Application of the method of kinetic equation to the Darcy–Stefan problem*

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Abstract

The Cauchy problem for the Darcy–Stefan model that describes freezing and thawing of a continuous medium filtering through a porous ground is considered. To study the problem, a version of the method of kinetic equation, which was proposed by P. I. Plotnikov, is applied, the kinetic formulation of the problem is constructed and, by its virtue, the existence theorem for weak entropy solutions is proved.

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1. INTRODUCTION

Mathematical modelling of freezing and thawing phenomena in porous grounds, with account of the convection of a liquid saturating phase, is a necessary part of the scientific support of many technologies in industry and agriculture [1]. The main feature of the majority of the relevant problems is that they combine the Stefan problem on phase transitions and the Darcy law of dynamics of viscous continuous media filtering through incompressible porous grounds.

In 1999, J. F. Rodrigues and J. M. Urbano [2] have proposed a multi-dimensional Darcy–Stefan model of rather general form in order to describe effects of freezing and thawing with account of buoyancy forces that depend on temperature nonlinearly. The density was supposed to be the same constant for solid and liquid phases.

In [2] the initial-boundary value problem in a bounded domain was considered. The definition of a weak generalized solution was proposed for the enthalpy formulation of the problem, i.e. for the formulation in which the unknown functions were the velocity field, the gradient of pressure, and the temperature distribution, and the enthalpy appeared as a subset of the monotone maximal graph of the multi-valued function of temperature. The main result of [2] consisted of the theorem on existence of a weak generalized solution to the problem, and the proof was based on the classical methods from the theory of elliptic and parabolic equations.

In the present work, in Section 2, a Cauchy problem for the above described Darcy–Stefan model (Problem D-S) is set such that one of the unknown functions is not temperature but enthalpy. This clearly brings the new difficulty to the problem since in such setting the heat equation has the degenerate hyperbolic-parabolic type. In Section 3, a definition of entropy solution for Problem D-S is introduced. The main result of the paper consists of justification of Theorem 3 (see in Section 6) on existence of entropy solutions to Problem D-S. To this end, in Section 4, the approximate problem is posed and the corresponding sequence of approximate solutions is constructed. Using the classical methods from the theory of elliptic and parabolic equations the relative compactness of families of approximate values of velocity and pressure

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gradients is proved. After this, it is necessary to show that the family of approximate values of enthalpy is relatively compact as well. To this end, in Sections 5–6 there is used the method of kinetic equation that was invented in the pioneering works of Y. Brenier [3], P.-L. Lions, B. Perthame, and E. Tadmor [4, 5], and modified and developed in the later works of these and other authors. Namely, we use in the present paper the version of the method, which was proposed by P. I. Plotnikov in [6] and then developed in [7, 8, 9, 10].

More precisely, in Section 5, the probability parametrized measures that are called the Young measures associated with the sequences of approximate values of enthalpy and temperature are introduced and the kinetic formulation (Problem K) of the considered problem is derived. It appears in the form of the set of the properties of the distribution functions of the Young measures. The peculiarity of Problem K is that the kinetic formulation of the heat equation is linear. It allows to fulfill for it the renormalization procedure (in Section 6) and then to show that from the structure of the resulting renormalized inequality it appears that the Young measures are the Dirac measures and, consequently, the family of values of approximate enthalpy is relatively compact.

2. THE PROBLEM FORMULATION

Problem D-S. Let a continuous medium be occupying an open domain $\Omega = (0,1)^d$ for all moments of time $t \in [0,T]$. Let the frozen (rigid) phase be occupying a volume $\mathcal{S}(t)$ $(t \in [0,T])$ and the liquid (thawed) phase be occupying the open complement $\mathcal{L}(t) := \Omega \setminus \overline{\mathcal{S}(t)}$ of $\mathcal{S}(t)$. Domains $\mathcal{S}(t)$ and $\mathcal{L}(t)$ and the interface $\Phi(t) := \overline{\mathcal{S}(t)} \cap \overline{\mathcal{L}(t)}$ a priori are not known.

