CONVERGENCE OF VISCOSITY SOLUTIONS FOR ISOTHERMAL GAS DYNAMICS

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Abstract. We study the hyperbolic system of Euler equations for an isothermal, compressible fluid. The *strong convergence theorem* of approximate solutions is proved by the theory of compensated compactness. The existence of weak entropy solution of Cauchy problems with large L^∞ initial data which may include vacuum is also obtained. We note that we establish the commutation relations not only for the *weak* entropies but also for the *strong* ones by using the *analytic extension theorem*.

Key Words: Isothermal gas dynamics, Compensated compactness, Analytic extension theorem

1. Introduction

The one dimension Euler equations of compressible fluid read

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \tag{1.1}$$

where the unknown variable $\rho \geq 0$ denotes the density of the mass, u the velocity. Usually, it is convenient to study (1.1) by a new variable, the momentum $m = \rho u$. Thus (1.1) becomes

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + (\frac{m^2}{\rho} + p(\rho))_x = 0. \end{cases} \tag{1.2}$$

For polytropic perfect gas, $p = p_0\rho^\gamma$ with γ the adiabatic exponent. (1.1) is called isentropic gas dynamics for $\gamma > 1$, while it is called isothermal gas dynamics for $\gamma = 1$. Without loss of generality, p_0 is normalized to be 1 here.

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One of the main difficulties for the mathematical analysis of (1.1) is the singularity at the vacuum $\rho = 0$. It is noted that the term ρu^2 is only Lipschitz continuous for $\gamma > 1$ near the vacuum, while it is not even Lipschitz continuous for $\gamma = 1$ due to the infiniteness of the velocity u. This shows that the isothermal case is completely different from the isentropic one. Another difficulty is the development of shock wave in solutions of (1.1) no matter how smooth the initial data are.

Away from vacuum, the first result on the existence of BV solution with large initial data in the BV space was obtained in Nishida [16] by using the Glimm scheme [9] for $\gamma = 1$. Poupaud, Rascle and Vila [18] made a great simplification and improved the results of [16] to the isothermal Euler-Poisson system. For the case of $\gamma > 1$, the existence of BV solution was established in Nishida and Smoller [17] for large total variation with small $\gamma - 1$.

When vacuum occurs, the first global existence for (1.1) with large initial data in L^∞ was established in Diperna [8] for $\gamma = 1+\frac{2}{2n+1}, n \geq 2$ by the theory of compensated compactness. For the interval $(1, \frac{5}{3}]$, the existence was solved by Ding, Chen and Luo [1, 5]. Lions, Perthame, Tadmor and Sougandis [12,13] treated this problem for $\gamma > \frac{5}{3}$. More recently, Chen and Le Floch [3] studied the existence problem for general pressure where p acts like γ-law $(1 < \gamma < 3)$ near the vacuum $\rho = 0$. The approach of [3] further simplified the proofs for the case of $\gamma > 1$.

The purpose of the present paper is to prove the existence of global weak L^∞ solution for isothermal gas dynamics with large initial data by the theory of compensated compactness. Our initial data are

$$(\rho(0,x), m(0,x)) = (\rho_0(x), m_0(x)) \in L^\infty, \tag{1.3}$$

where the vacuum may occur. It is known that all effective approaches for isentropic gas dynamics are based on the subtle analysis for Euler-Poisson-Darboux (EPD) equation, while the entropy equation of $\gamma = 1$ is not governed by EPD equation. Therefore all previous approaches [1,3,5,8,12,13] fail here. We shall use a new approach to achieve our goal. We establish the commutation relations for some strong entropies even though we do not know whether these strong waves satisfy the H^{-1} compact condition or not. It should be noted here that the strong entropy is useless for isentropic gas dynamics (see Lions, Perthame and Tadmor [12]).

In addition, we note that the vacuum may disappear in the solution to the Riemann problem for isothermal flow. This fact does not mean that studying vacuum is not meaningful. It is well known that the centered rarefaction wave disappears in the solution immediately for scalar conservation laws. But people investigate the initial value problems when the initial value problem includes the centered rarefaction wave for scalar equation.

Now we recall the definition of the weak entropy solution.

Definition 1. $(\rho, m)(x,t) \in L^\infty(R^2_+)$ *is called a weak entropy solution of (1.2), if it holds, for any test function* $\phi \in C_0^\infty(R^2_+)$, *that*

$$\begin{cases} \displaystyle\iint_{t>0} (\rho\phi_t + m\phi_x)\ dxdt + \int_R \rho_0(x)\phi(x,0)\ dx = 0, \\ \displaystyle\iint_{t>0} m\phi_t + (\frac{m^2}{\rho} + p(\rho))\phi_x\ dxdt + \int_R m_0(x)\phi(x,0)\ dx = 0, \end{cases} \tag{1.4}$$

and for any weak entropy pair $(\eta, q)(\rho, m)$ *with convex* $\eta(\rho, m)$,

$$\eta(\rho, m)_t + q(\rho, m)_x \leq 0, \tag{1.5}$$

holds in the sense of distributions. Here the entropy pair (η, q) *is determined by the additional conservation law*

$$\eta(\rho, m)_t + q(\rho, m)_x = 0, \tag{1.6}$$

for any smooth solution of (1.2) and weak entropy is an entropy that vanishes at the vacuum.

Our main results are

Theorem 1 (Existence theorem).

Let $\gamma = 1$, *assume that the initial data satisfy*

$$0 \leq \rho_0(x) \leq M, \quad |m_0(x)| \leq \rho_0(x)(M + |\log\rho_0(x)|), \quad a.e. \tag{1.7}$$

then there exists a global weak entropy solution of (1.2), (1.3) satisfying

$$0 \leq \rho(x,t) \leq C, \quad |m(x,t)| \leq \rho(x,t)(C + |\log\rho(x,t)|), \quad a.e., \tag{1.8}$$

where C *only depends on* M.

Remark 1. It is known that Nishida [16] had established the existence of global BV solution for the isothermal gas dynamics if the initial data $(\rho_0, u_0) \in BV(R)$ and $\rho_0 > c > 0$. But here our initial data is more rough, i.e. $(\rho_0, u_0) \in L^\infty(R)$. Furthermore, our initial data may include the vacuum.

Theorem 2 (Compactness framework).

Let $\gamma = 1$ and $(\rho^\varepsilon, m^\varepsilon)$ be a sequence of approximate solutions of (1.2) satisfying (1.8) uniformly in ε. Assume that,

$$\partial_t \eta(\rho^\varepsilon, m^\varepsilon) + \partial_x q(\rho^\varepsilon, m^\varepsilon) \text{ is compact in } H^{-1}_{loc}, \tag{1.9}$$

holds for some (not all) weak entropies (η, q) with

$$\eta = \rho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2} u}, \quad q = (u+\xi)\eta, \quad \xi \in (-1,1), \tag{1.10}$$

then there exists a function $(\rho(x,t), m(x,t))$ satisfying (1.8) such that, extracting a subsequence if necessary,

$$(\rho^\varepsilon(x,t), m^\varepsilon(x,t)) \to (\rho(x,t), m(x,t)), \quad \text{in } L^p_{loc}(R^2_+), \tag{1.11}$$

for all $p \in [1, +\infty)$.

