$\Phi(\rho,u,\xi,v) = \frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}\mathbb{1}_{\omega_1<\xi<\omega_2}\mathbb{1}_{\omega_1<v<\omega_2}|\xi - v|^{2\lambda-1}\Upsilon_{\lambda-1}(z), \tag{2.1}$$

where

$$z = \frac{(\xi+v)(\omega_1+\omega_2) - 2(\omega_1\omega_2+\xi v)}{(\omega_2-\omega_1)|\xi-v|}, \tag{2.2}$$

and

$$\Upsilon_{\lambda-1}(z) = \int_1^z (y^2-1)^{\lambda-1}\,dy, \qquad z \ge 1. \tag{2.3}$$

The z from (2.2) is indeed greater than 1, because

$$\begin{aligned} (\xi+v)(\omega_1+\omega_2) - 2(\omega_1\omega_2+\xi v) - (\omega_2-\omega_1)(\xi-v) &= 2(\omega_2-\xi)(v-\omega_1), \\ (\xi+v)(\omega_1+\omega_2) - 2(\omega_1\omega_2+\xi v) + (\omega_2-\omega_1)(\xi-v) &= 2(\xi-\omega_1)(\omega_2-v), \end{aligned}$$

which are both nonnegative in $\{\omega_1 < \xi, v < \omega_2\}$. Also,

$$z^2 - 1 = \frac{4(\omega_2-\xi)(\xi-\omega_1)(\omega_2-v)(v-\omega_1)}{(\omega_2-\omega_1)^2(\xi-v)^2}. \tag{2.4}$$

One can check that $\Phi \ge 0$,

$$\begin{aligned} \int \Phi(\rho,u,\xi,v)\,dv &= \chi(\rho,\xi-u), \\ \int v\Phi(\rho,u,\xi,v)\,dv &= [(1-\theta)u + \theta\xi)]\chi(\rho,\xi-u), \end{aligned} \tag{2.5}$$

and similar relations also hold for integration in ξ since Φ is symmetric in ξ, v. We now define for any convex function S the kinetic entropy

$$H_S(f,\xi) = \int_{\mathbb{R}} \Psi(f,\xi,v) S(v)\, dv, \text{ for } f \neq 0, \qquad H_S(0,\xi) = 0, \tag{2.6}$$

where

$$\Psi(f,\xi,v) = \Phi(\rho(f,\xi), u(f,\xi), \xi, v), \quad f \neq 0, \tag{2.7}$$

and its macroscopic version

$$G_S(\rho,u,\xi) = \int_{\mathbb{R}} \Phi(\rho,u,\xi,v) S(v)\, dv = H_S(M(\rho,u,\xi),\xi). \tag{2.8}$$

Then, the entropy and entropy flux for (1.1) are defined by

$$\eta_S(\rho,u) = \int_{\mathbb{R}} \chi(\rho, v-u) S(v)\, dv = \int G_S(\rho,u,\xi)\, d\xi, \tag{2.9}$$

$$G_S(\rho,u) = \int_{\mathbb{R}} [(1-\theta)u + \theta v)] \chi(\rho, v-u) S(v)\, dv = \int \xi\, G_S(\rho,u,\xi)\, d\xi. \tag{2.10}$$

Let us remark that for $S(v) = v^2/2$, we get $H_S = H$ and $\eta_S = \eta$. We shall prove the differentiability of H_S for $f \in D \setminus \{0\}$, and we use the convention $H_S'(0,\xi) = 0$ for $S \in C^1(\mathbb{R},\mathbb{R})$ and not identically equal to $v \mapsto v^2/2$, while $H'(0,\xi) = (\frac{\theta}{1-\theta}\frac{\xi^2}{2}, -\frac{\theta}{1-\theta}\xi)$ (but this is not really necessary, according to (1.25)). We finally define

$$A_\gamma = \frac{4\gamma\kappa}{(\gamma-1)^2}, \tag{2.11}$$

$$l(z) = u + \sqrt{A_\gamma} \rho^{(\gamma-1)/2}\, z, \tag{2.12}$$

$$m(z) = -u + \frac{\gamma-1}{2}\sqrt{A_\gamma} \rho^{(\gamma-1)/2}\, z, \tag{2.13}$$

and

$$T(\rho,u) = (\frac{\gamma\kappa}{\gamma-1}\rho^{\gamma-1} - u^2/2, u), \tag{2.14}$$

which coincides with $\eta'(\rho,u)$ whenever $\rho > 0$. Here and all throughout the paper, the prime denotes differentiation with respect to conservative variables, $(\rho, q \equiv \rho u)$ here.

3 Preliminary Convexity results

In this section we prove that H_S is convex in D, and some technical regularity results. We notice that

$$M_0(\rho,u,\xi) = \chi(\rho, \xi-u) = c_{\gamma,\kappa} \left[(\omega_2 - \xi)(\xi - \omega_1)\right]_+^\lambda, \tag{3.1}$$

with $\omega_1 = u - \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$, $\omega_2 = u + \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}$.

Let us first invert the maxwellian function. Consider the relation $f = M(\rho, u, \xi)$ for some fixed $\xi \in \mathbb{R}$. Whenever $f_0 > 0$, this relation can be inverted in $u = u(f, \xi)$, $\rho = \rho(f, \xi)$ from (1.29), (1.30), and then $\omega = F(f)$ with

$$F(f) = \begin{pmatrix} \frac{f_1/f_0 - \theta\xi}{1-\theta} - \sqrt{\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda}} \\ \frac{f_1/f_0 - \theta\xi}{1-\theta} + \sqrt{\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda}} \end{pmatrix}. \tag{3.2}$$

The inverse relation is given by $f = R(\omega)$, with

$$R(\omega) = \begin{pmatrix} c_{\gamma,\kappa}\left(\left(\frac{\omega_2-\omega_1}{2}\right)^2 - \left(\xi - \frac{\omega_1+\omega_2}{2}\right)^2\right)_+^\lambda \\ \left((1-\theta)\frac{\omega_1+\omega_2}{2} + \theta\xi\right)c_{\gamma,\kappa}\left(\left(\frac{\omega_2-\omega_1}{2}\right)^2 - \left(\xi - \frac{\omega_1+\omega_2}{2}\right)^2\right)_+^\lambda \end{pmatrix}. \tag{3.3}$$

Lemma 3.1 *By these formulas, $\{f_0 > 0\}$ is in bijection with $\{\omega_1 < \xi < \omega_2\}$.*

Proof. Whenever $f_0 > 0$, we can check that $R(F(f)) = f$. Obviously, we have $|\frac{\omega_2-\omega_1}{2}| > |\xi - \frac{\omega_2+\omega_1}{2}| \Leftrightarrow \omega_1 < \xi < \omega_2$, thus $F(R(\omega)) = \omega$ in $\{\omega_1 < \xi < \omega_2\}$. Furthermore, $R(\omega)_0 > 0$ in $\{\omega_1 < \xi < \omega_2\}$ and

$$F(f)_1 - \xi = \frac{1}{1-\theta}(f_1/f_0 - \xi) - \sqrt{\left(\frac{f_1/f_0 - \xi}{1-\theta}\right)^2 + (f_0/c_{\gamma,\kappa})^{1/\lambda}} < 0 \text{ if } f_0 > 0.$$

Finally we get a C^∞ change of variable. $\square$

Lemma 3.2 *i) If $S : \mathbb{R} \to \mathbb{R}$ is of class C^k then the functions $(\rho, u) \mapsto \eta_S(\rho, u)$ and $(\rho, q) \mapsto \eta_S(\rho, u)$ with $q = \rho u$ are C^k in $\{\rho > 0\}$.*
ii) If $S : \mathbb{R} \to \mathbb{R}$ is bounded on compact sets, then the function $(\omega_1, \omega_2) \mapsto G_S(\rho, u, \xi)$ is continuously differentiable in $\{\omega_1 < \xi < \omega_2\}$, with

$$\frac{\partial G_S}{\partial \omega_i}(\rho, u, \xi) = \int_{\mathbb{R}} \frac{\partial \Phi}{\partial \omega_i}(\rho, u, \xi, v) S(v)\, dv \qquad \text{for } i = 1, 2.$$

iii) If $S : \mathbb{R} \to \mathbb{R}$ is bounded on compact sets, then $H_S(., \xi)$ is continuous at 0 in $\{f \in D\ ;\ |f_1| \leq A f_0\}$, for any $A > 0$.

