On maximal and minimal generalized entropy solutions to Cauchy problem for a first-order quasilinear equation.

E.Yu. Panov (pey@novsu.ac.ru) [1]

Abstract

We prove existence of maximal and minimal generalized entropy solutions (g.e.s.) of the Cauchy problem for a first-order quasilinear equation in the case of only continuous flux vector, and give some usefull applications. In particular we establish uniqueness of g.e.s. for input data which are periodic with respect to some linear independent $n-1$ spatial vectors (n is number of spatial variables).

1. Introduction.

Consider Cauchy problem for a scalar first-order quasilinear equation

$$u_t + \operatorname{div}_x \varphi(u) = g(t,x), \tag{1}$$

$\varphi = (\varphi_1, \dots, \varphi_n)$, $u = u(t,x)$, $(t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n$, $T > 0$, with an initial condition

$$u(0,x) = u_0(x) \in L^\infty(\mathbb{R}^n). \tag{2}$$

The source term $g = g(t,x) \in L^1_{loc}(\Pi_T)$ in (1) must satisfy the following condition:

$$g(t,\cdot) \in L^\infty(\mathbb{R}^n) \text{ for almost all } t \in (0,T) \text{ and } \int\limits_0^T \|g(t,\cdot)\|_\infty dt < \infty. \tag{3}$$

Flux functions $\varphi_i(u)$ are supposed to be only continuous: $\varphi_i(u) \in C(\mathbb{R})$, $i = 1, \dots, n$.

Definition 1 (S.N.Kruzhkov, [8, 9]). A bounded measurable function $u = u(t,x)$ is called a generalized entropy solution (g.e.s. for short) to Cauchy problem (1), (2) if:

a) $\forall k \in \mathbb{R}$

$$|u-k|_t + \operatorname{div}_x[\operatorname{sign}(u-k)(\varphi(u) - \varphi(k))] \le \operatorname{sign}(u-k) g \tag{4}$$

in the sense of distributions on Π_T (in $\mathcal{D}'(\Pi_T)$);

b) $\operatorname*{ess\,lim}\limits_{t \to 0+} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$, i.e. there exists a set $\mathcal{E} \subset (0,T)$ of full Lebesgue measure such that $u(t,\cdot) \in L^1_{loc}(\mathbb{R}^n)$, $t \in \mathcal{E}$ and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0+$, $t \in \mathcal{E}$.

The condition (4) means that for any test function $f = f(t,x) \in C_0^\infty(\Pi_T)$, $f \ge 0$

$$\int_\Pi \{|u-k| f_t + \operatorname{sign}(u-k)[(\varphi(u) - \varphi(k), \nabla_x f) + g f]\} dt dx \ge 0,$$

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here and below $(\cdot,\cdot)$ denotes scalar product in $\mathbb{R}^n$.

It is well-known (see for example [4, 9]) that condition (4) can be written in equivalent form: for any convex function $\eta(u)\in C^1(\mathbb{R})$ (entropy)

$$\eta(u)_t+\mathrm{div}_x\psi(u)\le\eta'(u)g \text{ in } \mathcal{D}'(\Pi_T), \tag{5}$$

where $\psi(u)=(\psi_1(u),\dots,\psi_n(u))$ is a corresponding vector of entropy flux defining by the equality

$$\psi(u)=\eta'(u)\varphi(u)-\int_0^u\varphi(\sigma)d\eta'(\sigma) \tag{6}$$

(so in the case of smooth $\varphi(u)\in C^1(\mathbb{R},\mathbb{R}^n)$ the vector $\psi(u)$ is determined up to an additive constant by the identity $\psi'(u)=\eta'(u)\varphi'(u)$).

Taking in (4) $k=\pm\|u\|_\infty$ we can easy derive that $u_t+\mathrm{div}_x\varphi(u)=g$ in $\mathcal{D}'(\Pi_T)$ and $u(t,x)$ satisfies to equation (1) in the sense of distributions (thus it is indeed a generalized solution).

Study of the problem (1), (2) in the case of only continuous flux $\varphi(u)$ was begun in [4, 10, 11]. Since then theory of the problem (1), (2) was considerably advanced.

Remind some notions and results which will be used below. It is known that a g.e.s. always exists (the explicit proof contains in [1, 14, 15]).

In the case under consideration when flux vector $\varphi(u)$ is assumed to be only continuous the effect of infinite speed of propagation of an initial perturbation generally appears which can imply even nonuniqueness of g.e.s. to the problem (1), (2) if space dimension $n>1$. The corresponding examples were constructed in [12]-[14] for the case $n=2$, $g\equiv 0$. Remark that for Lipschitz-continuous flux vector uniqueness of g.e.s. is well-known (see [8, 9]). Some sufficient condition of uniqueness were found in [12]-[14]. Moreover under these conditions *comparison principle* for generalized entropy sub- and super-solutions was obtained in [13] (see also [5, 6]).

Recall the definitions of generalized entropy sub- and super-solutions (see [2]-[7],[12]). Let $f^+=\max(f,0)$, $f^-=\max(-f,0)$. Denote by $\chi_{u>v}$ a characteristic function of the set of independent variables on which the relation $u>v$ is satisfied for two given functions u and v . In the same way we use notations $\chi_{u<v}$, $\chi_{u=v}$ and so on. Let $\mathrm{sign}^+(r)=\chi_{r>0}$ be the Heaviside function and $\mathrm{sign}^-(r)=-\mathrm{sign}^+(-r)$.

Definition 2. A bounded measurable function $u(t,x)$ on Π_T is called a generalized entropy sub-solution (briefly g.e.sub-s.) to the Cauchy problem (1), (2) if

1) $\forall k\in\mathbb{R}$

$$((u-k)^+)_t+\mathrm{div}_x[\mathrm{sign}^+(u-k)(\varphi(u)-\varphi(k))]\le\mathrm{sign}^+(u-k)g \text{ in } \mathcal{D}'(\Pi_T); \tag{7}$$

2) $\operatorname*{ess\,lim}_{t\to0+}(u(t,x)-u_0(x))^+=0$ in $L^1_{loc}(\mathbb{R}^n)$.

Definition 3. A bounded measurable function $u(t,x)$ on Π_T is called a generalized entropy super-solution (briefly g.e.super-s.) to the Cauchy problem (1), (2) if

1) $\forall k \in \mathbb{R}$

$$((u-k)^-)_t + \mathrm{div}_x[\mathrm{sign}^-(u-k)(\varphi(u)-\varphi(k))] \le \mathrm{sign}^-(u-k)g \ \text{ in } \mathcal{D}'(\Pi_T); \qquad (8)$$

2) $\operatorname*{ess\,lim}_{t\to 0+}(u(t,x)-u_0(x))^- = 0$ in $L^1_{loc}(\mathbb{R}^n)$.