Remark 2. Theorem 2 is useful to study the existence of global weak L^∞ solution of (1.1) with source term for $\gamma = 1$, for instance, the compressible Euler equations with damping, the Euler-Poisson system, etc.

Before we explain our ideas, it is worthwhile to briefly recall the theory of compensated compactness. To prove the existence, one usually constructs a sequence of approximate solutions $(\rho^\varepsilon, m^\varepsilon)$ by using viscosity perturbation or a finite difference scheme, then extracts a strong subsequence of $(\rho^\varepsilon, m^\varepsilon)$, if necessary, to get the desired results. However, it is very difficult to get a strong convergent subsequence for (1.2). Usually it is easy to obtain the uniform boundness estimates for $(\rho^\varepsilon, m^\varepsilon)$, which indicates that extracting a weak convergent subsequence is available. It is well known that only weak convergence is not sufficient to imply the existence of weak solution due to the nonlinearity of (1.2). So some information on the derivative of $(\rho^\varepsilon, m^\varepsilon)$ is needed. Tartar [20] first applied Young measure to introduce the commutation relations

$$< \nu_{x,t}, q_1\eta_2 - q_2\eta_1 >=< \nu_{x,t}, q_1 >< \nu_{x,t}, \eta_2 > - < \nu_{x,t}, q_2 >< \nu_{x,t}, \eta_1 >, \tag{1.12}$$

with any two entropy pairs $(\eta_i, q_i), i = 1, 2$ for a.e. (x,t), if

$$\eta_i(\rho^\varepsilon, m^\varepsilon) + q_i(\rho^\varepsilon, m^\varepsilon), \tag{1.13}$$

lie in a compact subset of H^{-1}_{loc} as ε vanishes. If Young measure satisfying (1.12) reduces to a point mass for a.e. (x,t), then the weak convergence becomes strong, and the existence of weak solution is established. Therefore the H^{-1} compact condition is essential to the theory of compensated compactness.

For strictly hyperbolic systems with smooth flux, the H^{-1} compact condition is easy due to the uniform boundness of approximate solutions and Murat's lemma [15] provided that the system has a strictly convex entropy. However, for isentropic gas dynamics, not all entropy pairs can be applied into Tartar commutation relations (1.12) because only weak entropy-pairs are known to satisfy the H^{-1} compact condition. As pointed out by Lions, Perthame and Tadmor [12], strong entropies are useless for isentropic case. Fortunately, since all weak entropies obey famous EPD equation, people (see [1,5,8,12,13]) are able to imply the Young measure is either a single point or a subset of the vacuum by careful entropy analysis for EPD equation, and then prove the existence of weak solution for the isentropic case. More precisely, in the proof of [1,5,8], the heart of the matter is to construct the special weak entropies and apply them into the commutation relations. This is possible because (1.12) represents an imbalance of regularity: the operator on the left is more regular than the one on the right due to cancellation. The novel idea of applying the technique of fractional derivatives was introduced in [5]. A new analysis of (1.12) was proposed by Lions, Perthame, Tadmor and Souganidis [12,13] for $\gamma > 1$. Motivated by a kinetic formulation of (1.1), they made the crucial observation that the use of special weak entropies could be bypassed and (1.12) be directly expressed with the entropy kernel of EPD equation. We refer to [1,5,8,12,13] for details. Even though only weak entropy pairs are used in the case $\gamma > 1$, Diperna [8] conjectured that it may be possible to establish the commutation relations for all entropy pairs, weak and strong ones. If it is true, the proof is quite simple (see [8]).

In general, to exploit the classical theory of compensated compactness, the following steps are necessary:

(1) To construct a sequence of approximate solutions and obtain the uniform boundness of approximate solutions,

(2) To establish the H^{-1} compact condition for infinite entropy pairs,

(3) To apply div-curl lemma into all entropy pairs satisfying (2) to establish the commutation relations,

(4) To apply the commutation relations to reduce Young measure to a point mass for a.e. (x,t).

Similar to the isentropic case, it is also difficult for $\gamma = 1$ to prove the H^{-1} compact condition for strong entropy pairs (step 2). In fact, we do not know whether the strong entropy pairs satisfy the H^{-1} compact condition or not. Compared with the isentropic case, the main difficulties for the isothermal flow arise on the following two aspects: the infiniteness of eigenvalues due to the presence of vacuum and the fact that the entropy equation is not of EPD type. Among them, the second one is essential. Thus all approaches of [1,5,8,12,13] fail here.

Since only weak entropies are known to satisfy H^{-1} compact condition for $\gamma = 1$, it seems that the strong entropies are useless like the isentropic case. However, the use of strong entropies is the key point of our proofs. The main novelty in this paper is that we establish the commutation relations not only for the *weak* entropies but also for the *strong* ones by using the *analytic extension theorem* even though we do not know whether strong entropies satisfy H^{-1} compact condition or not.

To achieve our goal, we first choose a special formula of entropies parameterized by a complex variable ξ. The formula includes both weak and strong entropies determined by the value of ξ. Then we prove that there exists a segment such that for any ξ belonging to the segment, the entropy pair is of weak type and satisfies the H^{-1} compact condition. Therefore the commutation relations are established for some weak entropies in this segment. It is observed that the two sides of (1.12) are regular for ξ. In fact, they are analytic functions with respect to ξ. So the commutation relations exactly hold for the whole complex space except two points (-1,0) and (1,0) due to the analytic extension theorem. It is noted that the entropies are strong if $|\xi| > 1$ (see (2.9)). Therefore Diperna's conjecture [8] is partially verified for isothermal flow, i.e. the commutation relations hold for some weak and strong entropy pairs. We note that the H^{-1} compact condition for strong entropies (step 2) can be bypassed. Since both weak and strong entropy pairs are applied into (1.12), we establish a strong convergence theorem of approximate solutions and prove the existence of weak entropy solution for isothermal gas dynamics. Finally, it is worthwhile to point out that our approximate solutions are constructed by adding the viscosity perturbation to (1.1) due to the infiniteness of eigenvalues. It is observed that the eigenvalues $\lambda_1 = u - 1$ and $\lambda_2 = u + 1$ increase with the speed of $|\ln \rho|$. This indicates the possibility of constructing approximate solutions by numerical scheme. We will discuss it in the future.