Proof. *i*) By the change of variable $v - u = \frac{2\sqrt{\gamma\kappa}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} z$, we get

$$\eta_S(\rho, u) = \frac{\rho}{J_\lambda}\int_{\mathbb{R}} (1 - z^2)_+^\lambda S\left(u + \frac{2\sqrt{\gamma\kappa}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}} z\right) dz, \tag{3.4}$$

and we can obviously differentiate under the integral.
ii) Let us first show that G_S in (2.8) is well defined. Since we need $\omega_1 < v < \omega_2$ for Φ to be different from zero, we only have to check that $\int_{\omega_1}^{\omega_2} \Phi\, dv < \infty$. Obviously this integral vanishes if $\xi \notin]\omega_1, \omega_2[$, thus let us assume that $\omega_1 < \xi < \omega_2$. The only possible singularities of Φ in v are at the points ω_1, ω_2, and ξ. We have $\Upsilon_{\lambda-1}(z) \underset{z\to 1}{\sim} 2^{\lambda-1}(z-1)^\lambda/\lambda$, hence $\Upsilon_{\lambda-1}(z) \underset{v\to\omega_2}{\sim} L(\omega_2 - v)^\lambda$ with L

a constant. Thus Φ is continuous at ω_2, and also at ω_1.
Next, if $\lambda > 1/2$, $\Upsilon_{\lambda-1}(z) \underset{z\to\infty}{\sim} z^{2\lambda-1}/(2\lambda-1)$, which yields $\Upsilon_{\lambda-1}(z) \underset{v\to\xi}{\sim} L'|\xi - v|^{-(2\lambda-1)}$ with L' a constant, and therefore $\lim\limits_{v\to\xi} |\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(z)$ is finite.
If $\lambda = 1/2$, we get $\Upsilon_{\lambda-1}(z) \underset{v\to\xi}{\sim} |\ln|\xi-v||$, which is integrable at ξ.
If $0 < \lambda < 1/2$, $\Upsilon_{\lambda-1}(z)$ has a finite limit when $v \to \xi$, and $|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}(z)$ is integrable in v at ξ.
Finally, in any case, G_S is defined by a convergent integral.
More generally, we have that for any $A > 0$,

$$v \mapsto |\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}\left(\frac{A}{|\xi-v|}\right) \text{ is locally integrable in } v. \tag{3.5}$$

In the set $\{\omega_1 < \xi < \omega_2\}$, we can locally upper estimate Φ by such a function, thus by Lebesgue's theorem, we obtain that G_S is continuous in $\{\omega_1 < \xi < \omega_2\}$.

Next, we have in distribution sense in $\{\omega_1 < \xi < \omega_2\}$, for any $v \neq \xi$,

$$\frac{\partial}{\partial\omega_1}(\mathbb{1}_{\omega_1<v<\omega_2}\Upsilon_{\lambda-1}(z)) = \mathbb{1}_{\omega_1<v<\omega_2}\frac{\partial}{\partial\omega_1}\Upsilon_{\lambda-1}(z),$$

because $\Upsilon_{\lambda-1}(z)$ vanishes for $\omega_1 = v$. Now

$$\frac{\partial}{\partial\omega_1}(\Upsilon_{\lambda-1}(z)) = (z^2-1)^{\lambda-1}\frac{\partial z}{\partial\omega_1},$$

with

$$\frac{\partial z}{\partial\omega_1} = -\frac{2(\omega_2-\xi)(\omega_2-v)}{|\xi-v|(\omega_2-\omega_1)^2},$$

thus

$$\begin{aligned}&\frac{\partial}{\partial\omega_1}(\mathbb{1}_{\omega_1<v<\omega_2}\Upsilon_{\lambda-1}(z))\\ =\ & -2\mathbb{1}_{\omega_1<v<\omega_2}\left[\frac{(\omega_2-\xi)(\omega_2-v)}{(\omega_2-\omega_1)^2}\right]^{\lambda}\frac{[4(\xi-\omega_1)(v-\omega_1)]^{\lambda-1}}{|\xi-v|^{2\lambda-1}}.\end{aligned}$$

Similarly, we have in distribution sense in $\{\omega_1 < \xi < \omega_2\}$, for any $v \neq \xi$,

$$\begin{aligned}&\frac{\partial}{\partial\omega_2}(\mathbb{1}_{\omega_1<v<\omega_2}\Upsilon_{\lambda-1}(z))\\ =\ & 2\mathbb{1}_{\omega_1<v<\omega_2}\left[\frac{(\xi-\omega_1)(v-\omega_1)}{(\omega_2-\omega_1)^2}\right]^{\lambda}\frac{[4(\omega_2-\xi)(\omega_2-v)]^{\lambda-1}}{|\xi-v|^{2\lambda-1}}.\end{aligned}$$

With this formula, we see that

$$\Lambda_i(\omega_1,\omega_2) \equiv \int_{\mathbb{R}}\frac{\partial\Phi}{\partial\omega_i}(\rho,u,\xi,v)S(v)\,dv$$

is well-defined for $\omega_1 < \xi < \omega_2$, and is locally bounded. Moreover, one can easily deduce that $\partial_{\omega_i} G_S = \Lambda_i$ in the sense of distributions in $\{\omega_1 < \xi < \omega_2\}$. Then, by the change of variable $\mu = v - \omega_1$, we see by Lebesgue's theorem that Λ_1 is continuous in $\{\omega_1 < \xi < \omega_2\}$. A similar result holds for Λ_2, thus we obtain the result.

iii) Let ξ be fixed. In $\{f \in D\ ;\ |f_1| \le Af_0 \textit{ and } f_0 \le B\}$, and if $f \neq 0$, setting $(\omega_1, \omega_2) = F(f)$ from (3.2), we have $D \le \omega_1 < \xi < \omega_2 \le E$ with D, E some constants. Then, $S(v)$ is bounded in $[D, E]$, and thus

$$f_0 \inf S \le H_S(f, \xi) \le f_0 \sup S,$$

which yields that $H_S(f, \xi) \to 0$ as $f \to 0$ with $|f_1| \le Af_0$. $\square$

Proposition 3.3 *If $S : \mathbb{R} \to \mathbb{R}$ is of class C^1 then we have $H_S'(M(\rho, u, \xi), \xi) = \eta_S'(\rho, u)$ whenever $(M(\rho, u, \xi))_0 > 0$.*

Proof. Since by Lemma 3.1 $H_S(f, \xi) = G_S(\rho(f, \xi), u(f, \xi), \xi)$ whenever $f_0 > 0$, we get from Lemma 3.2(*ii*) that H_S is C^1 in $\{f_0 > 0\}$. Furthermore, for $\omega_1 < \xi < \omega_2$,

$$\begin{aligned}
\frac{\partial G_S}{\partial \omega_1}(\rho, u, \xi) = -2\frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}4^{\lambda-1}\frac{(\omega_2 - \xi)^\lambda (\xi - \omega_1)^{\lambda-1}}{(\omega_2 - \omega_1)^{2\lambda}}& \\
\times \int_{\mathbb{R}} \mathbb{1}_{\omega_1 < v < \omega_2} (\omega_2 - v)^\lambda (v - \omega_1)^{\lambda-1} S(v)\, dv&, \\
\frac{\partial G_S}{\partial \omega_2}(\rho, u, \xi) = 2\frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}4^{\lambda-1}\frac{(\omega_2 - \xi)^{\lambda-1} (\xi - \omega_1)^{\lambda}}{(\omega_2 - \omega_1)^{2\lambda}}& \\
\times \int_{\mathbb{R}} \mathbb{1}_{\omega_1 < v < \omega_2} (\omega_2 - v)^{\lambda-1} (v - \omega_1)^{\lambda} S(v)\, dv&.
\end{aligned}$$