Remark 1 (see [1, 5, 13, 17]).

a) The conditions (7) and (8) are equivalent to requirements that (5) is satisfied for any nondecreasing (respectively - nonincreasing) convex function $\eta(u) \in C^1(\mathbb{R})$;

b) A function $u = u(t,x) \in L^\infty(\Pi)$ is a g.e.super-s. of the problem (1), (2) if and only if the function $-u$ is a g.e.sub-s. of the Cauchy problem for the equation $u_t - \mathrm{div}_x\varphi(-u) = -g$ with the initial function $-u_0(x)$;

c) A function $u(t,x) \in L^\infty(\Pi)$ is a g.e.s. of the problem (1), (2) if and only if it is a g.e.sub-s. and a g.e.super-s. of this problem simultaneously;

d) Let $y = Ax$ be a nondegenerate linear transform on $\mathbb{R}^n$. Then $u(t,x)$ is a g.e.sub-s. (g.e.super-s., g.e.s.) of the problem (1), (2) if and only if the function $\bar{u}(t,y) = u(t,A^{-1}y)$ is a g.e.sub-s. (respectively, g.e.super-s., g.e.s.) of the problem

$$u_t + \mathrm{div}_y\bar{\varphi}(u) = \bar{g}, \quad u(0,y) = u_0(A^{-1}y),$$

where $\bar{g}(t,y) = g(t,A^{-1}y)$ and the flux vector $\bar{\varphi}(u) = A^*\varphi(u)$, $A^* : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a conjugate transform.

We also need the following result which is contained in [1, 6].

Proposition 1. *Let $u(t,x)$ and $\tilde{u}(t,x)$ be a g.e.sub-s. and a g.e.super-s. of the problem* (1), (2) *with initial functions $u_0(x)$ and $\tilde{u}_0(x)$ and source terms $g(t,x)$ and $\tilde{g}(t,x)$ respectively.*

Assume in addition that one of the following condition A) or B) is fulfilled:

A) *One of the function u or $\tilde{u}$ doesn't depend on x , i.e. $u \equiv u(t)$ or $\tilde{u} \equiv \tilde{u}(t)$ a.e. on Π_T ;*

B) *$\mu_{n+1}\{ (t,x) \in \Pi_T \mid u(t,x) > \tilde{u}(t,x) \} < \infty$, where μ_r , $r \in \mathbb{N}$ denotes r -dimensional Lebesgue measure.*

Then for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n}(u(t,x)-\tilde{u}(t,x))^+dx \le \int_{\mathbb{R}^n}(u_0(x)-\tilde{u}_0(x))^+dx + \int_{\Pi_t}(g(\tau,x)-\tilde{g}(\tau,x))^+\chi_{u\ge\tilde{u}}d\tau dx. \qquad (9)$$

In particular if $u_0 \le \tilde{u}_0$ a.e. on $\mathbb{R}^n$, $g \le \tilde{g}$ a.e. on Π_T then $u \le \tilde{u}$ a.e. on Π_T (comparison principle).

Applying the comparison principle to a g.e.s. u of the problem (1), (2), g.e.sub-s. $u_-(t) = \operatorname{ess\,inf} u_0(x) + \int_0^t \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} g(\tau,x)d\tau$ and g.e.super-s. $u_+(t) = \operatorname{ess\,sup} u_0(x) + \int_0^t \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} g(\tau,x)d\tau$ of the same problem (notice that by (3) the functions $u_-(t)$ and $u_+(t)$ are bounded) we easy derive the following *maximum principle* (see [5, 6]):

Corollary 1. *If $u = u(t,x)$ is a g.e.s. of the problem* (1), (2) *then for a.a.* $(t,x) \in \Pi_T$

$$\operatorname{ess\,inf} u_0(x) + \int_0^t \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} g(\tau,x)d\tau \le u(t,x) \le$$
$$\le \operatorname{ess\,sup} u_0(x) + \int_0^t \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} g(\tau,x)d\tau.$$

For two g.e.s. u and $\tilde{u}$ of the problem (1), (2) satisfying requirement A) of Proposition 1 we can change places of these functions in the inequality (9) and deduce the following

Corollary 2 (see [5, 6]). *Let $u = u(t,x)$, $\tilde{u} = \tilde{u}(t)$ be g.e.s. of the problem* (1), (2) *with initial functions $u_0(x)$ and $\tilde{u}_0(x) \equiv \tilde{u}(0)$ and source terms $g(t,x)$ and $\tilde{g}(t,x) \equiv \tilde{g}(t)$ respectively. Then for a.a. $t \in (0,T)$*

$$\int_{\mathbb{R}^n} |u(t,x) - \tilde{u}(t)|dx \le \int_{\mathbb{R}^n} |u_0(x) - \tilde{u}(0)|dx + \int_{\Pi_t} |g(\tau,x) - \tilde{g}(\tau)|d\tau dx.$$

In [1, 5, 6] some special cases of uniqueness were investigated in the class of summarable with respect to spatial variables solutions. Established in these papers existence of the maximal and minimal g.e.s. playes crucial role for study of the problem. Existence of the maximal and minimal g.e.s. was proved under the requirement that initial data and source function in some sense vanish as $x \to \infty$.

In section 2 below we prove existence of the maximal and minimal solutions for a general situation and give some usefull application. In particular we show that the maximal (minimal) g.e.s. coincides the maximal g.e.sub-s. (respectively, the minimal g.e.super-s.), existence of which were recently established in [17]. In section 3 we prove uniqueness of g.e.s. to the problem (1), (2) in the case when input data of the problem are periodic functions with respect to some $n-1$ spatial variables. Thus we present a generalization of the corresponding result in [17].

Remark that results similar mentioned above are true also for the stationary problem

$$u + \operatorname{div}_x \varphi(u) = g(x), \ u = u(x), \ x \in \mathbb{R}^n,$$

studied in papers [1, 5, 6], in parallel with investigation of the Cauchy problem.

In the present work we intend to consider only the case of the Cauchy problem but emphasize that proofs of the main results remain valid (with corresponding changes) for the stationary problem as well.

2. The main theorems.

Firstly, we give the following usefull reformulation of Definition 1.

Let $\bar{\Pi}_T = [0,T) \times \mathbb{R}^n$, $C_0^\infty(\bar{\Pi}_T)$ be a subspace of functions in $C^\infty(\bar{\Pi}_T)$ with compact support (which may have nonempty intersection with initial hyperplane $\{t = 0\}$).

Proposition 2. *A function* $u = u(t,x) \in L^\infty(\Pi_T)$ *is a g.e.s. of the problem* (1), (2) *if and only if for any* $k \in \mathbb{R}$ *and any test function* $f = f(t,x) \in C_0^\infty(\bar\Pi_T)$, $f \geq 0$

$$\int_{\Pi_T} \{|u-k|f_t + \operatorname{sign}(u-k)[(\varphi(u)-\varphi(k), \nabla_x f) + gf]\}dtdx + \int_{\mathbb{R}^n} |u_0(x)-k|f(0,x)dx \geq 0. \qquad (10)$$

Proof. Let $u = u(t,x)$ be a g.e.s. of the problem (1), (2); $k \in \mathbb{R}$, $f = f(t,x) \in C_0^\infty(\bar\Pi_T)$, $f \geq 0$. Choose a function $\beta(s) \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\beta(s) \subset [0,1]$, $\beta(s) \geq 0$, $\int \beta(s)ds = 1$ and set for $\nu \in \mathbb{N}$ $\delta_\nu(s) = \nu\beta(\nu s)$, $\theta_\nu(t) = \int_{-\infty}^t \delta_\nu(s)ds$. It's clear that the sequence $\delta_\nu(s)$ converges to the Dirac δ -measure in $\mathcal{D}'(\mathbb{R})$ as $\nu \to \infty$; $0 \leq \theta_\nu(t) \leq 1$, $\theta_\nu(t) = 0$ for $t \leq 0$, $\theta_\nu(t) = 1$ for $t \geq 1/\nu$ and therefore the sequence $\theta_\nu(t)$ converges to the Heaviside function $\operatorname{sign}^+(t)$ pointwise as $\nu \to \infty$.