It is necessary to find the distribution of enthalpy $e = e(\boldsymbol{x},t)$, the velocity field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t)$, and the pressure gradient $\nabla_x p = \nabla_x p(\boldsymbol{x},t)$, satisfying the equations

$$\partial_t e + \mathrm{div}_x(\vec{v}e) = \varDelta_x b, \quad \boldsymbol{x} \in \Omega \setminus \Phi(t),\ t \in (0,T), \tag{1a}$$

$$b = \begin{cases} b_s(e) & \text{for } e < 0, \\ \quad 0 & \text{for } e \in [0,l], \\ b_l(e) & \text{for } e > l, \end{cases} \qquad 0 < b_s'(e), b_l'(e) < +\infty, \quad \forall\, e \in \mathbb{R}, \tag{1b}$$

$$\boldsymbol{v} = 0, \quad \boldsymbol{x} \in \mathcal{S}(t),\ t \in (0,T), \tag{1c}$$

$$\mathrm{div}_x \boldsymbol{v} = 0, \quad \boldsymbol{x} \in \mathcal{L}(t),\ t \in (0,T), \tag{1d}$$

$$\boldsymbol{v} = -\nabla_x p + \boldsymbol{g}(b), \quad \boldsymbol{x} \in \mathcal{L}(t),\ t \in (0,T), \tag{1e}$$

initial data

$$e(\boldsymbol{x},0) = e_0(\boldsymbol{x}), \quad \text{where} \quad |e_0(\boldsymbol{x})| \le c_0 = \mathrm{const}, \quad \boldsymbol{x} \in \mathbb{R}^d, \tag{1f}$$

and conditions on $\Phi(t)$,

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{x} \in \Phi(t), \quad t \in (0,T), \tag{1g}$$

$$[\nabla_x b(e)]_s^l \cdot \boldsymbol{n} = l \boldsymbol{w} \cdot \boldsymbol{n}, \quad \boldsymbol{x} \in \Phi(t),\ t \in (0,T). \tag{1h}$$

The system (1a)–(1h) is endowed by means of the following periodic structure. The whole space $\mathbb{R}^d$ is divided into the periods, $\mathbb{R}^d = \bigcup\limits_{\boldsymbol{z} \in \mathbb{Z}^d} (\boldsymbol{z} + [0,1)^d)$, where $\mathbb{Z}^d$ is the set of all vectors with integer components, and the unknown functions and the initial data are extended onto $\mathbb{R}^d$ by the formulas

$$e(\boldsymbol{x} + \boldsymbol{\tau}_i, t) = e(\boldsymbol{x},t), \quad \boldsymbol{v}(\boldsymbol{x} + \boldsymbol{\tau}_i, t) = \boldsymbol{v}(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \mathbb{R}^d \times [0,T), \tag{1i}$$

$$\nabla_x p(\boldsymbol{x} + \boldsymbol{\tau}_i, t) = \nabla_x p(\boldsymbol{x},t), \quad \boldsymbol{x} \in \mathcal{L}(t),\ t \in (0,T), \tag{1j}$$

where $\boldsymbol{\tau}_i$ ($i = 1, \dots, d$) are the standard basis vectors.

Let us explain the physical meaning of the components of the formulation of Problem D-S.

- (1a) is the heat equation.
- b is temperature.
- (1b) is the state equation of the continuous medium. The condition that temperature is constant within the entire interval $[0, l]$ expresses the effect of the phase transition (thawing). The values $b = 0$ and $e = l$ are the temperature of thawing and the latent heat of thawing, respectively.
- Condition (1c) postulates that the frozen phase is at rest.
- (1d) is the incompressibility condition for the liquid phase.
- (1e) is the Darcy law. $\boldsymbol{g} \in C^1(\mathbb{R})$ is the density of buoyancy forces. From the physical point of view, it is, in general, a nonlinear vector-function of temperature (see, for example, [11]).
- (1g)–(1h) are the Renkine–Hugoniot conditions on the strong discontinuity $\Phi(t)$. (1h) is historically also called the Stefan condition. $\boldsymbol{n}$ is the unit normal to $\Phi(t)$ pointing into $\mathcal{S}(t)$. $\boldsymbol{w}$ is the speed of $\Phi(t)$. The bracket $[\varphi]_s^l := \varphi^l - \varphi_s$ denotes the jump of a function φ on $\Phi(t)$, where φ^l and φ_s are the limit values on $\Phi(t)$ from $\mathcal{L}(t)$ and from $\mathcal{S}(t)$, respectively, [12], [13].

3. DEFINITION OF ENTROPY SOLUTIONS

We are going to study the question about whether weak generalized solutions of Problem D-S exist. Although we do not study the question about uniqueness of solutions (if any), in line with this question, it is sound to propose the definition of *entropy* solutions because the notion of entropy solutions is more restrictive than the notion of (merely) weak solutions and because, in view of observations from [14, 15], the demand that an entropy condition should hold looks to be necessary for the uniqueness.