This paper is organized as follows: in section 2, we give a formula of entropy pairs, parameterized by a complex variable ξ. In section 3, we study the viscosity solutions $(\rho^\varepsilon, m^\varepsilon)$ and prove the weak entropy fields lie in a compact subset of H^{-1}_{loc} when $\xi \in (-1, 1)$ as ε vanishes. In section 4, we prove a strong convergence theorem of approximate solutions and obtain the existence of weak solution for isothermal gas dynamics.

2. Entropy waves

This section is devoted to the entropy for isothermal flow. We recall that (η, q) is an entropy-flux pair if for any smooth solutions of (1.1), it satisfies an additional equation,

$$\eta_t(\rho, u) + q_x(\rho, u) = 0. \tag{2.1}$$

By definition, weak entropy is an entropy η that vanishes at the vacuum.

Let $\gamma = 1$, (2.1) yields $\nabla\eta\nabla f = \nabla q$ with the flux $f = (\rho u, \rho u^2 + \rho)^T$, i.e.

$$q_\rho = u\eta_\rho + \frac{1}{\rho}\eta_u,\ q_u = \rho\eta_\rho + u\eta_u, \tag{2.2}$$

which indicates,

$$\eta_{\rho\rho} = \frac{1}{\rho^2}\eta_{uu}. \tag{2.3}$$

We choose the form $\eta = h(\rho)e^{ku}$, then (2.3) infers

$$h'' - \frac{k^2}{\rho^2}h = 0. \tag{2.4}$$

Thus, we have $h(\rho) = \rho^m$ with $m(m-1) = k^2$. Now we consider the parameter k in the complex space. Let $k = \frac{\xi}{1-\xi^2}$, $\xi \in \mathbf{C}$, then $m = \frac{1}{1-\xi^2}$. Therefore we have the following formula of entropy pairs

$$\begin{aligned} \eta &= \rho^{\frac{1}{1-\xi^2}} e^{\frac{\xi}{1-\xi^2}u}, \\ q &= (u+\xi)\eta. \end{aligned} \tag{2.5}$$

We note that these entropies are analytic functions with respect to ξ. It is easy to see the points (-1,0) and (1,0) are singular for (η, q).

On the other hand, it is convenient to introduce a coordinate system of Riemann invariants (w, z) with

$$\nabla w \cdot r_1 = 0,\ \nabla z \cdot r_2 = 0, \tag{2.6}$$

where $r_1 = (1, u-1)^T$, $r_2 = (1, u+1)^T$ are the right eigenvectors of the Jacobian matrix of f: $\nabla f r_i = \lambda_i r_i$, $i = 1, 2$. In the setting of isothermal flow, the Riemann invariants read

$$w = \rho e^u,\ z = \rho e^{-u}. \tag{2.7}$$

Thus we rewrite (2.5) as

$$\begin{aligned} \eta &= w^{\frac{1}{2(1-\xi)}} z^{\frac{1}{2(1+\xi)}}, \\ q &= (u+\xi)\eta. \end{aligned} \tag{2.8}$$

It is observed that the formula (2.8) includes two kinds of entropies determined by the new complex variable ξ. By definition, η is a weak entropy if and only if the following holds

$$\mathrm{Re}\frac{1}{2(1-\xi)} > 0, \ \mathrm{Re}\frac{1}{2(1+\xi)} > 0, \tag{2.9}$$

i.e. $\xi \in \Omega_w = \{\xi \in \mathbf{C}; -1 < \mathrm{Re}\xi < 1\}$.

In fact, we can get more information from (2.8). If we consider ξ in the real space, the entropies defined in (2.8) form a fundamental set of weak entropies for isothermal flow, i.e.

Lemma 2.1. *Let $\gamma = 1$, then for any $\varphi(\xi) \in C_0^\infty(-1,1)$,*

$$\begin{aligned} \eta &= \int_{-1}^{1} \varphi(\xi) w^{\frac{1}{2(1-\xi)}} z^{\frac{1}{2(1+\xi)}} d\xi, \\ q &= \int_{-1}^{1} \varphi(\xi)(u+\xi) w^{\frac{1}{2(1-\xi)}} z^{\frac{1}{2(1+\xi)}} d\xi \end{aligned} \tag{2.10}$$

is a weak entropy-flux pair of (1.1).

3. Viscosity solutions

We consider the following viscous perturbation of the isothermal flow

$$\begin{cases} \rho_t^\varepsilon + m_x^\varepsilon = \varepsilon \rho_{xx}^\varepsilon, \\ m_t^\varepsilon + (\frac{(m^\varepsilon)^2}{\rho^\varepsilon} + \rho^\varepsilon)_x = \varepsilon m_{xx}^\varepsilon, \end{cases} \tag{3.1}$$

with initial data

$$(\rho^\varepsilon, m^\varepsilon)|_{t=0} = (\rho_0^\varepsilon(x), m_0^\varepsilon(x)), \tag{3.2}$$

where $(\rho_0^\varepsilon(x)$,$m_0^\varepsilon(x))$ satisfy

$$\varepsilon \le \rho_0^\varepsilon(x) \le M, \ |m_0^\varepsilon(x)| \le M. \tag{3.3}$$

It is easy to see (3.3) holds if $\rho_0^\varepsilon(x)$ is given by smoothing out $\rho_0(x)$ with a standard mollifier and adding ε.

In terms of the theory of positive invariant region in Chueh, Conley and Smoller [4], it is easy to see $\{(w,z);\ w \le \text{const},\ z \le \text{const}\}$ is the invariant region of (3.1), which indicates that the Riemann invariants w^ε, z^ε are uniformly bounded in L^∞. This implies that $(\rho^\varepsilon, m^\varepsilon)$ are also uniformly bounded in L^∞. It is noted that there always exists a local smooth solution for (3.1) due to Diperna [8]. In order to prove the existence of smooth solution to (3.1) and (3.2), it is also important to obtain a priori estimate of lower bound for the density ρ^ε. Diperna first gave the lower bound by his lemma 4.1 (see [8]) even though this lemma was stated in an incorrect way. Chen [2] fixed this lemma. On the other hand, Lu [14] also studied the lower bound for general pressure $p(\rho)$ by maximum principle, in which, the restriction of initial data on the infinity was not needed. Thus the uniform L^∞ estimates and the lower bound of ρ^ε adding the local existence theorem gives the following global existence result.

Lemma 3.1. *If the initial data satisfy the condition (3.3), then for any fixed $\varepsilon > 0$, there exists a smooth solution for the Cauchy problem (3.1),(3.2) in $R_T = R \times [0,T]$ (for arbitrary T), which satisfies*

$$0 < c(\varepsilon, t) \le \rho^\varepsilon(x,t) \le C, \ \ |m^\varepsilon(x,t)| \le \rho^\varepsilon(x,t)(C + |\log \rho^\varepsilon(x,t)|), \tag{3.4}$$

where $c(\varepsilon,t)$ is an appropriate function and C only depends on M.