Then, we have by Lemma 3.2(*i*) that for $\rho > 0$, η_S is C^1, and by differentiating directly (2.9), we get for $\rho > 0$

$$\begin{aligned}
\frac{\partial \eta_S}{\partial \rho}(\rho, u) \;=\; \frac{1}{\rho}\int c_{\gamma,\kappa}[(\omega_2 - v)(v - \omega_1)]_+^{\lambda-1}\Big[&(1-\theta)(\omega_2 - \omega_1)^2/4 \\
&-2\lambda u(v - u)\Big] S(v) dv,
\end{aligned}$$

$$\frac{\partial \eta_S}{\partial q}(\rho, u) = \frac{1}{\rho}\int c_{\gamma,\kappa}[(\omega_2 - v)(v - \omega_1)]_+^{\lambda-1} 2\lambda (v - u) S(v) dv.$$

Next, we write

$$\begin{aligned}
M_0(\rho, u, \xi) &= c_{\gamma,\kappa}\left[(\omega_2 - \xi)(\xi - \omega_1)\right]_+^\lambda, \\
M_1(\rho, u, \xi) &= \left[(1-\theta)\frac{\omega_1 + \omega_2}{2} + \theta\xi\right] M_0(\rho, u, \xi),
\end{aligned}$$

and for $\omega_1 < \xi < \omega_2$, we get

$$\begin{aligned}
\frac{\partial M_0}{\partial \omega_1} &= -\lambda c_{\gamma,\kappa}(\omega_2 - \xi)^\lambda (\xi - \omega_1)^{\lambda-1}, \\
\frac{\partial M_0}{\partial \omega_2} &= \lambda c_{\gamma,\kappa}(\omega_2 - \xi)^{\lambda-1} (\xi - \omega_1)^{\lambda}, \\
\frac{\partial M_1}{\partial \omega_1} &= -\lambda c_{\gamma,\kappa}(\omega_2 - \xi)^\lambda (\xi - \omega_1)^{\lambda-1}\left[u - \theta(\omega_2 - \omega_1)/2\right], \\
\frac{\partial M_1}{\partial \omega_2} &= \lambda c_{\gamma,\kappa}(\omega_2 - \xi)^{\lambda-1} (\xi - \omega_1)^{\lambda}\left[u + \theta(\omega_2 - \omega_1)/2\right].
\end{aligned}$$

Finally, a computation gives that for $\omega_1 < \xi < \omega_2$,

$$\frac{\partial \eta_S}{\partial \rho}(\rho,u)\frac{\partial M_0}{\partial \omega_i}(\rho,u,\xi) + \frac{\partial \eta_S}{\partial q}(\rho,u)\frac{\partial M_1}{\partial \omega_i}(\rho,u,\xi) = \frac{\partial G_S}{\partial \omega_i}(\rho,u,\xi), \quad i=1,2.$$

This formula can be written equivalently

$$\eta_S'(\rho,u)M'(\rho,u,\xi) = G_S'(\rho,u,\xi), \quad \omega_1 < \xi < \omega_2.$$

Since for $\omega_1 < \xi < \omega_2$, $G_S(\rho,u,\xi) = H_S(M(\rho,u,\xi),\xi)$ and $M'(\rho,u,\xi)$ is invertible, this gives the result. $\square$

Corollary 3.4 *If $S:\mathbb{R}\to\mathbb{R}$ is of class C^k then $H_S(.,\xi)$ is C^k in $\{f_0>0\}$.*

The main result of this section is the following.

Proposition 3.5
i) If $S:\mathbb{R}\to\mathbb{R}$ is convex, of class C^2, then η_S is convex in $\{\rho>0\}$, and if $S''>0$ then $\eta_S''>0$.
ii) If $S:\mathbb{R}\to\mathbb{R}$ is convex and continuous, then $H_S(.,\xi)$ is convex in D.

Proof. The $i)$ was proved in [10]. Indeed, from (3.4) we get

$$\begin{aligned}&\eta_S''(\rho,u)\cdot X\cdot X\\ &= \frac{1}{J_\lambda}\int_{-1}^{1}(1-z^2)^\lambda\left\{\gamma\kappa\rho^{\gamma-2}(1-z^2)X_0^2 + \frac{(X_1+m(z)X_0)^2}{\rho}\right\}S''(l(z))\,dz,\end{aligned} \tag{3.6}$$

which yields the result.

We turn to the $ii)$. We can assume that S is C^2 and $S''>0$, since the general case of a continuous convex S can easily be obtained by approximating S by smooth strictly convex functions.

Let us first prove that H_S is convex in $\{f_0>0\}$. Since by Corollary 3.4, H_S is C^2 there, and according to Proposition 3.3, $H_S'(f,\xi) = \eta_S'(M^{-1}(f,\xi))$, we get by differentiation that $H_S''(f,\xi) = \eta_S''(M^{-1}(f,\xi))(M^{-1})'(f,\xi)$ for $f_0>0$. Then, we can apply Lemma 3.6 below with $B=(M^{-1})'$, $C=\eta_S''$, $D=H_S''$. We need only to prove that the eigenvalues of M' are positive, and this was done in [4].

Finally, we need to extend this convexity around 0. For $f,g\in D\setminus\{0\}$, $0<\vartheta<1$, we have the convexity inequality $H_S((1-\vartheta)f+\vartheta g,\xi) \le (1-\vartheta)H_S(f,\xi)+\vartheta H_S(g,\xi)$. Now letting f tend to 0 in the cones, $iii)$ of Lemma 3.2 gives $H_S(\vartheta g,\xi)\le\vartheta H_S(g,\xi)$, and it yields that H_S is convex in D. $\square$

Lemma 3.6 *If $D=CB$ in $\mathcal{M}_n(\mathbb{R})$ with D symmetric, C symmetric positive definite, and $\sigma(B)\subset\mathbb{R}^+$, then D is positive semidefinite.*

Proof. C defines a scalar product in $\mathbb{R}^n$ by $(X|Y)=X^tCY$. Since $D=CB$ is symmetric, we have $(Y|BX)=Y^tDX=X^tDY=(X|BY)$, hence B is self-adjoint for $(.|.)$ and thus diagonalizable. Then, $(X|BX)=X^tDX$ gives that $\sigma(B)\subset\mathbb{R}^+$ if and only if D is positive semidefinite. $\square$

4 The Key convexity inequality

We need now to generalize Proposition 2.4 of [3].

Proposition 4.1 *If $S : \mathbb{R} \to \mathbb{R}$ is convex, of class C^1, then for every $f \in D$, $\rho \geq 0$ and $u, \xi \in \mathbb{R}$, we have*

$$H_S(f,\xi) \geq H_S(M(\rho,u,\xi),\xi) + T_S(\rho,u) \cdot (f - M(\rho,u,\xi)), \tag{4.1}$$

with

$$T_S(\rho,u) = \frac{1}{J_\lambda} \int_{-1}^{1} (1-z^2)^\lambda \begin{pmatrix} S(l(z)) + m(z)S'(l(z)) \\ S'(l(z)) \end{pmatrix} dz, \tag{4.2}$$

which coincides with $\eta'_S(\rho,u)$ when $\rho > 0$.

We recall from (2.12) and (2.13) that $l(z) = u + \sqrt{A_\gamma}\rho^{(\gamma-1)/2}z$, and $m(z) = -u + \frac{\gamma-1}{2}\sqrt{A_\gamma}\rho^{(\gamma-1)/2}z$. We remark that $T_{v\mapsto v^2/2} = T$. Let us first state two lemmas that are left to the reader.