Definition 1. *The pair $(e, \nabla_x p)$ is called an entropy solution of Problem D-S if it satisfies the conditions*

$$e \in L^\infty(Q), \quad B(e) \in L^2(0, T; H^1(\Omega)), \tag{2a}$$

where $B'(e) = \sqrt{b'(e)}$,

$$\nabla_x p \in L^r(0, T; H^1(\Omega)), \quad \forall\, r < +\infty, \tag{2b}$$

the periodicity conditions (1i)–(1j)*, the integral inequality*

$$\int_Q \big(\varphi(e)\zeta_t + \varphi^+(e)(-\nabla_x p + \boldsymbol{g}(b(e))) \cdot \nabla_x \zeta) + \omega(e)\Delta_x \zeta - \varphi''(e)|\nabla_x B(e)|^2 \zeta\big) d\boldsymbol{x}dt$$
$$+ \int_\Omega \varphi(e_0)\zeta(\boldsymbol{x}, 0) d\boldsymbol{x} \geq 0 \tag{2c}$$

for all functions φ, φ^+, and ω such that

$$\varphi \in C^2_{loc}(\mathbb{R}), \quad \varphi''(e) \geq 0, \quad (\varphi^+)'(e) = 1_{e>l}\varphi'(e), \quad \omega'(e) = \varphi'(e)b'(e), \tag{2d}$$

and for any nonnegative 1-periodic in $\boldsymbol{x}$ function $\zeta \in C^2_{loc}(\mathbb{R}^d \times [0,T])$ such that $\zeta|_{t=T} = 0$, and the integral equality

$$\int_Q \big(\nabla_x p - \boldsymbol{g}(b(e))\big) \cdot \nabla_x \psi d\boldsymbol{x} dt = 0, \tag{2e}$$

where the function ψ is arbitrary smooth and 1-periodic in $\boldsymbol{x}$.

Remark 1. *Once an entropy solution is built, the velocity field can be reconstructed by the formula*

$$\boldsymbol{v}(\boldsymbol{x},t) = 1_{e(x,t)>l}\big(-\nabla_x p(\boldsymbol{x},t) + \boldsymbol{g}(b(e(\boldsymbol{x},t)))\big).$$

Also let us notice that, from the physical point of view, values of the pressure gradient $\nabla_x p$ make sense only for $\boldsymbol{x} \in \mathcal{L}(t)$, although $\nabla_x p$ by the Definition 1 should be defined in the whole domain Ω.

4. APPROXIMATE SOLUTIONS OF PROBLEM D-S

Alongside Problem D-S we consider its parabolic approximation

$$(\boldsymbol{x},t) \in \mathbb{R}^d \times [0,T]: \qquad e_t + \boldsymbol{v} \cdot \nabla_x e^+ = \Delta_x b(e) + \varepsilon \Delta_x e, \tag{3a}$$

$$(\boldsymbol{x},t) \in \mathbb{R}^d \times [0,T]: \qquad \boldsymbol{v} = -\nabla_x p + \boldsymbol{g}(b(e)), \quad \operatorname{div}_x \boldsymbol{v} = 0, \tag{3b}$$

$$(\boldsymbol{x},t) \in \mathbb{R}^d \times [0,T]: \qquad e(\boldsymbol{x} + \boldsymbol{\tau}_i, t) = e(\boldsymbol{x},t), \quad e(\boldsymbol{x},0) = e_0(\boldsymbol{x}). \tag{3c}$$

From the well-known facts of the mathematical theory of filtration [16, Chapter 5] it follows that, for any fixed $\varepsilon > 0$, there exists a solution $(e_\varepsilon,\, \nabla_x p_\varepsilon)$ of the problem (3).

The maximum principle and the energy estimate yield that

$$-c_0 \le e_\varepsilon \le c_0 \quad \text{a.e. n } Q, \tag{4}$$

$$\|\nabla_x b(e_\varepsilon)\|_{L^2(Q)} + \|\nabla_x B(e_\varepsilon)\|_{L^2(Q)} + \varepsilon \|\nabla_x e_\varepsilon\|_{L^2(Q)} + \|\nabla_x p_\varepsilon\|_{L^r(Q)} \le c_1, \tag{5}$$

where $r \in [1,+\infty)$ is arbitrary and c_1 does not depend on ε.