In order to apply the theory of compensated compactness, it is necessary to prove the divergence of weak entropy-flux pair is in a compact subset of H^{-1} as ε vanishes.

Take the form (η, q) as in (2.8), let $\xi \in (-1,1)$, we compute,

$$\begin{aligned}
\eta_{\rho\rho} &= \frac{\xi^2}{(1-\xi^2)^2}(1 - 2\xi u + u^2)\rho^{\frac{\xi^2}{1-\xi^2}-1} e^{\frac{\xi}{1-\xi^2}u} > 0, \\
\eta_{\rho m} &= \frac{\xi^2}{(1-\xi^2)^2}(\xi - u)\rho^{\frac{\xi^2}{1-\xi^2}-1} e^{\frac{\xi}{1-\xi^2}u}, \\
\eta_{mm} &= \frac{\xi^2}{(1-\xi^2)^2}\rho^{\frac{\xi^2}{1-\xi^2}-1} e^{\frac{\xi}{1-\xi^2}u} > 0,
\end{aligned} \tag{3.5}$$

and

$$\eta_{\rho\rho}\eta_{mm} - \eta_{\rho m}^2 = \frac{\xi^4}{(1-\xi^2)^3}\rho^{\frac{2\xi^2}{1-\xi^2}-2} e^{\frac{2\xi}{1-\xi^2}u} > 0, \tag{3.6}$$

which indicates that η is strictly convex for any $\xi \in (-1,1)$. This implies that

$$\eta(\rho^\varepsilon, m^\varepsilon)_t + q(\rho^\varepsilon, m^\varepsilon)_x$$

is compact in H^{-1}_{loc} due to Diperna [7,8]. Therefore we have the following lemma.

Lemma 3.2. *Assume that $(\rho^\varepsilon, m^\varepsilon)$ are the solutions of (3.1), (3.2), then for any $\xi \in (-1,1)$,*

$$\eta_t(\rho^\varepsilon, m^\varepsilon) + q_x(\rho^\varepsilon, m^\varepsilon) \text{ is compact in } H^{-1}_{loc}, \tag{3.7}$$

where (η, q) is defined as in (2.8).

4. Convergence of approximate solutions

This section is devoted to the existence of weak solution of isothermal gas dynamics. Choose $(\rho^\varepsilon, m^\varepsilon)$ as in (3.4), there exists a subsequence of $(\rho^\varepsilon, m^\varepsilon)$

(still denoted by $(\rho^\varepsilon, m^\varepsilon)$), such that, as $\varepsilon \to 0$,

$$\rho^\varepsilon(x,t) \rightharpoonup \rho(x,t), \;\; m^\varepsilon(x,t) \rightharpoonup m(x,t), \tag{4.1}$$

in $L^\infty((0,T)\times R)$ weak star, for some measurable functions $\rho(x,t), m(x,t)$.

Let us denote $\nu_{x,t}$ the Young measure associated to the weak limits (4.1). For any two entropy pairs in (2.8),

$$\eta_1 = w^{\frac{1}{2(1-\xi_1)}} z^{\frac{1}{2(1+\xi_1)}}, \qquad \eta_2 = w^{\frac{1}{2(1-\xi_2)}} z^{\frac{1}{2(1+\xi_2)}},$$

$$q_1 = (u+\xi_1)\eta_1, \qquad q_2 = (u+\xi_2)\eta_2, \qquad \xi_1, \xi_2 \in (-1,1),$$

Lemma 3.2 gives

$$\begin{aligned} < \nu_{x,t}, q_1\eta_2 - q_2\eta_1 >= & < \nu_{x,t}, q_1 >< \nu_{x,t}, \eta_2 > \\ & - < \nu_{x,t}, q_2 >< \nu_{x,t}, \eta_1 >, \quad \text{a.e.} \quad x,t, \end{aligned} \tag{4.2}$$

i.e.

$$\begin{aligned} (\xi_1 - \xi_2) < \nu_{x,t}, \eta_1\eta_2 >= & < \nu_{x,t}, (u+\xi_1)\eta_1 >< \nu_{x,t}, \eta_2 > \\ & - < \nu_{x,t}, (u+\xi_2)\eta_2 >< \nu_{x,t}, \eta_1 >, \quad \text{a.e.} \quad x,t. \end{aligned} \tag{4.3}$$

We shall show $\nu_{x,t}$ is either a point mass or concentrated in the vacuum. To this end, we show (4.2) holds for any $\xi_1, \xi_2 \in \mathbf{C}$ except two points $(-1,0)$ and $(1,0)$ through analytic extension theorem. In another word, we establish the commutation relations for both weak and strong waves.

Since w, z is bounded, it is convenient to study (4.2) in $w-z$ plane. Let

$$\Omega = \{(w,z); 0 \le w_- \le w \le w_+, \; 0 \le z_- \le z \le z_+\}$$

be the smallest rectangle containing the support of a fixed $\nu_{x,t}$. It is easy to see, if $w_+ = 0$ or $z_+ = 0$, Ω is supported in the vacuum. So we assume that $w_- < w_+$, $z_- < z_+$ in what follows.

Let $\xi_1 = 1 - \frac{1}{2n}$, then $\eta_1 = \eta_n = w^n z^{\frac{n}{4n-1}}$. From (4.2), we have

$$< \nu_{x,t}, q_n\eta - q\eta_n >=< \nu_{x,t}, q_n >< \nu_{x,t}, \eta > - < \nu_{x,t}, q >< \nu_{x,t}, \eta_n > . \tag{4.4}$$

We define probability measure μ_z as follows: for any $h \in C_0(R^2)$,

$$< \mu_z, h >= \lim_{n\to\infty} \frac{< \nu_{x,t}, h w^n z^{\frac{n}{4n-1}} >}{< \nu_{x,t}, w^n z^{\frac{n}{4n-1}} >}. \tag{4.5}$$

Similar to [7], it is easy to check that the support of the probability measure μ_z is contained in the line $w = w_+$.