Lemma 4.2 $H_{v\mapsto 1}(f,\xi) = f_0$, $H_{Id}(f,\xi) = f_1$, $H_{v\mapsto v^2/2}(f,\xi) = H(f,\xi)$.

Lemma 4.3 *For $S : \mathbb{R} \to \mathbb{R}$ of class C^1, we have for any fixed $u \in \mathbb{R}$, $\eta'_S(\rho,u) = T_S(\rho,u) \underset{\rho\to 0+}{\to} T_S(0,u)$.*

Proof of Proposition 4.1. We only treat the case when S is of class C^2 such that $S'' \geq \mathcal{C} > 0$, since the general case can easily be deduced by approximation of S.

*If $M_0(\rho,u,\xi) \neq 0$.
Then $T_S(\rho,u) = \eta'_S(\rho,u)$. The convexity of H_S (Proposition 3.5) together with the identity of Proposition 3.3 give the result.

*If $M_0(\rho,u,\xi) = 0$ and $\rho \neq 0$.
Let us consider $\psi : D \mapsto \mathbb{R}$ defined by

$$\begin{aligned} \psi(f) &= H_S(f,\xi) - H_S(M(\rho,u,\xi),\xi) - T_S(\rho,u)\cdot(f - M(\rho,u,\xi)) \\ &= H_S(f,\xi) - T_S(\rho,u)\cdot f. \end{aligned}$$

We need to prove that ψ is nonnegative.
– At infinity,
We have $S(v) \geq a + bv + \mathcal{C}v^2/2$, thus from the definition (2.6) of H_S and since $\Phi \geq 0$, we obtain with Lemma 4.2

$$H_S(f,\xi) \geq af_0 + bf_1 + \mathcal{C}H(f,\xi). \tag{4.3}$$

Now $f \mapsto af_0 + bf_1 - T_S(\rho,u)\cdot f$ is linear and $\mathcal{C} > 0$ thus by the superlinearity of H (Lemma 2.3 of [3]),

$$H(f,\xi) \geq \varepsilon_0 f_0^{p_0} + \varepsilon_1 |f_1|^{p_1},$$

with $p_0 = 1 + 1/\lambda$, $p_1 = 2(1+\lambda)/(1+2\lambda)$, we obtain that $\psi(f) \underset{f\to\infty}{\to} \infty$. Moreover, since $H \geq 0$, we deduce with (4.3) that ψ is lower bounded in D.

– At the frontier $\{f_0 = 0,\ f_1 \neq 0\}$,
The inequality (4.3) gives that $\psi(f) \to \infty$.
– At 0,
From (4.3) again and since $H \geq 0$, we obtain $\underline{\lim}_{f \to 0} \psi(f) \geq 0$.
– Inside D,
It only remains to study the possible minima of ψ in $int(D)$. By Proposition 3.3, ψ is C^1 there, and $\psi'(f) = \eta_S'(\rho(f,\xi), u(f,\xi)) - \eta_S'(\rho, u)$. From the identity

$$\Big(\eta_S'(Y) - \eta_S'(X)\Big)(Y - X) = \int_0^1 \eta_S''((1-\vartheta)X + \vartheta Y) \cdot (Y - X) \cdot (Y - X)\, d\vartheta,$$

and since η_S'' is positive definite, we conclude that if at some point f with $f_0 > 0$, $\psi'(f) = 0$, we must have $\rho(f,\xi) = \rho$, $u(f,\xi) = u$, and therefore $f = M(\rho, u, \xi)$. But since $M(\rho, u, \xi) = 0$, we get a contradiction. Therefore, there is no interior point that is a minimum, and we conclude that $\psi \geq 0$ in D.

*If $\rho = 0$.
We now know from the previous steps that (4.1) holds as soon as $\rho > 0$. Taking f, u and ξ fixed, we let $\rho \to 0$. We see obviously that $M(\rho, u, \xi) \to 0$, while remaining in a cone, and with Lemma 3.2 we get that $H_S(M(\rho, u, \xi), \xi) \to 0$. With Lemma 4.3, we can pass to the limit $\rho \to 0$ and get the result. □

We deduce obviously as in Proposition 2.5 of [3] an entropy minimization principle.

Corollary 4.4 *Assume that $S : \mathbb{R} \to \mathbb{R}$ is convex, of class C^1 and such that $|S(v)| \leq B(1 + v^2)$ for some $B \geq 0$. Consider $f \in L^1(\mathbb{R}_\xi)$ such that $f \in D$ a.e. and*

$$\int_{\mathbb{R}} H(f(\xi), \xi)\, d\xi < \infty. \tag{4.4}$$

Then $H_S(f(\xi), \xi)$ and $H_S(M[f](\xi), \xi)$ lie in $L^1(\mathbb{R}_\xi)$ with

$$\int_{\mathbb{R}} H_S(M[f](\xi), \xi)\, d\xi \leq \int_{\mathbb{R}} H_S(f(\xi), \xi)\, d\xi. \tag{4.5}$$

Proof. Since $\Phi \geq 0$, we get from definition (2.6) that

$$|H_S(f, \xi)| \leq B(f_0 + 2H(f, \xi)), \tag{4.6}$$

thus $\int |H_S(f(\xi), \xi)| d\xi < \infty$, and $\int |H_S(M[f](\xi), \xi)| d\xi < \infty$. Then, integrating the inequality of Proposition 4.1 with respect to ξ, with $(\rho, \rho u) = \int f(\xi) d\xi$ gives the result. □

Another important consequence of Proposition 4.1 is

Proposition 4.5 *Assume that $S : \mathbb{R} \to \mathbb{R}$ is convex, of class C^1 and such that $|S(v)| \leq B(1 + v^2)$ for some $B \geq 0$. Then, with f the solution of Theorem 1.1,*

$$\iint_{\mathbb{R}\times\mathbb{R}} H_S(f(t, x, \xi), \xi)\, dx d\xi \leq \iint_{\mathbb{R}\times\mathbb{R}} H_S(f^0(x, \xi), \xi)\, dx d\xi. \tag{4.7}$$

Proof. We apply Corollary 4.4 at fixed t and for a.e. x. We obtain

$$\int_{\mathbb{R}} H_S(M[f](t,x,\xi),\xi)\,d\xi \le \int_{\mathbb{R}} H_S(f(t,x,\xi),\xi)\,d\xi \qquad \forall t \ge 0,\ a.e.\ x \in \mathbb{R}.$$

Then, because of (4.6), we have obviously

$$\iint |H_S(f(t,x,\xi),\xi)| dx d\xi < \infty, \qquad \iint |H_S(M[f](t,x,\xi),\xi)| dx d\xi < \infty,$$

and therefore

$$\iint_{\mathbb{R}\times\mathbb{R}} H_S(M[f](t,x,\xi),\xi)\,dxd\xi \le \iint_{\mathbb{R}\times\mathbb{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi, \qquad \forall t \ge 0. \quad (4.8)$$

But we have a representation of the solution f,

$$f(t,x,\xi) = f^0(x-t\xi,\xi)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} M[f](t-s,x-s\xi,\xi)\,ds.$$

According to Jensen's inequality with the convex function H_S, we deduce as in [3] that

$$\begin{aligned} H_S(f(t,x,\xi),\xi) \le{} & H_S(f^0(x-t\xi,\xi),\xi)e^{-t/\varepsilon} \\ & + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} H_S(M[f](t-s,x-s\xi,\xi),\xi)\,ds. \end{aligned}$$