Applying the entropy inequality (4) to the test function $f_\nu = f_\nu(t,x) = f(t,x)\theta_\nu(t-t_0) \in C_0^\infty(\Pi_T)$, $t_0 \in (0,T)$ we obtain that

$$\int_0^T \delta_\nu(t-t_0)\int_{\mathbb{R}^n} |u(t,x)-k|f(t,x)dxdt + \int_{\Pi_T} \{|u-k|f_t + \operatorname{sign}(u-k)[(\varphi(u)-\varphi(k), \nabla_x f) + gf]\}\theta_\nu(t-t_0)dtdx \geq 0. \qquad (11)$$

Let $\mathcal{E}_1$ be the set of common Lebesgue points of functions $t \mapsto F(k,t) = \int_{\mathbb{R}^n} |u(t,x)-k|f(t,x)dx$ for all rational $k \in \mathbb{Q}$. Because $\mathbb{Q}$ is countable then $\mathcal{E}_1$ is a set of full Lebesgue measure in $(0,T)$. Evidently the map $k \mapsto F(k,t) \in L^\infty([0,T])$ is continuous uniformly by $t \in [0,T]$ and because $\mathbb{Q}$ is dense in $\mathbb{R}$ we derive that points $t \in \mathcal{E}_1$ are Lebesgue points of the functions $F(k,t)$ for all $k \in \mathbb{R}$. Let $\mathcal{E}_2 \subset (0,T)$ be the set of full measure from Definition 1 such that $u(t,\cdot) \in L^1_{loc}(\mathbb{R}^n)$, $t \in \mathcal{E}_2$ and $u(t,\cdot) \to u_0$ in $L^1_{loc}(\mathbb{R}^n)$ as $t \to 0+$, $t \in \mathcal{E}_2$. Let $\mathcal{E} = \mathcal{E}_1 \cap \mathcal{E}_2$. It's clear that $\mathcal{E}$ is a set of full measure in $(0,T)$ and if $t_0 \in \mathcal{E}$ then in the limit as $\nu \to \infty$ the relation (11) is transformed to the inequality

$$\int_{\mathbb{R}^n} |u(t_0,x)-k|f(t_0,x)dxdt + \int_{\Pi_T} \{|u-k|f_t + \operatorname{sign}(u-k)[(\varphi(u)-\varphi(k), \nabla_x f) + gf]\}\operatorname{sign}^+(t-t_0)dtdx \geq 0,$$

which after passage to the limit as $t_0 \to 0$, $t_0 \in \mathcal{E}$ (and taking into account requirement b) in Definition 1) implies (10).

For the proof of the inverse implication suppose that a function $u = u(t,x) \in L^\infty(\Pi_T)$ satisfies the condition (10). Taking in this condition test functions from $C_0^\infty(\Pi_T)$ we immediately derive that

$$|u-k|_t + \operatorname{div}_x[\operatorname{sign}(u-k)(\varphi(u)-\varphi(k))] \leq \operatorname{sign}(u-k)g \ \text{ in } \mathcal{D}'(\Pi_T)$$

and u satisfies the entropy requirement (4). It only remains to verify the initial condition. Choose some function $\rho(x) \in C_0^\infty(\mathbb{R}^n)$, $\rho(x) \geq 0$ and take in (10) $f = f_\nu(t,x) =$

$\rho(x)(1-\theta_\nu(t-t_0))$, where $\nu\in\mathbb{N}$, $t_0\in(0,T)$, $1/\nu<T-t_0$. We obtain that for any $k\in\mathbb{R}$

$$\int_{\mathbb{R}^n}|u_0(x)-k|\rho(x)dx-\int_0^T\delta_\nu(t-t_0)\int_{\mathbb{R}^n}|u(t,x)-k|\rho(x)dxdt+$$
$$\int_{\Pi_T}\mathrm{sign}\,(u-k)[(\varphi(u)-\varphi(k),\nabla_x\rho)+g\rho](1-\theta_\nu(t-t_0))dtdx\ge 0. \qquad (12)$$

Let $\mathcal{E}$ be the set of common Lebesgue points for functions

$$t\mapsto\int_{\mathbb{R}^n}|u(t,x)-k|\rho(x)dx,\ k\in\mathbb{R}.$$

In the same way as for the set $\mathcal{E}_1$ above one could easy prove that $\mathcal{E}$ has full Lebesgue measure on $(0,T)$ and if $t_0\in\mathcal{E}$ then passing to the limit in (12) as $\nu\to\infty$ we derive a relation

$$\int_{\mathbb{R}^n}|u_0(x)-k|\rho(x)dx-\int_{\mathbb{R}^n}|u(t_0,x)-k|\rho(x)dx+$$
$$\int_{\Pi_{t_0}}\mathrm{sign}\,(u-k)[(\varphi(u)-\varphi(k),\nabla_x\rho)+g\rho]dtdx\ge 0,$$

which implies an estimate

$$\int_{\mathbb{R}^n}|u(t_0,x)-k|\rho(x)dx\le\int_{\mathbb{R}^n}|u_0(x)-k|\rho(x)dx+C(t_0),$$

where, taking into account the condition (3),

$$C(t_0)=\int_{\Pi_{t_0}}[|\varphi(u)-\varphi(k)|\cdot|\nabla_x\rho|+\|g(t,\cdot)\|_\infty|\rho|]dtdx\underset{t_0\in\mathcal{E},t_0\to 0+}{\to}0$$

(here we use the notation $|v|$ for Euclidean norm of a vector v). Hence $\forall k\in\mathbb{R}$ and $\forall\rho(x)\in C_0^\infty(\mathbb{R}^n)$, $\rho(x)\ge 0$

$$\limsup_{t_0\in\mathcal{E},t_0\to 0+}\int_{\mathbb{R}^n}|u(t_0,x)-k|\rho(x)dx\le\int_{\mathbb{R}^n}|u_0(x)-k|\rho(x)dx. \qquad (13)$$

By continuous dependence of the integrals $\int_{\mathbb{R}^n}|u(t_0,x)-k|\rho(x)dx$ on the function $\rho(x)\in L^1(\mathbb{R}^n)$ and density of $C_0^\infty(\mathbb{R}^n)$ in $L^1(\mathbb{R}^n)$ the limit relation (13) is fulfilled for any nonnegative function $\rho(x)\in L^1(\mathbb{R}^n)$.