On the other hand, (4.4) yields

$$\frac{< \nu_{x,t}, q_n\eta - q\eta_n >}{< \nu_{x,t}, \eta_n >} = \frac{< \nu_{x,t}, q_n >}{< \nu_{x,t}, \eta_n >} < \nu_{x,t}, \eta > - < \nu_{x,t}, q > . \tag{4.6}$$

Let $n \to \infty$, we have

$$< \mu_z, q - \lambda_2 \eta > = < \nu_{x,t}, q - \lambda_2^+ \eta >, \tag{4.7}$$

where $\lambda_2^+ = < \mu_z, \lambda_2 >$ is finite even though $\lambda_2 = u + 1 = \frac{\ln w - \ln z}{2} + 1$ may go to infinity. In fact, since the left term of (4.7) is finite and $< \nu_{x,t}, \eta >$ is positive, λ_2^+ must be bounded. In the same way, let $\xi = -1 + \frac{1}{2n}$, we have

$$< \mu_w, q - \lambda_1 \eta > = < \nu_{x,t}, q - \lambda_1^+ \eta >, \tag{4.8}$$

with

$$\begin{aligned} &< \mu_w, h > = \lim_{n\to\infty} \frac{<\nu_{x,t}, h w^{\frac{n}{4n-1}} z^n>}{<\nu_{x,t}, w^{\frac{n}{4n-1}} z^n>}, \ \forall h \in C_0(R^2), \\ &\lambda_1^+ = < \mu_w, \lambda_1 > . \end{aligned} \tag{4.9}$$

Combining (4.7) and (4.8), we have

Lemma 4.1. *Let* $\eta = w^{\frac{1}{2(1-\xi)}} z^{\frac{1}{2(1+\xi)}}, q = (u+\xi)\eta, \xi \in (-1,1)$*, then the following holds*

$$\begin{aligned} (\lambda_2^+ - \lambda_1^+) \tfrac{<\nu_{x,t},\eta>}{\eta(w^+,z^+)} &= (1+\xi) < \mu_w, (\tfrac{w}{w^+})^{\frac{1}{2(1-\xi)}} > \\ &+ (1-\xi) < \mu_z, (\tfrac{z}{z^+})^{\frac{1}{2(1+\xi)}} > . \end{aligned} \tag{4.10}$$

In view of Lemma 4.1, we can prove $\nu_{x,t}(\{\rho > 0\}) = 1,\ a.e.\ x, t$. This means that there is no positive mass concentrated in the vacuum for the measure $\nu_{x,t}$. In fact, by the definitions of μ_w and μ_z, it is easy to check that $\mu_w(\{w > 0\}) = \mu_z(\{z > 0\}) = 1$, thus (4.10) can be rewritten as

$$\begin{aligned} (\lambda_2^+ - \lambda_1^+) \frac{< \nu_{x,t}, \eta >}{\eta(w^+, z^+)} \ &= \ < \mu_w, (1+\xi)\big[(\frac{w}{w^+})^{\frac{1}{2(1-\xi)}} - 1\big] > \\ &+ < \mu_z, (1-\xi)\big[(\frac{z}{z^+})^{\frac{1}{2(1+\xi)}} - 1\big] > +2. \end{aligned} \tag{4.11}$$

Letting Im $\xi \to \infty$, we get

$$\begin{aligned} (\lambda_2^+ - \lambda_1^+) \nu_{x,t}(\{\rho > 0\}) \ &= -\tfrac{1}{2} < \mu_w, \ln(\tfrac{w}{w^+}) > -\tfrac{1}{2} < \mu_z, \ln(\tfrac{z}{z^+}) > +2 \\ &= (\lambda_2^+ - \lambda_1^+), \end{aligned} \tag{4.12}$$

due to $< \mu_w, \ln w_+ > = < \mu_z, \ln w_+ > = \ln w_+$ and $0 \le \frac{w}{w_+}, \frac{z}{z_+} \le 1$. This implies $\nu_{x,t}(\{\rho > 0\}) = 1$.

Since the functions $< \nu_{x,t}, \eta >$, $< \mu_w, w^{\frac{1}{2(1-\xi)}} >$ and $< \mu_z, z^{\frac{1}{2(1+\xi)}} >$ are analytic in the domain Ω_w and (4.10) holds in the segment $(-1, 1)$, (4.10) must hold for all $\xi \in \Omega_w = \{\xi;\ -1 < \text{Re}\ \xi < 1\}$ due to analytic extension theorem. We now establish the commutation relations for the whole complex space except two singular points (-1,0) and (1,0), i.e. (4.10) holds not only for weak entropies but also for the strong ones.

For any function $h \in C_0(R^2)$, we define a probability measure as follows

$$< \overline{\mu}_z, h >= \lim_{n\to\infty} \frac{< \nu_{x,t}, hw^n >}{< \nu_{x,t}, w^n >}. \tag{4.13}$$

In the same way as in [7], it is easy to check supp $\overline{\mu}_z \subset \{(w,z); w = w^+\}$. We compute that, $\forall h \in C_0(R^2)$,

$$\begin{aligned} < \mu_z, h > &= \lim_{n\to\infty} \frac{< \nu_{x,t}, hw^n z^{\frac{n}{4n-1}} >}{< \nu_{x,t}, w^n >} \lim_{n\to\infty} \frac{< \nu_{x,t}, w^n >}{< \nu_{x,t}, w^n z^{\frac{n}{4n-1}} >} \\ &= \frac{< \overline{\mu}_z, hz^{\frac{1}{4}} >}{< \overline{\mu}_z, z^{\frac{1}{4}} >}, \end{aligned} \tag{4.14}$$

which implies that $< \mu_z, z^{\frac{1}{2(1+\xi)}} >$ is analytic in the domain Re $\frac{1}{2(1+\xi)} > -\frac{1}{4}$. In the same way, $< \mu_w, w^{\frac{1}{2(1-\xi)}} >$ is also analytic in the domain Re $\frac{1}{2(1-\xi)} > -\frac{1}{4}$. Thus, the right term of (4.10) is analytic in the domain $\Omega_0 = \{\xi \in \mathbf{C}; \mathrm{Re}\frac{1}{2(1+\xi)} > -\frac{1}{4},\ \mathrm{Re}\frac{1}{2(1-\xi)} > -\frac{1}{4}\}$. Next we show (4.10) holds for any $\xi \in \bar{\Omega}_0 = \{\xi \in \mathbf{C}; |\xi| > 3\} \subset \Omega_0$ by analytic extension theorem. Since $\eta = w^{\frac{1}{2(1-\xi)}} z^{\frac{1}{2(1+\xi)}}$ may be unbounded in $\bar{\Omega}_0$, it is necessary to show that $< \nu_{x,t}, \eta >$ is well defined for any $\xi \in \bar{\Omega}_0$.