By integration in x and ξ, we obtain

$$\begin{aligned} & \iint_{\mathbb{R}\times\mathbb{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi \\ \le & \iint_{\mathbb{R}\times\mathbb{R}} H_S(f^0(x-t\xi,\xi),\xi)\,dxd\xi\, e^{-t/\varepsilon} \\ & \qquad + \frac{1}{\varepsilon}\int_0^t e^{-s/\varepsilon} \iint_{\mathbb{R}\times\mathbb{R}} H_S(M[f](t-s,x-s\xi,\xi),\xi)\,dxd\xi\,ds \\ = & \iint_{\mathbb{R}\times\mathbb{R}} H_S(f^0(x,\xi),\xi)\,dxd\xi\, e^{-t/\varepsilon} \\ & \qquad + \frac{1}{\varepsilon}\int_0^t e^{-(t-s)/\varepsilon} \iint_{\mathbb{R}\times\mathbb{R}} H_S(M[f](s,x,\xi),\xi)\,dxd\xi\,ds, \end{aligned}$$

and setting

$$F_S(t) = \iint_{\mathbb{R}\times\mathbb{R}} H_S(f(t,x,\xi),\xi)\,dxd\xi,$$

we get with (4.8)

$$F_S(t) \le F_S(0)e^{-t/\varepsilon} + \frac{1}{\varepsilon}\int_0^t e^{-(t-s)/\varepsilon} F_S(s)\,ds.$$

Gronwall's lemma finally gives $F_S(t) \le F_S(0)$ and our result. $\square$

5 Maximum principle

This short section is devoted to the proof of Theorem 1.4 giving the maximum principle on the Riemann invariants.

Proof of Theorem 1.4. We have first that $\tilde{D}_\xi$ is convex. Indeed, using $S_M(v) = (v - \omega_{max})_+^2$ and $S_m(v) = (\omega_{min} - v)_+^2$ which are convex and C^1, we notice that $H_{S_M} \geq 0$, $H_{S_m} \geq 0$, and we have for a fixed ξ

$$\begin{aligned} H_{S_M}(f,\xi) \leq 0 \quad &\Leftrightarrow \quad f = 0 \textit{ or } \Phi(\rho(f,\xi), u(f,\xi), \xi, v)(v - \omega_{max})_+^2 = 0 \textit{ a.e. } v \\ &\Leftrightarrow \quad f = 0 \textit{ or } (v - \omega_{max})_+ = 0 \textit{ a.e. in }]\omega_1(f,\xi); \omega_2(f,\xi)[\\ &\Leftrightarrow \quad f = 0 \textit{ or } \omega_2(f,\xi) \leq \omega_{max}. \end{aligned}$$

A similar result holds for S_m, thus finally

$$(H_{S_M}(f,\xi) \leq 0 \text{ and } H_{S_m}(f,\xi) \leq 0) \Leftrightarrow f \in \tilde{D}_\xi, \tag{5.1}$$

and we conclude with Proposition 3.5 that $\tilde{D}_\xi$ is convex.

Let us now denote by $f(t,x,\xi)$ the solution of Theorem 1.1, with $f^0(x,\xi) \in \tilde{D}_\xi$. Then by Proposition 4.5 $H_{S_M}(f(t,x,\xi),\xi) = H_{S_m}(f(t,x,\xi),\xi) = 0$ $\forall t \geq 0$, *a.e.* x, ξ. This proves that $f \in \tilde{D}_\xi$ $\forall t \geq 0$ and $\tilde{D}_\xi$ is kinetic invariant.

We turn now to properties (1.33) and (1.34). Let $(\rho, u) \in \tilde{D}$. Then either $M(\rho,u,\xi) = 0 \in \tilde{D}_\xi$, or $M(\rho,u,\xi) \neq 0$, and in this case obviously $\omega_i(M(\rho,u,\xi),\xi) = \omega_i(\rho,u)$ for $i = 1; 2$. Therefore, $M(\rho,u,\xi) \in \tilde{D}_\xi$ also since $(\rho,u) \in \tilde{D}$ and $\rho > 0$. Thus (1.33) is proved.

Let now $f(\xi) \in L^1_\xi$ such that $f(\xi) \in \tilde{D}_\xi$ *a.e.* ξ. Setting $(\rho, \rho u) = \int_{\mathbb{R}} f(\xi)\, d\xi$, we have by integrating (4.1) for $S = S_M$ or S_m

$$0 \leq \int_{\mathbb{R}} H_S(M[f](\xi),\xi)\, d\xi \leq \int_{\mathbb{R}} H_S(f(\xi),\xi)\, d\xi = 0,$$

thus $H_S(M[f](\xi),\xi) = 0$ *a.e.* ξ. Therefore $M[f](\xi) = M(\rho,u,\xi) \in \tilde{D}_\xi$ for a.e. ξ. Then, either $\rho = 0$ and $(\rho,u) \in \tilde{D}$, or $\rho > 0$, and in this case, for $\omega_1(\rho,u) < \xi < \omega_2(\rho,u)$ we have $\omega_i(M[f](\xi),\xi) = \omega_i(\rho,u)$ for $i = 1; 2$, thus $\omega_{min} \leq \omega_1(\rho,u) \leq \omega_2(\rho,u) \leq \omega_{max}$, and $(\rho,u) \in \tilde{D}$, that is (1.34).

Finally, the compact support property comes directly from the fact that if $\xi \notin]\omega_{min}, \omega_{max}[$, we have $\tilde{D}_\xi = \{0\}$. Indeed, let us suppose that $\xi \notin]\omega_{min}, \omega_{max}[$, and let $f \in \tilde{D}_\xi$. If $f \neq 0$ then according to Lemma 3.1 $\omega_{min} \leq \omega_1(f,\xi) < \xi < \omega_2(f,\xi) \leq \omega_{max}$, which is impossible, thus $f = 0$. $\square$

6 Relaxation limit

Let f_ε be the solution of Theorem 1.1 with the same initial data $f^0(x,\xi)$ and $(\rho_\varepsilon, u_\varepsilon)$ be the approximate solutions to (1.1) defined by (1.6). In order to prove the convergence, as ε tends to 0, of ρ_ε and u_ε, we are going to use the stability result of [9]. Thus we need to prove that

$$\partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)] \text{ is compact in } H^{-1}_{loc}, \tag{6.1}$$

for any suitable entropy η_S and entropy flux G_S (see Proposition 6.5 for a more precise result). In order to obtain this, we shall need to prove first that $M[f_\varepsilon]-f_\varepsilon$ tends to 0 strongly. We refer to [1] and [2] and the references therein for other situations where a similar result can be obtained. Let us begin by an a priori bound.

Proposition 6.1 *If we suppose that $f^0(x,\xi)\in\tilde{D}_\xi$ a.e. x,ξ, then f_ε, $M[f_\varepsilon]\in\tilde{D}_\xi$ for any $t>0$ and a.e. x,ξ, and ρ_ε, u_ε, f_ε, $M[f_\varepsilon]$ are uniformly bounded in L^∞.*

Proof. This is an immediate consequence of the maximum principle stated in Theorem 1.4 and of formula (3.3). □

In all this section, we shall suppose that

$$f^0(x,\xi)\in\tilde{D}_\xi \text{ for a.e. } x,\ \xi, \tag{6.2}$$

and we shall denote by F the a priori L^∞ bound. Thus we have

$$|f_\varepsilon|\le F, \qquad |M[f_\varepsilon]|\le F, \tag{6.3}$$

$$\mathrm{supp}_\xi f_\varepsilon\subset[\omega_{min},\omega_{max}], \qquad \mathrm{supp}_\xi M[f_\varepsilon]\subset[\omega_{min},\omega_{max}]. \tag{6.4}$$

From the definition of $\tilde{D}_\xi$, we see that we have also for some constant A the bounds

$$|(f_\varepsilon)_1|\le A(f_\varepsilon)_0, \qquad |(M[f_\varepsilon])_1|\le A(M[f_\varepsilon])_0. \tag{6.5}$$

Proposition 6.2 *After extraction of a subsequence, $f_\varepsilon-M[f_\varepsilon]\underset{\varepsilon\to 0}{\to}0$ a.e. t,x,ξ.*