On the strength of (4) and (5), it follows from (3b) that

$$\|\Delta_x p_\varepsilon(t)\|_{L^2(\Omega)} \le c_2 \quad \text{for a.e. } t \in [0,T], \tag{6}$$

and from (3a) that

$$\|\partial_t e_\varepsilon\|_{L^2(0,T;H^{-2}(\Omega))} \le c_3. \tag{7}$$

This and formula (3b) yield the estimate

$$\|\partial_t \Delta_x p_\varepsilon\|_{L^2(0,T;H^{-3}(\Omega))} \le c_4. \tag{8}$$

On the strength of (6) and (8), we conclude that

$$\|\nabla_x p_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t \nabla_x p_\varepsilon\|_{L^2(0,T;H^{-2}(\Omega))} \le c_5, \tag{9}$$

and on the strength of (4), (9), and (3b), that

$$\|\boldsymbol{v}_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\boldsymbol{v}_\varepsilon\|_{L^2(0,T;H^{-2}(\Omega))} \le c_6. \tag{10}$$

In (6)–(10), the constants c_2–c_6 do not depend on ε. From the bounds (9) and (10) and from the Simon compactness lemma [17] it follows that it is possible to extract a subsequence of approximate solutions such that

$$\boldsymbol{v}_\varepsilon \to \boldsymbol{v}, \quad \nabla_x p_\varepsilon \to \nabla_x p \quad \text{strongly in } L^2(Q), \quad \text{as } \varepsilon \searrow 0. \tag{11}$$

Also, the bounds (4) and (5) yield that there exist a subsequence from ε such that

$$e_\varepsilon \to e \text{ weakly* in } L^\infty(Q), \quad \nabla_x B(e_\varepsilon) \to \beta_* \text{ weakly in } L^2(Q), \tag{12}$$

as $\varepsilon \searrow 0$.

5. KINETIC FORMULATION OF PROBLEM D-S

On the strength of (12) and the Tartar [18] and the Ball [19] theorems, we can introduce into considerations the measure-valued functions $\nu \in L^\infty_w(\mathbb{R}^d_x \times [0,T];\, Prob(\mathbb{R}_\lambda))$ and $\sigma \in L^\infty_w(\mathbb{R}^d_x \times [0,T];\, Prob(\mathbb{R}_\lambda \times \mathbb{R}^d_q))$ defined by the formulas

$$\varphi(e_\varepsilon) \underset{\varepsilon\searrow 0}{\longrightarrow} \int_{\mathbb{R}_\lambda} \varphi(\lambda) d\nu_{x,t}(\lambda) \text{ weakly* in } L^\infty(Q), \tag{13}$$

for any $\varphi \in C(\mathbb{R}_\lambda)$, and

$$\varphi(e_\varepsilon,\, \nabla_x B(e_\varepsilon)) \underset{\varepsilon\searrow 0}{\longrightarrow} \int_{\mathbb{R}_\lambda \times \mathbb{R}^d_q} \varphi(\lambda, \boldsymbol{q}) d\sigma_{x,t}(\lambda, \boldsymbol{q}) \text{ weakly* in } L^\infty(Q), \tag{14}$$

for any $\varphi \in C(\mathbb{R}_\lambda \times \mathbb{R}^d_q)$, such that $|\varphi(\lambda, \boldsymbol{q})| \le c(1 + |\lambda| + |\boldsymbol{q}|)^r$, $0 \le r < 2$.

$Prob(Y)$ denotes by the definition a set of probability measures on a Euclidean space Y. $Prob(Y)$ is the subset of the space of Radon measures $\mathbb{M}(Y)$ provided with the standard norm. $L^\infty_w(\mathbb{R}^d_x \times [0,T];\, Prob(Y))$ is the space of bounded weakly measurable mappings from $\mathbb{R}^d_x \times [0,T]$ into $Prob(Y)$ provided with the norm $\|\varphi\|_{L^\infty_w(\mathbb{R}^d_x \times [0,T];\,(Y))} = \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d_x \times [0,T]} \|\varphi_\xi\|_{\mathbb{M}(Y)}$.

Measures $\nu_{x,t}$ and $\sigma_{x,t}$ are defined for a.e. $(\boldsymbol{x}, t) \in \mathbb{R}^d_x \times [0,T]$ and are called the Young measures associated with the weakly convergent subsequences e_ε and $(e_\varepsilon,\, \nabla_x B(e_\varepsilon))$, respectively.