We compute

$$\begin{aligned} f(\xi; w, z) =\ & \Big(\frac{w}{w_+}\Big)^{\frac{1}{2(1-\xi)}} \Big(\frac{z}{z_+}\Big)^{\frac{1}{2(1+\xi)}} + \Big(\frac{w}{w_+}\Big)^{\frac{1}{2(1+\xi)}} \Big(\frac{z}{z_+}\Big)^{\frac{1}{2(1-\xi)}} \\ =\ & \mathrm{e}^{\frac{\ln\frac{w}{w_+} + \ln\frac{z}{z_+}}{2(1-\xi^2)}} \Big[\mathrm{e}^{\frac{(\ln\frac{w}{w_+} - \ln\frac{z}{z_+})\xi}{2(1-\xi^2)}} + \mathrm{e}^{\frac{-(\ln\frac{w}{w_+} - \ln\frac{z}{z_+})\xi}{2(1-\xi^2)}}\Big] \\ =\ & 2\left\{1 + \sum_{k=1}^{\infty} \frac{\big(-\ln\frac{w}{w_+} - \ln\frac{z}{z_+}\big)^k}{2^k k!} \Big(\frac{\frac{1}{\xi^2}}{1-\frac{1}{\xi^2}}\Big)^k\right\} \\ & \times \left\{1 + \sum_{j=1}^{\infty} \frac{\big(\ln\frac{w}{w_+} - \ln\frac{z}{z_+}\big)^{2j}}{4^j (2j)!} \Big(\frac{\frac{1}{\xi}}{1-\frac{1}{\xi^2}}\Big)^{2j}\right\} \\ =\ & 2\left\{1 + \sum_{m=1}^{\infty} \Big[\sum_{k=1}^{m} \frac{\big(-\ln\frac{w}{w_+} - \ln\frac{z}{z_+}\big)^k}{2^k k!} \binom{m-1}{k-1}\Big] \frac{1}{\xi^{2m}}\right\} \\ & \times \left\{1 + \sum_{l=1}^{\infty} \Big[\sum_{j=1}^{l} \frac{\big(\ln\frac{w}{w_+} - \ln\frac{z}{z_+}\big)^{2j}}{4^j (2j)!} \binom{l+j-1}{2j-1}\Big] \frac{1}{\xi^{2l}}\right\} \\ =\ & \sum_{p=0}^{\infty} \frac{c_{2p}(w,z)}{\xi^{2p}}. \end{aligned} \tag{4.15}$$

It is observed that $\ln\frac{w}{w_+} + \ln\frac{z}{z_+} \le 0$, we have $c_{2p}(w,z) \ge 0$ in supp $\nu_{x,t}$.

From (4.10), we have

$$\begin{aligned} & (\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, f(\xi; w, z) > \\ = \; & < \mu_w, f(\xi; w, z) > + < \mu_z, f(\xi; w, z) > \\ & + < \mu_w, \xi[(\frac{w}{w_+})^{\frac{1}{2(1-\xi)}} - (\frac{w}{w_+})^{\frac{1}{2(1+\xi)}}] > \\ & + < \mu_z, \xi[(\frac{z}{z_+})^{\frac{1}{2(1-\xi)}} - (\frac{z}{z_+})^{\frac{1}{2(1+\xi)}}] > . \end{aligned} \tag{4.16}$$

Similar to (4.15), we expand the last two terms of (4.16) to Laurent series. We compute

$$\begin{aligned} g(\xi; w) &= \xi[(\frac{w}{w_+})^{\frac{1}{2(1-\xi)}} - (\frac{w}{w_+})^{\frac{1}{2(1+\xi)}}] \\ &= \xi[\mathrm{e}^{\frac{\ln \frac{w}{w_+}}{2(1-\xi^2)}}(\mathrm{e}^{\frac{\xi \ln \frac{w}{w_+}}{2(1-\xi^2)}} - \mathrm{e}^{\frac{-\xi \ln \frac{w}{w_+}}{2(1-\xi^2)}})] \\ &= \left\{1 + \sum_{m=1}^{\infty}\Big[\sum_{k=1}^{m} \frac{\big(-\ln \frac{w}{w_+}\big)^k}{2^k k!}\binom{m-1}{k-1}\Big]\frac{1}{\xi^{2m}}\right\} \\ & \quad \times \left\{\sum_{l=0}^{\infty}\Big[\sum_{j=0}^{l} \frac{-\big(\ln \frac{w}{w_+}\big)^{2j+1}}{4^j (2j+1)!}\binom{l+j}{2j}\Big]\frac{1}{\xi^{2l}}\right\} \\ &= \sum_{p=0}^{\infty} \frac{h_{2p}(\frac{w}{w_+})}{\xi^{2p}}. \end{aligned} \tag{4.17}$$

It is obvious that $h_{2p}(\frac{w}{w_+}) \geq 0$. We note that $< \mu_w, f(\xi; w, z) >, < \mu_w, g(\xi; w) >$, $< \mu_z, g(\xi; z) >$ and $< \mu_z, f(\xi; w, z) >$ are analytic in Ω_0 due to (4.14). Thus $< \mu_w, c_{2p}(w, z) >, < \mu_w, h_{2p}(\frac{w}{w_+}) >, < \mu_z, h_{2p}(\frac{z}{z_+}) >$ and $< \mu_z, c_{2p}(w, z) >$ are bounded. Now we choose $\xi \in R$ and $|\xi| > 3$, then Levi's lemma gives

$$\begin{aligned} < \mu_w, f(\xi; w, z) > &= \sum_{p=0}^{\infty} \frac{< \mu_w, c_{2p}(w, z) >}{\xi^{2p}}, \\ < \mu_z, f(\xi; w, z) > &= \sum_{p=0}^{\infty} \frac{< \mu_z, c_{2p}(w, z) >}{\xi^{2p}}, \\ < \mu_w, g(\xi; w) > &= \sum_{p=0}^{\infty} \frac{< \mu_w, h_{2p}(\frac{w}{w_+}) >}{\xi^{2p}}, \\ < \mu_z, g(\xi; z) > &= \sum_{p=0}^{\infty} \frac{< \mu_w, h_{2p}(\frac{z}{z_+}) >}{\xi^{2p}}, \end{aligned} \tag{4.18}$$

due to the fact that $c_{2p}(w, z), h_{2p}(\frac{w}{w_+}), h_{2p}(\frac{z}{z_+}) \geq 0$. This yields that

$$\sum_{p=0}^{\infty} \frac{< \mu_w, c_{2p}(w, z) >}{\xi^{2p}} \tag{4.19}$$

is absolutely convergent for any $\xi \in \bar{\Omega}_0$ because $< \mu_w, f(\xi; w, z) >$ is analytic in $\bar{\Omega}_0$. In the same way,

$$\sum_{p=0}^{\infty} \frac{< \mu_z, c_{2p}(w,z) >}{\xi^{2p}}, \quad \sum_{p=0}^{\infty} \frac{< \mu_w, h_{2p}(\frac{w}{w_+}) >}{\xi^{2p}} \quad \text{and} \quad \sum_{p=0}^{\infty} \frac{< \mu_z, h_{2p}(\frac{z}{z_+}) >}{\xi^{2p}} \tag{4.20}$$

are also convergent in $\bar{\Omega}_0$.