Proof. From the entropy dissipation identity

$$\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f_\varepsilon(T,x,\xi),\xi)\,dxd\xi+\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}}(H'(f_\varepsilon,\xi)-T(\rho_\varepsilon,u_\varepsilon))\cdot\frac{f_\varepsilon-M[f_\varepsilon]}{\varepsilon}\,dtdxd\xi$$
$$=\iint\limits_{\mathbb{R}\times\mathbb{R}} H(f^0(x,\xi),\xi)\,dxd\xi,$$

and since $H\ge 0$ and $(H'(f_\varepsilon,\xi)-T(\rho_\varepsilon,u_\varepsilon))\cdot(f_\varepsilon-M[f_\varepsilon])\ge 0$, we get

$$\iiint\limits_{]0,T[\times\mathbb{R}\times\mathbb{R}}(H'(f_\varepsilon,\xi)-T(\rho_\varepsilon,u_\varepsilon))\cdot(f_\varepsilon-M[f_\varepsilon])\,dtdxd\xi\le C_H^0\varepsilon,$$

which yields that $(H'(f_\varepsilon,\xi)-T(\rho_\varepsilon,u_\varepsilon))\cdot(f_\varepsilon-M[f_\varepsilon])\to 0$ in $L^1(]0,T[\times\mathbb{R}\times\mathbb{R})$. Then, extracting a subsequence if necessary,

$$(H'(f_\varepsilon,\xi)-T(\rho_\varepsilon,u_\varepsilon))\cdot(f_\varepsilon-M[f_\varepsilon])\underset{\varepsilon\to 0}{\to}0 \quad \textit{a.e. } t,x,\xi. \tag{6.6}$$

Since by Proposition 3.3 $T(\rho_\varepsilon,u_\varepsilon)=H'(M[f_\varepsilon],\xi)$ whenever $M[f_\varepsilon]\neq 0$, in order to deduce the result, we need to get somehow a lower estimate on H''. We notice that for $X=(X_0,X_1)\in D\setminus\{0\}$, $Y=(Y_0,Y_1)\in\mathbb{R}^2$,

$$H''(X,\xi)\cdot Y\cdot Y=K_1X_0^{1/\lambda-1}Y_0^2+K_2(X_1Y_0-X_0Y_1)^2/X_0^3, \tag{6.7}$$

with $K_1 = \theta/2\lambda c_{\gamma,\kappa}^{1/\lambda}$ and $K_2 = 1/(1-\theta)$.

First step: let us prove that for $\lambda \geq 1$, $(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0$ *a.e.* t, x, ξ. Consider a point (t,x,ξ) where $(M[f_\varepsilon])_0(t,x,\xi) \neq 0$ and $(f_\varepsilon)_0(t,x,\xi) \neq 0$. We set $X(\vartheta) = (1-\vartheta)f_\varepsilon + \vartheta M[f_\varepsilon]$. For every $\vartheta \in [0,1]$, $X(\vartheta)_0^{1/\lambda - 1} \geq F^{1/\lambda-1}$ because $\lambda \geq 1$, thus with (6.7)

$$\begin{aligned}
&(H'(M[f_\varepsilon],\xi) - H'(f_\varepsilon,\xi)) \cdot (M[f_\varepsilon] - f_\varepsilon) \\
&= \int_0^1 H''(X(\vartheta),\xi) \cdot (M[f_\varepsilon] - f_\varepsilon)^2 \, d\vartheta \\
&\geq K_1 F^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2 .
\end{aligned} \tag{6.8}$$

Taking into account (1.25), we deduce that

$$\begin{aligned}
& \mathbb{1}_{(M[f_\varepsilon])_0 \neq 0}(H'(M[f_\varepsilon],\xi) - H'(f_\varepsilon,\xi)) \cdot (M[f_\varepsilon] - f_\varepsilon) \\
\geq \; & \mathbb{1}_{(M[f_\varepsilon])_0 \neq 0} K_1 F^{1/\lambda-1}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2 \quad \text{a.e.,}
\end{aligned}$$

and together with the entropy dissipation (6.6) we conclude that

$$\mathbb{1}_{(M[f_\varepsilon])_0 \neq 0}((M[f_\varepsilon])_0 - (f_\varepsilon)_0)^2 \underset{\varepsilon\to 0}{\to} 0 \quad \textit{a.e. } t,x,\xi. \tag{6.9}$$

But $\int_{\mathbb{R}} (f_\varepsilon)_0 \, d\xi = \int_{\mathbb{R}} (M[f_\varepsilon])_0 \, d\xi$ *a.e.* t, x, thus for any bounded domain B in (t,x),

$$\begin{aligned}
& \iiint\limits_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 \neq 0\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \, dt dx d\xi \\
= & \iiint\limits_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} ((f_\varepsilon)_0 - (M[f_\varepsilon])_0) \, dt dx d\xi \\
= & \iiint\limits_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} (f_\varepsilon)_0 \, dt dx d\xi \\
= & \iiint\limits_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0| \, dt dx d\xi .
\end{aligned}$$

But by dominated convergence and with (6.9),

$$\iiint\limits_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 \neq 0\}} ((M[f_\varepsilon])_0 - (f_\varepsilon)_0) \, dt dx d\xi \underset{\varepsilon\to 0}{\to} 0,$$

and we get

$$\iiint\limits_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0 = 0\}} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0| \, dt dx d\xi \underset{\varepsilon\to 0}{\to} 0.$$

Extracting a subsequence if necessary, we conclude that

$$\mathbb{1}_{(M[f_\varepsilon])_0=0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t,x,\xi. \tag{6.10}$$

Finally, (6.9) and (6.10) give that

$$(f_\varepsilon)_0-(M[f_\varepsilon])_0\underset{\varepsilon\to 0}{\to} 0 \qquad a.e.\ t,x,\xi.$$

Second step: we prove that for $\lambda<1$, $(f_\varepsilon)_0-(M[f_\varepsilon])_0\underset{\varepsilon\to 0}{\to} 0$ $a.e.\ t,x,\xi$. Consider a point (t,x,ξ) where $(M[f_\varepsilon])_0(t,x,\xi)\neq 0$ and $(f_\varepsilon)_0(t,x,\xi)\neq 0$. We set $X(\vartheta)=(1-\vartheta)f_\varepsilon+\vartheta M[f_\varepsilon]$. For every $\vartheta\in[\frac{1}{2},1]$, $X(\vartheta)_0^{1/\lambda-1}\geq(1/2)^{1/\lambda-1}(M[f_\varepsilon])_0^{1/\lambda-1}$ because $\lambda<1$. Thus

$$\begin{aligned}
&(H'(M[f_\varepsilon],\xi)-H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon)\\
&\geq\int_{\frac{1}{2}}^{1}K_1X(\vartheta)_0^{1/\lambda-1}((M[f_\varepsilon])_0-(f_\varepsilon)_0)^2\,d\vartheta\\
&\geq K_1(1/2)^{1/\lambda}(M[f_\varepsilon])_0^{1/\lambda-1}((M[f_\varepsilon])_0-(f_\varepsilon)_0)^2,
\end{aligned}$$

and we deduce with (1.25) that for any $\alpha>0$

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0>\alpha}(H'(M[f_\varepsilon],\xi)-H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon)\\
&\geq K_1(1/2)^{1/\lambda}\alpha^{1/\lambda-1}((M[f_\varepsilon])_0-(f_\varepsilon)_0)^2\mathbb{1}_{(M[f_\varepsilon])_0>\alpha}\quad\text{a.e.}
\end{aligned}\tag{6.11}$$