Choose for some positive ε a step-function $u_\varepsilon=\sum_{r=1}^N\alpha_r\chi_{A_r}(x)$ such that $\|u_0-u_\varepsilon\|_\infty<\varepsilon$. Here $\chi_{A_r}(x)$ are characteristic functions of measurable sets $A_r\subset\mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n}|u(t_0,x)-u_0(x)|\rho(x)dx\le\varepsilon\|\rho\|_1+\int_{\mathbb{R}^n}|u(t_0,x)-u_\varepsilon(x)|\rho(x)dx=$$
$$\varepsilon\|\rho\|_1+\sum_{r=1}^N\int_{\mathbb{R}^n}|u(t_0,x)-\alpha_r|\rho\chi_{A_r}(x)dx$$

and by the limit relation (13)

$$\limsup_{t_0\in\mathcal{E},t_0\to 0+}\int_{\mathbb{R}^n}|u(t_0,x)-u_0(x)|\rho(x)dx\le$$
$$\varepsilon\|\rho\|_1+\sum_{r=1}^N\int_{\mathbb{R}^n}|u_0(x)-\alpha_r|\rho\chi_{A_r}(x)dx=$$
$$\varepsilon\|\rho\|_1+\int_{\mathbb{R}^n}|u_0(x)-u_\varepsilon(x)|\rho(x)dx\le 2\varepsilon\|\rho\|_1$$

for arbitrary $\varepsilon > 0$. Therefore $\forall \rho(x) \in L^1(\mathbb{R}^n)$, $\rho(x) \ge 0$

$$\limsup_{t_0 \in \mathcal{E}, t_0 \to 0+} \int_{\mathbb{R}^n} |u(t_0, x) - u_0(x)| \rho(x) dx = 0$$

and $\operatorname{ess}\lim_{t \to 0+} u(t, \cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$. Thus u is a g.e.s. of the problem (1), (2). The proof is complete.

Corollary 3. *If a function $u = u(t,x) \in L^\infty(\Pi_T)$ is a g.e.s. of the problem* (1), (2) *then for any test function $f = f(t,x) \in C_0^\infty(\bar{\Pi}_T)$*

$$\int_{\Pi_T} [u f_t + (\varphi(u), \nabla_x f) + g f] dt dx + \int_{\mathbb{R}^n} u_0(x) f(0,x) dx = 0. \tag{14}$$

Proof. Consider the relation (10) with $k = \pm M$, $M \ge \|u\|_\infty$. Taking into account that evidently

$$\int_{\Pi_T} [k f_t + (\varphi(k), \nabla_x f)] dt dx + \int_{\mathbb{R}^n} k f(0,x) dx = 0$$

we obtain the relation: for any $f = f(t,x) \in C_0^\infty(\bar{\Pi}_T)$, $f \ge 0$

$$\pm \left(\int_{\Pi_T} [u f_t + (\varphi(u), \nabla_x f) + g f] dt dx + \int_{\mathbb{R}^n} u_0(x) f(0,x) dx \right) \ge 0, \quad \text{i.e.}$$

$$\int_{\Pi_T} [u f_t + (\varphi(u), \nabla_x f) + g f] dt dx + \int_{\mathbb{R}^n} u_0(x) f(0,x) dx = 0$$

for any nonnegative test function f . For the end of the proof it only remains to notice that any function $f \in C_0^\infty(\bar{\Pi}_T)$ can be present as a difference of two nonnegative functions from $C_0^\infty(\bar{\Pi}_T)$. Indeed, $f = (f + C)\theta - C\theta$, where a constant $C > \|f\|_\infty$ and the nonnegative function $\theta \in C_0^\infty(\bar{\Pi}_T)$ equals 1 on a support of the function f .

Remark 2. In the same way as in Proposition 2 above one could establish the following statements:

1) a function $u = u(t,x) \in L^\infty(\Pi_T)$ is a g.e.sub-s. of the problem (1), (2) if and only if for any $k \in \mathbb{R}$ and any test function $f = f(t,x) \in C_0^\infty(\bar{\Pi}_T)$, $f \ge 0$

$$\int_{\Pi_T} \{(u-k)^+ f_t + \operatorname{sign}^+(u-k)[(\varphi(u) - \varphi(k), \nabla_x f) + g f]\} dt dx + \int_{\mathbb{R}^n} (u_0(x) - k)^+ f(0,x) dx \ge 0;$$

2) a function $u = u(t,x) \in L^\infty(\Pi_T)$ is a g.e.super-s. of the problem (1), (2) if and only if for any $k \in \mathbb{R}$ and any test function $f = f(t,x) \in C_0^\infty(\bar{\Pi}_T)$, $f \ge 0$

$$\int_{\Pi_T} \{(u-k)^- f_t + \operatorname{sign}^-(u-k)[(\varphi(u) - \varphi(k), \nabla_x f) + g f]\} dt dx + \int_{\mathbb{R}^n} (u_0(x) - k)^- f(0,x) dx \ge 0.$$

Theorem 1. *There exists the maximal g.e.s. $u = u(t,x) = J^+(u_0, g)$ of the problem* (1), (2). *Besides*

1) *$u(t,x) \ge v(t,x)$ a.e. on Π_T for any g.e.sub-s. $v(t,x)$ of the problem* (1), (2)*;*

2) *If* $u = J^+(u_0, g)$, $v = J^+(v_0, h)$ *then for a.a.* $t \in (0,T)$

$$\int_{\mathbb{R}^n} (u(t,x) - v(t,x))^+ dx \le \int_{\mathbb{R}^n} (u_0(x) - v_0(x))^+ dx + \int_{\Pi_t} (g(\tau,x) - h(\tau,x))^+ d\tau dx. \qquad (15)$$

Proof. Let $M = \|u_0\|_\infty$, $M_1(t) = \|g(t,\cdot)\|_\infty$. Choose a sequence of constants d_ν , $\nu \in \mathbb{N}$ such that $\forall \nu \in \mathbb{N} \quad M < d_{\nu+1} < d_\nu$, set for $\nu \in \mathbb{N}$

$$u_{0,\nu}(x) = \begin{cases} u_0(x) & , \quad |x| \le \nu \\ d_\nu & , \quad |x| > \nu \end{cases}, \quad g_\nu(t,x) = \begin{cases} g(t,x) & , \quad |x| \le \nu \\ M_1(t) & , \quad |x| > \nu \end{cases}$$

and let $u_\nu = u_\nu(t,x)$ be a g.e.s. to the problem (1), (2) with an initial data $u_{0,\nu}(x)$ and a source function $g_\nu(t,x)$ (such g.e.s. really can be found by the existence theorem). Evidently the function $\tilde{u}_\nu(t) = d_\nu + \int_0^t M_1(s)ds$ is a g.e.s. of the problem (1), (2) with a constant initial data $u_0 \equiv d_\nu$ and a source term $g \equiv M_1(t)$ and this g.e.s. doesn't depend on x . By Corollary 2 for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n} |u_\nu(t,x) - \tilde{u}_\nu(t)| dx \le \int_{\mathbb{R}^n} |u_{0,\nu}(x) - d_\nu| dx + \int_{\Pi_T} |g_\nu(t,x) - M_1(t)| dt dx \le C_\nu = \left[(M + d_\nu) + 2\int_0^T M_1(t)dt \right] \mu_n(B_\nu) < +\infty, \qquad (16)$$