Thus, from (4.10) and (4.16-4.20), we have

$$\begin{aligned}(\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, f(\xi; w, z) > \;=\; & \sum_{p=0}^{\infty} \frac{< \mu_w, c_{2p}(w,z) + h_{2p}(\frac{w}{w_+}) >}{\xi^{2p}} \\ & + \sum_{p=0}^{\infty} \frac{< \mu_z, c_{2p}(w,z) + h_{2p}(\frac{z}{z_+}) >}{\xi^{2p}}.\end{aligned} \tag{4.21}$$

if $\xi \in \Omega_w$. We define

$$\begin{aligned} g_p^{(n)}(w,z) &= (-1)^p n^{2p} (f(ni; w, z) - \textstyle\sum_{s=0}^{p-1} (-1)^s \frac{c_{2s}(w,z)}{n^{2s}}), \\ &= \textstyle\sum_{s=p}^{\infty} (-1)^{p+s} \frac{c_{2s}(w,z)}{n^{2(s-p)}}, \qquad p = 1, 2, \cdots . \end{aligned} \tag{4.22}$$

It is easy to see that $g_p^{(n)}(w,z) \geq 0$ and it converges to $c_{2p}(w,z)$ a.e. $\nu_{x,t}$ as $n \to \infty$. We note that $c_0(w,z) = 2$, $< \nu_{x,t}, c_0(w,z) >$ is well defined. Furthermore, direct computation yields

$$\begin{aligned} & (\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, c_0(w,z) > \\ = \; & < \mu_w, c_0(w,z) + h_0(\tfrac{w}{w_+}) > + < \mu_z, c_0(w,z) + h_0(\tfrac{z}{z_+}) > . \end{aligned} \tag{4.23}$$

Now we assume that $< \nu_{x,t}, c_{2s}(w,z) >$ are well defined and

$$\begin{aligned} & (\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, c_{2s}(w,z) > \\ = \; & < \mu_w, c_{2s}(w,z) + h_{2s}(\tfrac{w}{w_+}) > \\ & + < \mu_z, c_{2s}(w,z) + h_{2s}(\tfrac{z}{z_+}) >, \end{aligned} \tag{4.24}$$

for $s = 0, 1, \cdots, p-1$. We shall show $< \nu_{x,t}, c_{2p}(w,z) >$ is also well defined and (4.24) holds for $s = p$.

From (4.21), (4.22) and (4.24), it is easy to check that $\lim\limits_{n\to\infty} < \nu_{x,t}, g_p^{(n)}(w,z) >$ exists. By Fatou's Lemma, we have

$$< \nu_{x,t}, c_{2p}(w,z) > \leq \lim_{n\to\infty} < \nu_{x,t}, g_p^{(n)}(w,z) >, \tag{4.25}$$

which, together with (4.24), implies

$$\begin{aligned} & (\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, c_{2p}(w,z) > \\ \leq \; & < \mu_w, c_{2p}(w,z) + h_{2p}(\tfrac{w}{w_+}) > + < \mu_z, c_{2p}(w,z) + h_{2p}(\tfrac{z}{z_+}) >, \end{aligned} \tag{4.26}$$

To prove (4.24) for $s = p$, we use the way of contradiction. We assume that

$$\begin{aligned} & (\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, c_{2p}(w,z) > \\ < \; & < \mu_w, c_{2p}(w,z) + h_{2p}(\tfrac{w}{w_+}) > + < \mu_z, c_{2p}(w,z) + h_{2p}(\tfrac{z}{z_+}) > . \end{aligned} \tag{4.27}$$

It is observed that

$$g_p^{(n)}(w,z) = c_{2p}(w,z) - \frac{1}{n^2} g_{p+1}^{(n)}(w,z). \tag{4.28}$$

Thus, we have

$$\begin{aligned}
&(\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, \tfrac{1}{n^2} g_{p+1}^{(n)}(w,z) > \\
= \ &(\lambda_2^+ - \lambda_1^+) < \nu_{x,t}, c_{2p}(w,z) > - < \mu_w, c_{2p}(w,z) + h_{2p}(\tfrac{w}{w_+}) > \\
&- < \mu_z, c_{2p}(w,z) + h_{2p}(\tfrac{z}{z_+}) > \\
&+ \sum_{s=p+1}^{\infty} (-1)^{p+s+1} \frac{< \mu_w, c_{2s}(w,z) + h_{2s}(\frac{w}{w_+}) >}{\xi^{2(s-p)}} \\
&+ \sum_{s=p+1}^{\infty} (-1)^{p+s+1} \frac{< \mu_z, c_{2s}(w,z) + h_{2s}(\frac{z}{z_+}) >}{\xi^{2(s-p)}},
\end{aligned} \tag{4.29}$$

which yields $\lim_{n\to\infty} < \nu_{x,t}, \frac{1}{n^2} g_{p+1}^{(n)}(w,z) >< 0$. This contradicts with the fact that $g_{p+1}^{(n)}(w,z) > 0$.

By the induction principle, $< \nu_{x,t}, c_{2p}(w,z) >$ are well defined and (4.24) holds for all $p = 0, 1, \cdots$. Therefore the Laurent series

$$\sum_{p=0}^{\infty} \frac{< \nu_{x,t}, c_{2p}(w,z) >}{\xi^{2p}} \tag{4.30}$$

is convergent for any $\xi \in \bar{\Omega}_0$.

Now we construct a sequence of $\nu_{x,t}$–measurable functions

$$f_m(\xi; w,z) = \sum_{p=0}^{m} \frac{c_{2p}(w,z)}{\xi^{2p}} \tag{4.31}$$

which converges to $f(\xi; w,z)$ in $\{w > 0, z > 0\}$. Since

$$< \nu_{x,t}, |f_m(\xi; w,z)| > \le \sum_{p=0}^{\infty} \frac{< \nu_{x,t}, c_{2p}(w,z) >}{|\xi|^{2p}} < \infty \tag{4.32}$$

for $\xi \in \bar{\Omega}_0$, again using Fatou's Lemma, we have $f(\xi; w,z) \in \mathbf{L}(d\nu_{x,t})$ if $\xi \in \bar{\Omega}_0$. In particular, $w^{-\alpha} z^{\frac{\alpha}{4\alpha+1}} + z^{-\alpha} w^{\frac{\alpha}{4\alpha+1}} \in \mathbf{L}(d\nu_{x,t})$ for any $0 < \alpha < \frac{1}{4}$. Therefore, again using the analytic extension theorem, (4.10) holds in the domain $\bar{\Omega}_0$. We note that η is exactly a strong entropy if $\xi \in \bar{\Omega}_0 / \Omega_w$, thus we establish (4.10) for some strong entropies.