Also, we introduce into considerations the distribution function of the measure $\nu_{x,t}$,

$$f(\boldsymbol{x}, t, \lambda) = \int_{\mathbb{R}_s} 1_{\lambda \ge s} d\nu_{x,t}(s). \tag{15}$$

Keeping track of [6, section 4] and [8, section 3] we derive the set of the properties of the Young measures that can be represented as the theorem on existence of solutions of the so-called kinetic formulation of Problem D-S.

Theorem 1. *For any initial data of the form* (16a) *Problem K has at least one solution* $(f, \nabla_x p, \boldsymbol{v}, \sigma, M)$.

Problem K. (Kinetic formulation of Problem D-S.) Let $f_0 : \mathbb{R}^d_x \times \mathbb{R}_\lambda \mapsto [-c_0, c_0]$ be a measurable function such that f_0 is 1-periodic in $\boldsymbol{x}$ monotonous and right continuous in λ, and

$$f_0(\boldsymbol{x}, \lambda) = 0 \;\text{ for }\; \lambda < -c_0 \;\text{ and }\; f_0(\boldsymbol{x}, \lambda) = 1 \;\text{ for }\; \lambda \ge c_0. \tag{16a}$$

It is necessary to find a distribution function $f \in L^\infty(\mathbb{R}^d \times (0,T) \times \mathbb{R})$, a pressure gradient $\nabla_x p \in L^2(0,T; H^1_{loc}(\mathbb{R}^d))$, a velocity field $\boldsymbol{v} \in L^2(0,T; H^1_{loc}(\mathbb{R}^d))$, a measure-valued function $\sigma \in L^\infty_w(\mathbb{R}^d \times (0,T);\, Prob(\mathbb{R} \times \mathbb{R}^d))$, and a nonnegative defect measure $M \in \mathbb{M}(\mathbb{R}^d \times (0,T) \times \mathbb{R})$ satisfying the following conditions.

(a) Function $f(\boldsymbol{x}, t, \lambda)$ is 1-periodic in $\boldsymbol{x}$, monotonous and right continuous in $\lambda \in \mathbb{R}$. Moreover,

$$f(\boldsymbol{x}, t, \lambda) = 0 \;\text{ for }\; \lambda < -c_0 \;\text{ and }\; f(\boldsymbol{x}, t, \lambda) = 1 \;\text{ for }\; \lambda \ge c_0. \tag{16b}$$

In particular, $0 \le f \le 1$ a.e. in $Q \times \mathbb{R}_\lambda$. This amounts to the assertion that the Stieltjes measure $\nu_{x,t} = d_\lambda f(\boldsymbol{x}, t, \lambda)$ is a probability measure on $\mathbb{R}_\lambda$, and spt $\nu_{x,t} \subset [-c_0, c_0]$.

(b) The velocity field $\boldsymbol{v}$ and the pressure gradient $\nabla_x p$ are 1-periodic in $\boldsymbol{x}$.

(c) $\sigma_{x,t}$ is 1-periodic in $\boldsymbol{x}$, its support entirely lays in the strip $[-c_0, c_0] \times \mathbb{R}^d$, and the condition

$$\int_Q \left\{ \int_{\mathbb{R}_\lambda \times \mathbb{R}^d_q} |\boldsymbol{q}|^2 d\sigma_{x,t}(\lambda, \boldsymbol{q}) \right\} d\boldsymbol{x} dt < \infty \tag{16c}$$

holds. In particular, function

$$\chi(\boldsymbol{x},t,s) := \int_{(-\infty,s]\times\mathbb{R}^d_q} |\boldsymbol{q}|^2 d\sigma_{x,t}(\lambda,\boldsymbol{q}) \tag{16d}$$

is 1-periodic in $\boldsymbol{x}$, monotonous and right continuous in s and the support of the Stieltjes measure $d_\lambda\chi(\boldsymbol{x},t,\lambda)$ entirely lays in the interval $[-c_0,c_0]$ for a.e. $(\boldsymbol{x},t)\in\mathbb{R}^d_x\times(0,T)$.