here B_ν is a ball in $\mathbb{R}^n$ of radius ν . Notice that $\tilde{u}_{\nu+1}(t) - \tilde{u}_\nu(t) \equiv d_{\nu+1} - d_\nu$. Then, using (16) we deduce that for any positive $h < d_\nu - d_{\nu+1}$ for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - u_\nu(t,x) + h)^+ dx = \int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - \tilde{u}_{\nu+1}(t) + \tilde{u}_\nu(t) - u_\nu(t,x) + h + d_{\nu+1} - d_\nu)^+ dx \le \int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - \tilde{u}_{\nu+1}(t) + \tilde{u}_\nu(t) - u_\nu(t,x))^+ dx \le \int_{\mathbb{R}^n} |u_{\nu+1}(t,x) - \tilde{u}_{\nu+1}(t)| dx + \int_{\mathbb{R}^n} |u_\nu(t,x) - \tilde{u}_\nu(t)| dx \le C_\nu + C_{\nu+1} < +\infty. \qquad (17)$$

This estimate yields

$$\mu_{n+1}\{\ (t,x) \in \Pi_T \mid u_{\nu+1}(t,x) > u_\nu(t,x)\ \} = \mu_{n+1}\{\ (t,x) \in \Pi_T \mid (u_{\nu+1}(t,x) - u_\nu(t,x) + h)^+ > h\ \} \le \frac{1}{h}\int_0^T \int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - u_\nu(t,x) + h)^+ dx dt \le (C_\nu + C_{\nu+1})T/h < \infty \qquad (18)$$

and by Proposition 1, B) and the fact that $u_{0,\nu+1}(x) \le u_{0,\nu}(x)$ a.e. on $\mathbb{R}^n$, $g_{\nu+1}(t,x) \le g_\nu(t,x)$ a.e. on Π_T we derive the estimate: for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - u_\nu(t,x))^+ dx \le \int_{\mathbb{R}^n} (u_{0,\nu+1}(x) - u_{0,\nu}(x))^+ dx + \int_{\Pi_T} (g_{\nu+1}(t,x) - g_\nu(t,x))^+ dt dx = 0.$$

Thus $u_{\nu+1}(t,x) \le u_\nu(t,x)$ a.e. on Π_T for all $\nu \in \mathbb{N}$. By Corollary 1 the sequence $u_\nu(t,x)$ is also essentially bounded from below: $u_\nu(t,x) \ge -M - \int_0^T M_1(t)dt$ a.e. on Π_T. Therefore for a.a. $(t,x) \in \Pi_T$ there exists the limit function $\lim_{\nu\to\infty} u_\nu(t,x) = u(t,x) \in L^\infty(\Pi_T)$. It's clear that $u_\nu(t,x) \to u(t,x)$ in $L^1_{loc}(\Pi_T)$ as $\nu \to \infty$. By definition of the sequences $u_{0,\nu}$, g_ν they converge as $\nu \to \infty$ to the functions u_0 and g respectively, pointwise and in the corresponding spaces $L^1_{loc}(\mathbb{R}^n)$, $L^1_{loc}(\Pi_T)$.

Since $u_\nu(t,x)$ is a g.e.s. to the problem (1), (2) with $u_0 = u_{0,\nu}$, $g = g_\nu$ it satisfies the condition (10): for any $k \in \mathbb{R}$ and any test function $f = f(t,x) \in C_0^\infty(\bar\Pi_T)$, $f \ge 0$

$$\int_{\Pi_T} \{|u_\nu - k|f_t + \operatorname{sign}(u_\nu - k)[(\varphi(u_\nu) - \varphi(k), \nabla_x f) + g_\nu f]\}dtdx +$$
$$\int_{\mathbb{R}^n} |u_{0,\nu}(x) - k|f(0,x)dx \ge 0.$$

Passing to the limit as $\nu \to \infty$ and taking into account the limit relations above we conclude that for any $k \in \mathbb{R}$ and any $f = f(t,x) \in C_0^\infty(\bar\Pi_T)$, $f \ge 0$

$$\int_{\Pi_T} \{|u - k|f_t + \operatorname{sign}(u - k)[(\varphi(u) - \varphi(k), \nabla_x f) + gf]\}dtdx +$$
$$\int_{\mathbb{R}^n} |u_0(x) - k|f(0,x)dx \ge 0$$

and by Proposition 2 $u = u(t,x)$ is a g.e.s. of the original problem (1), (2).

Show that $u(t,x)$ satisfies the requirement 1). Let $v = v(t,x)$ be a g.e.sub-s. of the problem (1), (2). Since the function $\tilde v = \tilde v(t) = M + \int_0^t M_1(s)ds$ is a g.e.s. (and hence a g.e.super-s.) of this problem with data $u_0 \equiv M$, $g \equiv M_1(t)$ then by Proposition 1, A) for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (v(t,x) - \tilde v(t))^+ dx \le \int_{\mathbb{R}^n} (u_0(x) - M)^+ dx + \int_{\Pi_T} (g(t,x) - M_1(t))^+ dtdx = 0$$

and $v(t,x) \le \tilde v(t)$ a.e. on Π_T. Using this unequality, the evident identity $\tilde v(t) - \tilde u_\nu(t) = M - d_\nu$ and the estimate (16) we establish that for any $\nu \in \mathbb{N}$, $h \in \mathbb{R}$, $0 < h < d_\nu - M$

$$\int_{\mathbb{R}^n} (v(t,x) - u_\nu(t,x) + h)^+ dx \le \int_{\mathbb{R}^n} (\tilde v(t) - u_\nu(t,x) + h)^+ dx \le$$
$$\int_{\mathbb{R}^n} (\tilde u_\nu(t) - u_\nu(t,x) + h + M - d_\nu)^+ dx \le$$
$$\int_{\mathbb{R}^n} (\tilde u_\nu(t) - u_\nu(t,x))^+ dx \le C_\nu.$$

It implies that

$$\mu_{n+1}\{\ (t,x) \in \Pi_T \mid v(t,x) > u_\nu(t,x)\ \} =$$
$$\mu_{n+1}\{\ (t,x) \in \Pi_T \mid (v(t,x) - u_\nu(t,x) + h)^+ > h\ \} \le$$
$$\frac{1}{h}\int_0^T \int_{\mathbb{R}^n} (v(t,x) - u_\nu(t,x) + h)^+ dxdt \le C_\nu T/h < \infty.$$

Applying again Proposition 1, B) we deduce that for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (v(t,x) - u_\nu(t,x))^+ dx \le \int_{\mathbb{R}^n} (u_0(x) - u_{0,\nu}(x))^+ dx +$$
$$\int_{\Pi_T} (g(t,x) - g_\nu(t,x))^+ dtdx = 0.$$

Therefore $v(t,x) \le u_\nu(t,x)$ a.e. on Π_T for all $\nu \in \mathbb{N}$ and passing to the limit as $\nu \to \infty$ yields $v(t,x) \le u(t,x)$ a.e. on Π_T. Thus the condition 1) is satisfied and in particular $u(t,x) = J^+(u_0, g)$ is the maximal g.e.s. of our problem. It remains only to prove the property 2).