Choosing any constant $\alpha \in (0, \frac{1}{4})$, we define a probability measure μ_{1z} like (4.13), i.e. for any $h \in C_0(R^2)$,

$$< \mu_{1z}, h > = \lim_{n\to\infty} \frac{< \nu_{x,t}, h w^n z^{-\alpha} >}{< \nu_{x,t}, w^n z^{-\alpha} >}. \tag{4.33}$$

We compute

$$
\begin{aligned}
< \mu_z, h > &= \lim_{n\to\infty} \frac{< \nu_{x,t}, hw^n z^{\frac{n}{4n-1}} >}{< \nu_{x,t}, w^n z^{-\alpha} >} \lim_{n\to\infty} \frac{< \nu_{x,t}, w^n z^{-\alpha} >}{< \nu_{x,t}, w^n z^{\frac{n}{4n-1}} >} \\
&= \frac{< \mu_{1z}, hz^{\frac{1}{4}+\alpha} >}{< \mu_{1z}, z^{\frac{1}{4}+\alpha} >}.
\end{aligned}
\tag{4.34}
$$

In terms of previous argument, we have (4.10) for any $\xi \in \bar{\Omega}_1 = \{\xi \in \mathbf{C}; |\xi| > \frac{2}{1+4\alpha} + 1\} \subset \Omega_1 = \{\xi \in \mathbf{C}; \mathrm{Re}\frac{1}{2(1+\xi)} > -\frac{1}{4} - \alpha \text{ and } \mathrm{Re}\frac{1}{2(1-\xi)} > -\frac{1}{4} - \alpha\}$. Thus, repeating the above arguments, we can establish (4.10) for any $\xi \in \bar{\Omega}_k = \{\xi \in \mathbf{C}; |\xi| > \frac{2}{1+4k\alpha} + 1\} \subset \Omega_k = \{\xi \in \mathbf{C}; \mathrm{Re}\frac{1}{2(1+\xi)} > -\frac{1}{4} - k\alpha \text{ and } \mathrm{Re}\frac{1}{2(1-\xi)} > -\frac{1}{4} - k\alpha\}$, where k is arbitrary positive integer. Therefore we have the following lemma.

Lemma 4.2. *For any $\xi_1, \xi_2 \in C/\{(-1,0),(0,1)\}$, the following holds*

$$
\begin{aligned}
< \nu_{x,t}, q_1\eta_2 - q_2\eta_1 >= &< \nu_{x,t}, q_1 >< \nu_{x,t}, \eta_2 > \\
&- < \nu_{x,t}, q_2 >< \nu_{x,t}, \eta_1 >, \quad a.e. \quad x, t.
\end{aligned}
\tag{4.35}
$$

where $\eta_i, q_i, i = 1, 2$ are chosen as in (2.8)

Remark 3. Lemma 4.2 indicates that the Tartar commutation relations hold for both weak and strong waves chosen in (2.8).

Proof of Theorem 2. To prove Theorem 2, it is sufficient to show that $\nu_{x,t}$ is a point mass.

Let $\xi_1 = 1 - \frac{1}{2n}$ and $\xi_2 = 1 + \frac{1}{2n}$ in (4.35), we have

$$
\frac{< \nu_{x,t}, q(\xi_1)\eta(\xi_2) - q(\xi_2)\eta(\xi_1) >}{< \nu_{x,t}, \eta(\xi_1) >< \nu_{x,t}, \eta(\xi_2) >} = \frac{< \nu_{x,t}, q(\xi_1) >}{< \nu_{x,t}, \eta(\xi_1) >} - \frac{< \nu_{x,t}, q(\xi_2) >}{< \nu_{x,t}, \eta(\xi_2) >}. \tag{4.36}
$$

Letting $n \to \infty$ gives

$$
\lim_{n\to\infty} \frac{< \nu_{x,t}, q(\xi_2) >}{< \nu_{x,t}, \eta(\xi_2) >} = \lambda_2^+. \tag{4.37}
$$

On the other hand, let $\xi \in \Omega_w, \xi_2 = 1 + \frac{1}{2n}$ in (4.35). Similar to (4.6) and (4.7), we have, as $n \to \infty$,

$$
< \mu_{-z}, q - \lambda_2\eta >=< \nu_{x,t}, q - \lambda_2^+\eta >, \tag{4.38}
$$

due to (4.37), where the probability measure μ_{-z} is defined by, for any $h \in C_0(R^2)$,

$$
< \mu_{-z}, h >= \lim_{n\to\infty} \frac{< \nu_{x,t}, hw^{-n} z^{\frac{n}{4n+1}} >}{< \nu_{x,t}, w^{-n} z^{\frac{n}{4n+1}} >}. \tag{4.39}
$$

It is easy to see the support of the measure μ^{-z} is contained in the line $\{w = w_-\}$.

By (4.7) and (4.38), we have

$$< \mu_z, q - \lambda_2\eta >=< \mu_{-z}, q - \lambda_2\eta >, \tag{4.40}$$

for any $\xi \in \Omega_w$.

Let $\eta = w^n z^{\frac{n}{4n-1}}, q = (u + 1 - \frac{1}{2n})\eta$, we compute

$$w_+^n < \mu_z, z^{\frac{n}{4n-1}} >= w_-^n < \mu_{-z}, z^{\frac{n}{4n-1}} >, \tag{4.41}$$

which infers $w_- = w_+$ as $n \to \infty$. In the same way, we also have $z_- = z_+$. Thus $\nu_{x,t}$ is either a point mass or supported in the vacuum. This indicates Theorem 2 due to the standard theory of compensated compactness.

Proof of Theorem 1. Choose $(\rho^\varepsilon, m^\varepsilon)$ as in (3.4), then Lemma 3.1 gives, for any test function $\phi \in C_0^\infty(R_+^2)$,

$$\begin{aligned}
&\iint_{t>0} (\rho^\varepsilon\phi_t + m^\varepsilon\phi_x)\ dxdt + \int_R \rho_0^\varepsilon(x)\phi(x,0)\ dx \\
&= -\iint_{t>0} \varepsilon\rho^\varepsilon\phi_{xx}dxdt, \\
&\iint_{t>0} m^\varepsilon\phi_t + (\frac{(m^\varepsilon)^2}{\rho^\varepsilon} + \rho^\varepsilon)\phi_x\ dxdt + \int_R m_0^\varepsilon(x)\phi(x,0)\ dx \\
&= -\iint_{t>0} \varepsilon m^\varepsilon\phi_{xx}dxdt.
\end{aligned} \tag{4.42}$$

By Theorem 2, there exists a strong convergent subsequence of $(\rho^\varepsilon, m^\varepsilon)$ (still denoted by $(\rho^\varepsilon, m^\varepsilon)$), such that

$$(\rho^\varepsilon(x,t), m^\varepsilon(x,t)) \to (\rho(x,t), m(x,t)), \quad \text{a.e.} \tag{4.43}$$

Let $\varepsilon \to 0$, (1.4) holds from (4.42). Thus $(\rho(x,t), m(x,t))$ is a weak solution of (1.1) and (1.3) with $\gamma = 1$. Since $(\rho(x,t), m(x,t))$ is the limit of the viscosity solutions $(\rho^\varepsilon, m^\varepsilon)$, it is easy to check that (1.5) holds for any weak convex entropy. Therefore we complete the proof of Theorem 1.

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