Together with (6.6), we deduce the convergence

$$\mathbb{1}_{(M[f_\varepsilon])_0>\alpha}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\underset{\varepsilon\to 0}{\to} 0 \quad a.e.\ t,x,\xi, \tag{6.12}$$

for every $\alpha>0$. In order to treat the domain where $(M[f_\varepsilon])_0\leq\alpha$, we use the conservation relation $\int_{\mathbb{R}}(f_\varepsilon)_0\,d\xi=\int_{\mathbb{R}}(M[f_\varepsilon])_0\,d\xi$ $a.e.\ t,x$, which gives

$$\begin{aligned}
&\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}}(f_\varepsilon)_0\,dtdxd\xi\\
=&\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0>\alpha\}}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\,dtdxd\xi+\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}}(M[f_\varepsilon])_0\,dtdxd\xi.
\end{aligned}$$

Now, since $(M[f_\varepsilon])_0\leq\alpha$ in the last integral, we can choose an α small enough so that this integral is less than an arbitrarily given $\mu>0$. Then, from (6.12), the first integral can also be made less than μ if ε is small enough. Therefore, for such α and ε,

$$\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}}(f_\varepsilon)_0\,dtdxd\xi\leq 2\mu,$$

$$\iiint_{\{(t,x)\in B\}\cap\{(M[f_\varepsilon])_0\leq\alpha\}}\Big((f_\varepsilon)_0+(M[f_\varepsilon])_0\Big)\,dtdxd\xi\leq 3\mu.$$

Putting this together with (6.12), we conclude that for ε small enough,

$$\iiint\limits_{(t,x)\in B} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \le 4\mu.$$

This means that

$$\iiint\limits_{(t,x)\in B} |(f_\varepsilon)_0 - (M[f_\varepsilon])_0|\, dtdxd\xi \longrightarrow 0 \qquad \text{as } \varepsilon\to 0,$$

and extracting a subsequence if necessary, we get that

$$(f_\varepsilon)_0 - (M[f_\varepsilon])_0 \underset{\varepsilon\to 0}{\to} 0 \qquad a.e.\ t,x,\xi. \tag{6.13}$$

Third step: we prove that $(f_\varepsilon)_1 - (M[f_\varepsilon])_1 \underset{\varepsilon\to 0}{\to} 0$ $a.e.\ t,x,\xi$, for every $\lambda>0$. Consider a point (t,x,ξ) where $(M[f_\varepsilon])_0(t,x,\xi)\neq 0$ and $(f_\varepsilon)_0(t,x,\xi)\neq 0$. By using the lower estimate from (6.7) $H''(X,\xi)\cdot Y\cdot Y \ge K_2(X_1Y_0-X_0Y_1)^2/X_0^3$ with $X=X(\vartheta)=(1-\vartheta)f_\varepsilon+\vartheta M[f_\varepsilon]$ and $Y=M[f_\varepsilon]-f_\varepsilon$, we get

$$\begin{aligned}&(H'(M[f_\varepsilon],\xi)-H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon)\\ \ge\ & K_2\int_0^1 \frac{1}{X(\vartheta)_0^3}(X(\vartheta)_1Y_0-X(\vartheta)_0Y_1)^2\,d\vartheta.\end{aligned}$$

But $X(\vartheta)_1Y_0-X(\vartheta)_0Y_1=(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1$, and for $\vartheta\in[0,1]$, we have the two upper estimates $X(\vartheta)_0\le (f_\varepsilon)_0+(M[f_\varepsilon])_0$ and $X(\vartheta)_0\le F$, thus $X(\vartheta)_0^3\le F((f_\varepsilon)_0+(M[f_\varepsilon])_0)^2$. Therefore we obtain

$$\begin{aligned}&\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;(f_\varepsilon)_0\neq 0}(H'(M[f_\varepsilon],\xi)-H'(f_\varepsilon,\xi))\cdot(M[f_\varepsilon]-f_\varepsilon)\\ \ge\ & \frac{K_2}{F}\left(\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0+(M[f_\varepsilon])_0}\right)^2\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;(f_\varepsilon)_0\neq 0},\end{aligned}$$

and we conclude with (6.6) that

$$\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0+(M[f_\varepsilon])_0} \underset{\varepsilon\to 0}{\to} 0 \ \ a.e.\ t,x,\xi. \tag{6.14}$$

Then, we remark that

$$\begin{aligned}&\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}((f_\varepsilon)_1-(M[f_\varepsilon])_1)\\ &=\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}\\ &\quad-\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\frac{(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0},\end{aligned}\tag{6.15}$$

and that

$$\begin{aligned}&\mathbb{1}_{(f_\varepsilon)_0\neq 0}((f_\varepsilon)_1-(M[f_\varepsilon])_1)\\ &=\mathbb{1}_{(f_\varepsilon)_0\neq 0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0}\\ &\quad-\mathbb{1}_{(f_\varepsilon)_0\neq 0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}.\end{aligned}\tag{6.16}$$

Multiplying (6.15) by $\mathbb{1}_{(f_\varepsilon)_0\neq 0}\mathbb{1}_{(f_\varepsilon)_0\leq (M[f_\varepsilon])_0}$, (6.16) by $\mathbb{1}_{(M[f_\varepsilon])_0\neq 0}\ \mathbb{1}_{(M[f_\varepsilon])_0<(f_\varepsilon)_0}$, and then adding the two, we get

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;(f_\varepsilon)_0\neq 0}((f_\varepsilon)_1-(M[f_\varepsilon])_1)\\
&=\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (f_\varepsilon)_0\leq (M[f_\varepsilon])_0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}\\
&\quad+\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (M[f_\varepsilon])_0<(f_\varepsilon)_0}\frac{(f_\varepsilon)_1(M[f_\varepsilon])_0-(f_\varepsilon)_0(M[f_\varepsilon])_1}{(f_\varepsilon)_0}\\
&\quad-\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (f_\varepsilon)_0\leq (M[f_\varepsilon])_0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\frac{(M[f_\varepsilon])_1}{(M[f_\varepsilon])_0}\\
&\quad-\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0;\ (M[f_\varepsilon])_0<(f_\varepsilon)_0}((M[f_\varepsilon])_0-(f_\varepsilon)_0)\frac{(f_\varepsilon)_1}{(f_\varepsilon)_0}.
\end{aligned}\tag{6.17}$$

From (6.5) and (6.13), the last two terms tend to 0 a.e. But according to (6.14), the two first terms also tend to 0 a.e., and we conclude that

$$\mathbb{1}_{(M[f_\varepsilon])_0\neq 0;\ (f_\varepsilon)_0\neq 0}((f_\varepsilon)_1-(M[f_\varepsilon])_1)\underset{\varepsilon\to 0}{\to} 0\quad a.e.\ t,x,\xi.$$

Then,

$$\begin{aligned}
&\mathbb{1}_{(M[f_\varepsilon])_0=0}|(f_\varepsilon)_1-(M[f_\varepsilon])_1|\\
&=\mathbb{1}_{(M[f_\varepsilon])_0=0}|(f_\varepsilon)_1|\\
&\leq A\mathbb{1}_{(M[f_\varepsilon])_0=0}(f_\varepsilon)_0\\
&=A\mathbb{1}_{(M[f_\varepsilon])_0=0}((f_\varepsilon)_0-(M[f_\varepsilon])_0)\underset{\varepsilon\to 0}{\to} 0\quad a.e.
\end{aligned}\tag{6.18}$$

We perform a similar computation for $\mathbb{1}_{(f_\varepsilon)_0=0}|(f_\varepsilon)_1-(M[f_\varepsilon])_1|$, and we obtain finally

$$(f_\varepsilon)_1-(M[f_\varepsilon])_1\underset{\varepsilon\to 0}{\to} 0\quad a.e.\ t,x,\xi,$$

which completes the proof of Proposition 6.2. □

Proposition 6.3 *For $S:\mathbb{R}\to\mathbb{R}$ convex, of class C^1, such that $|S(v)|\leq B(1+v^2)$ for some $B\geq 0$, we have*

$$\partial_t\int_{\mathbb{R}}H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi+\partial_x\int_{\mathbb{R}}\xi H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi\leq 0,\tag{6.19}$$

with equality if $S(v)=1$ or v.