Let $M = \max(\|u_0\|_\infty, \|v_0\|_\infty)$, $M_1(t) = \max(\|g(t,\cdot)\|_\infty, \|h(t,\cdot)\|_\infty)$. In the same way as above choose the sequence d_ν, $\nu \in \mathbb{N}$ such that $\forall \nu \in \mathbb{N}$ $M < d_{\nu+1} < d_\nu$ and set for $\nu \in \mathbb{N}$

$$u_{0,\nu}(x) = \begin{cases} u_0(x) & , \ |x| \le \nu \\ d_\nu & , \ |x| > \nu \end{cases}, \quad g_\nu(t,x) = \begin{cases} g(t,x) & , \ |x| \le \nu \\ M_1(t) & , \ |x| > \nu \end{cases},$$

$$v_{0,\nu}(x) = \begin{cases} v_0(x) & , \ |x| \le \nu \\ d_\nu & , \ |x| > \nu \end{cases}, \quad h_\nu(t,x) = \begin{cases} h(t,x) & , \ |x| \le \nu \\ M_1(t) & , \ |x| > \nu \end{cases}.$$

Let $u_\nu = u_\nu(t,x)$, $v_\nu = v_\nu(t,x)$ be g.e.s. of the problem (1), (2) with data $u_{0,\nu}, g_\nu$ and $v_{0,\nu}, h_\nu$ respectively, and $\tilde{u}_\nu(t)$ be defined above. Then for any positive $h < d_\nu - d_{\nu+1}$ for a.a. $t \in (0,T)$

$$\int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - v_\nu(t,x) + h)^+ dx =$$
$$\int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - \tilde{u}_{\nu+1}(t) + \tilde{u}_\nu(t) - v_\nu(t,x) + h + d_{\nu+1} - d_\nu)^+ dx \le$$
$$\int_{\mathbb{R}^n} |u_{\nu+1}(t,x) - \tilde{u}_{\nu+1}(t)| dx +$$
$$\int_{\mathbb{R}^n} |v_\nu(t,x) - \tilde{u}_\nu(t)| dx \le C_\nu + C_{\nu+1} < +\infty,$$

with the constants C_ν defined at (16). As it was already shown this inequality yields (cf. (18)) that

$$\mu_{n+1}\{ (t,x) \in \Pi_T \mid u_{\nu+1}(t,x) > v_\nu(t,x) \} < \infty$$

and by Proposition 1, B)

$$\int_{\mathbb{R}^n} (u_{\nu+1}(t,x) - v_\nu(t,x))^+ dx \le \int_{\mathbb{R}^n} (u_{0,\nu+1}(x) - v_{0,\nu}(x))^+ dx +$$
$$\int_{\Pi_t} (g_{\nu+1}(\tau,x) - h_\nu(\tau,x))^+ d\tau dx =$$
$$\int_{|x|\le\nu} (u_0(x) - v_0(x))^+ dx + \int_0^t \int_{|x|\le\nu} (g(\tau,x) - h(\tau,x))^+ d\tau dx. \tag{19}$$

In the limit as $\nu \to \infty$ the sequence $u_{\nu+1}(t,x) - v_\nu(t,x)$ converges to $u(t,x) - v(t,x)$ and passing to the limit in (19) with using the Fatou lemma we derive the estimate (15). Theorem is proved.

Suppose that a function $u(t,x) \in L^\infty(\Pi_T)$ is the maximal g.e.s. of the Cauchy problem

$$u_t - \operatorname{div}_x \varphi(-u) = -g, \ u(0,x) = -u_0(x)$$

Then by Remark 1, b) the function $-u(t,x)$ is the minimal g.e.s. of the original problem and it satisfies properties similar to ones in Theorem 1. More precisely we obtain the following

Theorem 1'. *There exists the minimal g.e.s.* $u = u(t,x) = J^-(u_0, g)$ *of the problem* (1), (2). *Moreover*

1) $u(t,x) \le v(t,x)$ *a.e. on* Π_T *for any g.e.super-s.* $v(t,x)$ *of the problem* (1), (2);

2) *If* $u = J^-(u_0, g)$, $v = J^-(v_0, h)$ *then for a.a.* $t \in (0,T)$

$$\int_{\mathbb{R}^n} (u(t,x) - v(t,x))^+ dx \le \int_{\mathbb{R}^n} (u_0(x) - v_0(x))^+ dx + \\ \int_{\Pi_t} (g(\tau,x) - h(\tau,x))^+ d\tau dx. \tag{20}$$

Corollary 4. *Suppose that* $u^\pm = J^\pm(u_0, g)$, $v^\pm = J^\pm(v_0, h)$. *Then*

1) *for a.a.* $t \in (0,T)$

$$\int_{\mathbb{R}^n} |u^\pm(t,x) - v^\pm(t,x)| dx \le \int_{\mathbb{R}^n} |u_0(x) - v_0(x)| dx + \\ \int_{\Pi_t} |g(\tau,x) - h(\tau,x))| d\tau dx. \tag{21}$$

(L^1 *-contraction property*);

2) *if* $u_0(x) \le v_0(x)$ *a.e. in* $\mathbb{R}^n$, $g(t,x) \le h(t,x)$ *a.e. in* Π_T *then* $u^\pm(t,x) \le v^\pm(t,x)$ *a.e. in* Π_T (*monotonicity*).

Proof. For the proof of 1) we only need to change places of the solutions u and v in the relation (15) (and, respectively, in (20)) and to add the obtained inequality to the original one. The assertion 2) is immediately follows from (15), (20) as well.

Denote by D the set of pairs (u_0, g) , where $u_0 \in L^\infty(\mathbb{R}^n)$, $g \in L^1_{loc}(\Pi_T)$ and satisfies the conditions (3). Let $\alpha > 0$. Consider the following functional on D

$$I_\alpha(u_0, g) = \operatorname*{ess\,sup}_{t \in (0,T)} \int_{\mathbb{R}^n} (u^+(t,x) - u^-(t,x)) e^{-\alpha|x|} dx, \quad u^\pm = J^\pm(u_0, g).$$

If $u_0 - v_0 \in L^1(\mathbb{R}^n)$, $g - h \in L^1(\Pi_T)$ then, as it follows from (21), $\forall \alpha > 0$

$$|I_\alpha(u_0, g) - I_\alpha(v_0, h)| \le 2(\|u_0 - v_0\|_{L^1(\mathbb{R}^n)} + \|g - h\|_{L^1(\Pi_T)}), \tag{22}$$

i.e. this functional is Lipschitz-continuous on D with respect to L^1 -distance. It easy implies the following result.

Theorem 2. *Let* D_u *be a subset of* D *consisting of pairs* (u_0, g) , *for which the problem* (1), (2) *has an unique g.e.s. Then the set* D_u *is closed with respect to* L^1 *-distance. In other words if sequence* $(u_k, g_k) \in D_u$, $k \in \mathbb{N}$ *and* $(u_0, g) \in D$ *such that* $u_k - u_0 \in L^1(\mathbb{R}^n)$, $g_k - g \in L^1(\Pi_T)$ *and* $u_k - u_0 \to 0$ *in* $L^1(\mathbb{R}^n)$, $g_k - g \to 0$ *in* $L^1(\Pi_T)$ *then* $(u_0, g) \in D_u$.

Proof. Let $\alpha > 0$. It is evident that $(u_0, g) \in D_u$ if and only if $I_\alpha(u_0, g) = 0$. Thus $I_\alpha(u_k, g_k) = 0$ for all $k \in \mathbb{N}$. Because $u_k - u_0 \to 0$ in $L^1(\mathbb{R}^n)$, $g_k - g \to 0$ in $L^1(\Pi_T)$ the estimate (22) yields $I_\alpha(u_0, g) = 0$.

As it follows from Theorem 2, nonuniqueness of the g.e.s. of the problem (1), (2) doesn't disappear after small enough perturbation of the input data. More precisely the following statement holds:

Theorem 3. *Let* $(u_0,g)\in D\setminus D_u$, $u^{\pm}=J^{\pm}(u_0,g)$, $C=\operatorname*{ess\,sup}_{t\in(0,T)}\int_{\mathbb{R}^n}(u^+(t,x)-u^-(t,x))dx\le+\infty$. *If* $(v_0,h)\in D$, $u_0-v_0\in L^1(\mathbb{R}^n)$, $g-h\in L^1(\Pi_T)$ *and* $2(\|u_0-v_0\|_{L^1(\mathbb{R}^n)}+\|g-h\|_{L^1(\Pi_T)})<C$ *then* $(v_0,h)\notin D_u$.

Proof. By Levi theorem $C_\alpha=I_\alpha(u_0,g)\to C$ as $\alpha\to0+$. Hence there exists $\alpha>0$ such that

$$2(\|u_0-v_0\|_{L^1(\mathbb{R}^n)}+\|g-h\|_{L^1(\Pi_T)})<C_\alpha$$

and by the estimate (22) $I_\alpha(v_0,h)>0$, i.e. $(v_0,h)\notin D_u$.

Notice that in examples of nonuniqueness in [12]-[14] the constant $C=+\infty$. Thus in these examples uniqueness fails for any L^1 -perturbation of input data.

3. Uniqueness of g.e.s. in the case of periodic input data.

Let $\xi=\{\xi_1,\dots,\xi_{n-1}\}$ be a system of linear independent vectors in $\mathbb{R}^n$. Functions $u(x)\in L^\infty(\mathbb{R}^n)$, $g(t,x)\in L^\infty(\Pi_T)$ will be called ξ -periodic if $u(x+\xi_i)=u(x)$ a.e. on $\mathbb{R}^n$ and, respectively, $g(t,x+\xi_i)=g(t,x)$ a.e. on Π_T for all $i=1,\dots,n-1$.

Assume that input data $u_0(x)$, $g(t,x)$ of the problem (1), (2) are ξ -periodic functions. Then the following lemma holds.

Lemma 1. *The maximal and minimal g.e.s. of the problem* (1), (2) *are* ξ *-periodic functions.*

Proof. Let $u^+(t,x)$ be the maximal g.e.s. of the problem (1), (2), $\xi=\xi_i$, $1\le i\le n-1$. Then the functions $u^+(t,x\pm\xi)$ are g.e.s. of the same problem due to ξ -periodicity of input functions $u_0(x)$, $g(t,x)$. Since $u^+(t,x)$ is the maximal g.e.s. we have inequalities $u^+(t,x+\xi)\le u^+(t,x)$, $u^+(t,x-\xi)\le u^+(t,x)$ a.e. on Π_T . Applying shift transform to the latter inequality we can write it in the form $u^+(t,x)\le u^+(t,x+\xi)$ a.e. on Π_T . Together with the former inequality it implies that $u^+(t,x+\xi)=u^+(t,x)$ a.e. on Π_T , $i=1,\dots,n-1$. Thus $u^+(t,x)$ is a ξ -periodic function. The proof of ξ -periodicity of the minimal g.e.s. is the same.

Theorem 4. *The g.e.s. of the problem* (1), (2) *is unique.*

Proof. Using nondegenerate linear transform of spatial variables (see Remark 1, d)) we firstly reduce the problem to the case when $\xi_i=e_{i+1}$, $i=1,\dots,n-1$ are the last $n-1$ basis vectors of the space $\mathbb{R}^n$. Let $u^+(t,x)$ and $u^-(t,x)$ be the maximal and the minimal g.e.s. of the problem (1), (2). We only need to prove that these g.e.s. coincide almost everythere on Π_T .

Choose functions $p(y)\in C_0^\infty(\mathbb{R}^{n-1})$, $q(t,s)\in C_0^\infty(\bar{Q}_T)$, $\bar{Q}_T=[0,T)\times\mathbb{R}$; $p,q\ge0$, $\int_{\mathbb{R}^{n-1}}p(y)dy=1$ and let $p_\nu(y)=\nu^{1-n}p(y/\nu)$, $\nu\in\mathbb{N}$. Since the functions u^+ , u^- are g.e.s. of the problem (1), (2) they satisfy the identity (14) in Corollary 3. After substructing these identities for the functions u^+ , u^- with a test function $h(t,x)=p_\nu(x')q(t,x_1)$, $x'=(x_2,\dots,x_n)\in\mathbb{R}^{n-1}$ we obtain the following equality

$$\int_{\Pi_T}[(u^+-u^-)p_\nu(x')q_t(t,x_1)+$$

$$(\varphi_1(u^+)-\varphi_1(u^-))p_\nu(x')q_{x_1}(t,x_1)]dtdx+$$
$$\int_{\Pi_T}\sum_{i=2}^n(\varphi_i(u^+)-\varphi_i(u^-))\nu^{-n}p_{y_i}(x'/\nu)q(t,x_1)dtdx=0. \tag{23}$$

Write (23) in the form

$$\int_{\mathbb{R}^{n-1}}J_1(x')p_\nu(x')dx'+\int_{\mathbb{R}^{n-1}}\sum_{i=2}^n J_i(x')\nu^{-n}p_{y_i}(x'/\nu)dx'=0, \quad \text{where}$$

$$J_1(x')=\int_{\bar{Q}_T}[(u^+-u^-)q_t(t,x_1)+(\varphi_1(u^+)-\varphi_1(u^-))q_{x_1}(t,x_1)]dtdx_1,$$
$$J_i(x')=\int_{\bar{Q}_T}(\varphi_i(u^+)-\varphi_i(u^-))q(t,x_1)dtdx_1,\ i=2,\dots,n$$

and change variables setting $y=x'/\nu$. We obtain the relation

$$\int_{\mathbb{R}^{n-1}}J_1(\nu y)p(y)dy+\frac{1}{\nu}\int_{\mathbb{R}^{n-1}}\sum_{i=2}^n J_i(\nu y)p_{y_i}(y)dy=0. \tag{24}$$