Proof. Proposition 4.5 gives that $H_S(f_\varepsilon,\xi)\in L^\infty(]0,T[,L^1(\mathbb{R}\times\mathbb{R}))$. According to Proposition 3.3 and Proposition 6.1, $H_S'(f_\varepsilon,\xi)\in L^\infty$, since $H_S'(f_\varepsilon,\xi)=\eta_S'(\rho(f_\varepsilon,\xi),u(f_\varepsilon,\xi))$ whenever $(f_\varepsilon)_0>0$ (see (4.2)). Thus, according to [5],

$$\partial_t(H_S(f_\varepsilon,\xi))+\xi\partial_x(H_S(f_\varepsilon,\xi))=H_S'(f_\varepsilon,\xi)\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}.\tag{6.20}$$

From Proposition 4.1, we have

$$H_S(f_\varepsilon,\xi)\geq H_S(M[f_\varepsilon],\xi)+T_S(\rho_\varepsilon,u_\varepsilon)\cdot(f_\varepsilon-M[f_\varepsilon]).\tag{6.21}$$

Besides, since H_S is convex (Proposition 3.5),

$$H_S(M[f_\varepsilon],\xi) \geq H_S(f_\varepsilon,\xi) + H_S'(f_\varepsilon,\xi)\cdot(M[f_\varepsilon]-f_\varepsilon) \tag{6.22}$$

for $f_\varepsilon \neq 0$, but it is still valid where $f_\varepsilon = 0$ because there $M[f_\varepsilon] = 0$ a.e., see (1.25). Thus by combining (6.21) and (6.22) we get

$$(H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon,u_\varepsilon))\cdot(M[f_\varepsilon]-f_\varepsilon) \leq 0. \tag{6.23}$$

Therefore, we obtain by integration of (6.20) with respect to ξ

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}} H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi + \partial_x \int_{\mathbb{R}} \xi H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi\\
&= \int_{\mathbb{R}} H_S'(f_\varepsilon,\xi)\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,d\xi\\
&= \int_{\mathbb{R}} (H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon,u_\varepsilon))\cdot\frac{M[f_\varepsilon]-f_\varepsilon}{\varepsilon}\,d\xi \;\leq\; 0.
\end{aligned} \tag{6.24}$$

If $S(v) = 1$ or v, then $H_S(f,\xi) = f_0$ or f_1, thus $H_S'(f,\xi) = (1,0)$ or $(0,1)$, and (6.24) becomes an equality. □

Proposition 6.4 *For $S : \mathbb{R} \to \mathbb{R}$ convex, of class C^1, such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$, we have*

$$\begin{aligned}
&\int_{\mathbb{R}} H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi - \int_{\mathbb{R}} H_S(M[f_\varepsilon](t,x,\xi),\xi)\,d\xi \underset{\varepsilon\to 0}{\to} 0,\\
&\int_{\mathbb{R}} \xi H_S(f_\varepsilon(t,x,\xi),\xi)\,d\xi - \int_{\mathbb{R}} \xi H_S(M[f_\varepsilon](t,x,\xi),\xi)\,d\xi \underset{\varepsilon\to 0}{\to} 0
\end{aligned} \tag{6.25}$$

in $L^1(]0,T[\times B_R)$ for any $R > 0$.

Proof. Integrating (6.21) and (6.22), we get

$$0 \leq \int_{\mathbb{R}} H_S(f_\varepsilon,\xi)\,d\xi - \int_{\mathbb{R}} H_S(M[f_\varepsilon],\xi)\,d\xi \leq \int_{\mathbb{R}} H_S'(f_\varepsilon,\xi)\cdot(f_\varepsilon - M[f_\varepsilon])\,d\xi,$$

which is bounded by $C\varepsilon$ in $L^1(]0,\infty[\times\mathbb{R})$ according to (6.24), and we get the first convergence of (6.25). From (6.21) and (6.22), we also have

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \xi H_S(f_\varepsilon,\xi)\,d\xi - \int_{\mathbb{R}} \xi H_S(M[f_\varepsilon],\xi)\,d\xi\right|\\
\leq\ & \int_{\mathbb{R}} \Big|\xi H_S(f_\varepsilon,\xi) - \xi H_S(M[f_\varepsilon],\xi)\Big|\,d\xi\\
\leq\ & \int_{\mathbb{R}} |\xi| \max\Big(|T_S(\rho_\varepsilon,u_\varepsilon)\cdot(f_\varepsilon - M[f_\varepsilon])|, |H_S'(f_\varepsilon,\xi)\cdot(f_\varepsilon - M[f_\varepsilon])|\Big)\,d\xi\\
\leq\ & \int_{\mathbb{R}} |\xi T_S(\rho_\varepsilon,u_\varepsilon)\cdot(f_\varepsilon - M[f_\varepsilon])|\,d\xi\\
&+ \int_{\mathbb{R}} |\xi|(H_S'(f_\varepsilon,\xi) - T_S(\rho_\varepsilon,u_\varepsilon))\cdot(f_\varepsilon - M[f_\varepsilon])\,d\xi.
\end{aligned}$$

Since ξ remains bounded, the second integral is of order ε in $L^1(]0,\infty[\times\mathbb{R})$ as before. For the first one, we conclude with Proposition 6.2 and by using that $T_S(\rho_\varepsilon,u_\varepsilon)$ is bounded in $L^\infty(]0,T[\times\mathbb{R})$ according to Proposition 6.1. Therefore we get the second convergence of (6.25). □

Proposition 6.5 *For $S : \mathbb{R} \to \mathbb{R}$ convex, of class C^1, such that $|S(v)| \leq B(1+v^2)$ for some $B \geq 0$, we have that*

$$\partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)] \leq R_{S,\varepsilon}(t,x) \tag{6.26}$$

with $R_{S,\varepsilon} \to 0$ in $W^{-1,p}_{loc}$ for any $1 < p < \infty$ as $\varepsilon \to 0$. Moreover, (6.26) becomes an equality if $S(v) = 1$ or v.

Proof. We have that

$$\int_{\mathbb{R}} H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi = \eta_S(\rho_\varepsilon, u_\varepsilon),$$

$$\int_{\mathbb{R}} \xi H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi = G_S(\rho_\varepsilon, u_\varepsilon).$$

Thus we can write

$$\begin{aligned}
& \partial_t[\eta_S(\rho_\varepsilon, u_\varepsilon)] + \partial_x[G_S(\rho_\varepsilon, u_\varepsilon)] \\
= \; & \partial_t \int_{\mathbb{R}} H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi + \partial_x \int_{\mathbb{R}} \xi H_S(M(\rho_\varepsilon, u_\varepsilon, \xi), \xi)\, d\xi \\
= \; & \partial_t W_\varepsilon + \partial_x Z_\varepsilon + \partial_t \int_{\mathbb{R}} H_S(f_\varepsilon, \xi)\, d\xi + \partial_x \int_{\mathbb{R}} \xi H_S(f_\varepsilon, \xi)\, d\xi,
\end{aligned}$$

where

$$W_\varepsilon = \int_{\mathbb{R}} H_S(M[f_\varepsilon], \xi)\, d\xi - \int_{\mathbb{R}} H_S(f_\varepsilon, \xi)\, d\xi,$$

$$Z_\varepsilon = \int_{\mathbb{R}} \xi H_S(M[f_\varepsilon], \xi)\, d\xi - \int_{\mathbb{R}} \xi H_S(f_\varepsilon, \xi)\, d\xi$$

tend to 0 in L^p_{loc} according to Proposition 6.4. With Proposition 6.3, we obtain the result. □

Finally, in order to obtain Theorem 1.5, we need only to apply the stability result of [9], which is valid for a sequence $\rho_\varepsilon, u_\varepsilon$ bounded in L^∞ satisfying the conclusion of Proposition 6.5.

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