By Lemma 1 u^+ and u^- are ξ-periodic functions that implies periodicity of the function J_1. It is evident also that $J_i\in L^\infty(\mathbb{R}^{n-1})$, $i=1,\dots,n$. Then as it's well-known the sequence $J_1(\nu y)$ converges $*$-weakly in $L^\infty(\mathbb{R}^{n-1})$ to a constant $C=\int_P J_1(y)dy$, where $P=[0,1]^{n-1}$, and passing to the limit as $\nu\to\infty$ in the relation (24) we deduce that $C=0$, i.e.

$$\int_{\bar{Q}_T\times P}[(u^+-u^-)q_t(t,x_1)+(\varphi_1(u^+)-\varphi_1(u^-))q_{x_1}(t,x_1)]dtdx=0. \tag{25}$$

Choose some constant $M\ge\max(\|u^+\|_\infty,\|u^-\|_\infty)$ and let

$$\omega(r)=\max\{\ |\varphi_1(u)-\varphi_1(v)|\ |\ u,v\in[-M,M],|u-v|\le r\ \}$$

be a modulus of continuity for the function $\varphi_1(u)$ on $[-M,M]$. The function $\omega(r)$ is nondecreasing and subadditive on $[0,+\infty)$ and $\lim\limits_{r\to 0+}\omega(r)=\omega(0)=0$. Let $\varepsilon>0$. Represent $r\ge 0$ in the form $r=k\varepsilon+s$ with $k\in\mathbb{N}\cup\{0\}$ and $0\le s<\varepsilon$. Since $\omega(r)$ is nondecreasing and subadditive we have

$$\omega(r)\le\omega((k+1)\varepsilon)\le(k+1)\omega(\varepsilon)\le(r+\varepsilon)\omega(\varepsilon)/\varepsilon=C_\varepsilon(r+\varepsilon), \tag{26}$$

for $C_\varepsilon=\omega(\varepsilon)/\varepsilon$. Now, let $\chi(s)\in C^\infty(\mathbb{R})$ be a nonincreasing function such that $\chi(s)=1$ for $s\le R$ and $\chi(s)=0$ for $s\ge R+1$ with some positive constant R, and a function $\theta(t)\in C_0^\infty([0,T))$ be such that $\theta(t)\ge 0$, $\theta'(t)\le 0$. Define the test function $q(t,x_1)=\chi(s)\theta(t)$, $s=|x_1|-C_\varepsilon(T-t)$. If we substitute this test function in (25) then we'll get the relation

$$\int_{\bar{Q}_T\times P}(u^+-u^-)\chi(s)\theta'(t)dtdx+\int_{\bar{Q}_T\times P}[C_\varepsilon(u^+-u^-)+$$
$$(\varphi_1(u^+)-\varphi_1(u^-))\operatorname{sign}(x_1)]\chi'(s)\theta(t)dtdx=0. \tag{27}$$

The values $u^+(t,x), u^-(t,x) \in [-M,M]$ for a.a. $(t,x) \in \Pi_T$ and by definition of the $\omega(r)$ and the relation (26)

$$|\varphi_1(u^+) - \varphi_1(u^-)| \le \omega(u^+ - u^-) \le C_\varepsilon(u^+ - u^- + \varepsilon) \ \text{ a.e. on } \Pi_T.$$

Therefore (27) implies the estimate (remind also that $\chi'(s) \le 0$, $\theta(t) \ge 0$, $\theta'(t) \le 0$)

$$\begin{aligned} &\int_{[0,T]\times[-R,R]\times P} (u^+ - u^-)|\theta'(t)|dtdx \le \\ &\int_{\bar{Q}_T\times P} (u^+ - u^-)\chi(s)|\theta'(t)|dtdx \le \int_{\bar{Q}_T\times P}[C_\varepsilon(u^+ - u^-) - \\ &\qquad C_\varepsilon(u^+ - u^- + \varepsilon)]\chi'(s)\theta(t)dtdx \le \\ &\varepsilon C_\varepsilon \cdot \|\theta(t)\|_\infty \int_{\bar{Q}_T} |\chi'(s)|dtdx_1 = 2T\varepsilon C_\varepsilon \cdot \|\theta(t)\|_\infty, \end{aligned} \tag{28}$$

because

$$\int_{\bar{Q}_T} |\chi'(s)|dtdx_1 = 2\int_0^T \int_0^{+\infty} -\chi'(x_1 - C_\varepsilon(T-t))dx_1dt = 2\int_0^T dt = 2T.$$

Since $\varepsilon C_\varepsilon = \omega(\varepsilon) \underset{\varepsilon\to 0+}{\to} 0$ passing to the limit as $\varepsilon \to 0+$ in (28) yields

$$\int_{[0,T]\times[-R,R]\times P} (u^+ - u^-)|\theta'(t)|dtdx = 0$$

for all $R > 0$ and for all choose of θ . It evidently implies identity $u^+ = u^-$ a.e. on Π_T as required. Theorem is proved.

Corollary 5 (comparison principle). *Let $v(t,x)$ be a g.e.sub-s and $w(t,x)$ be a g.e.super-s of the problem* (1), (2) *with input data $(v_0(x), g(t,x))$ and $(w_0(x), h(t,x))$ respectively, and suppose that $v_0(x) \le w_0(x)$ a.e. on $\mathbb{R}^n$, $g(t,x) \le h(t,x)$ a.e. on Π_T . If at least one of the data $(v_0(x), g(t,x))$ or $(w_0(x), h(t,x))$ are ξ -periodic functions then $v(t,x) \le w(t,x)$ a.e. on Π_T .*

Proof. Let $v^\pm(t,x) = J^\pm(v_0, g)$, $w^\pm(t,x) = J^\pm(w_0, h)$. Assume that the functions $w_0(x), h(t,x)$ are ξ -periodic. Then, by Corollary 4 and Theorem 4

$$v(t,x) \le v^+(t,x) \le w^+(t,x) = w^-(t,x) \le w(t,x) \ \text{ a.e. on } \Pi_T.$$

The case of ξ -periodic functions $v_0(x), g(t,x)$ is treated in the similar way:

$$v(t,x) \le v^+(t,x) = v^-(t,x) \le w^-(t,x) \le w(t,x) \ \text{ a.e. on } \Pi_T.$$

In addition to Theorem 4 remark that in the case of ξ -periodical data the problem (1), (2) can be considered as the Cauchy problem with spatial variables on a manifold $T^{n-1} \times \mathbb{R}$, where T^{n-1} is a $n-1$ -dimensional torus. One of the consequences of Theorem 4 is existence and uniqueness of g.e.s. of this problem which is regarded in the sence of [16] (where the general case of arbitrary manifold was investigated).

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