(d) For any function $G\in C^1_{loc}(Q\times\mathbb{R}_z)$, measures $\nu_{x,t}$ and $\sigma_{x,t}$ have the relation

$$\begin{aligned}\int_Q\int_{\mathbb{R}_\lambda\times\mathbb{R}^d_q}\int_{\mathbb{R}_z} G_z(\boldsymbol{x},t,z)q_i d\gamma_{B(\lambda)} d\sigma_{x,t}(\lambda,\boldsymbol{q})d\boldsymbol{x}dt&\\ = -\int_Q\int_{\mathbb{R}_\lambda}\int_{\mathbb{R}_z} G_{x_i}(\boldsymbol{x},t,z)d\gamma_{B(\lambda)}d\nu_{x,t}(\lambda)d\boldsymbol{x}dt \quad \forall\, i=1,\dots,d,&\end{aligned} \tag{16e}$$

where $\gamma_{B(\lambda)}$ is the parametrized Dirac measure on $\mathbb{R}_z$ concentrated at the point $z=B(\lambda)$.

(e) Defect measure $M\in\mathbb{M}(\mathbb{R}^d_x\times(0,T)\times\mathbb{R}_\lambda)$ is 1-periodic in $\boldsymbol{x}$.

(f) f, $\boldsymbol{v}$, and $\nabla_x p$ satisfy the following equations and initial data.

$$Q\times\mathbb{R}_\lambda: \quad \partial_t f + 1_{\lambda>l}\,\boldsymbol{v}\cdot\nabla_x f - b'(\lambda)\Delta_x f + \partial_\lambda(\partial_\lambda\chi + M) = 0, \tag{16f}$$
$$Q\times\mathbb{R}_\lambda: \quad (\boldsymbol{v}+\nabla_x p - \boldsymbol{g}(b(\lambda)))f_\lambda = 0, \tag{16g}$$
$$\Omega\times\mathbb{R}_\lambda: \quad f(\boldsymbol{x},0,\lambda) = f_0(\boldsymbol{x},\lambda). \tag{16h}$$

(16f), (16g), and (16h) are understood in the distributions sense and can be equivalently collected in the form of the system of the following integral equalities.

$$\begin{aligned}&\int_{Q\times\mathbb{R}_\lambda}\Big\{\partial_t\zeta + 1_{\lambda>l}\,\boldsymbol{v}\cdot\nabla_x\zeta + b'(\lambda)\Delta_x\zeta\Big\}f(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dtd\lambda + \int_{Q\times\mathbb{R}_\lambda}\partial_\lambda\zeta dM\\ &+\int_{Q\times\mathbb{R}_\lambda}\zeta_\lambda d_\lambda\chi(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dt + \int_{\Omega\times\mathbb{R}_\lambda}\zeta(\boldsymbol{x},0,\lambda)f_0(\boldsymbol{x},\lambda)d\boldsymbol{x}d\lambda = 0.\end{aligned} \tag{16i}$$

In (16i), $\zeta(\boldsymbol{x},t,\lambda)$ is an arbitrary 1-periodic in $\boldsymbol{x}$ smooth function vanishing in the neighbourhood of the plane $\{t=T\}$ and for sufficiently large $|\lambda|$.

$$\int_{Q\times\mathbb{R}_\lambda}\Big\{\big(\boldsymbol{v}+\nabla_x p - \boldsymbol{g}(b(\lambda))\big)\cdot\boldsymbol{\eta}_\lambda - \boldsymbol{g}(b(\lambda))_\lambda\cdot\boldsymbol{\eta}\Big\}f(\boldsymbol{x},t,\lambda)d\boldsymbol{x}dtd\lambda = 0 \tag{16j}$$

In (16j), $\boldsymbol{\eta}(\boldsymbol{x},t,\lambda)$ is an arbitrary 1-periodic in $\boldsymbol{x}$ smooth vector-function vanishing for sufficiently large $|\lambda|$.

6. RENORMALIZED INEQUALITY. THEOREM ON EXISTENCE OF ENTROPY SOLUTIONS

In [6, section 5] there is invented and then in [7] (see also [8]) there is developed a renormalization technique for kinetic equations of the form (16f) that keeps rooting to the idea of R. J. DiPerna and P.-L. Lions [20] about renormalization procedures for linear transport equations. In the present paper, using this technique we establish the following.

Theorem 2. *For any smooth and convex on the interval* $[0,1]$ *function* φ *there exists a Radon measure* $H_\varphi \in C(\mathbb{R}_\lambda \times Q)^*$ *with a supporter entirely laying in the strip* $-c_0 \le \lambda \le c_0$ *such that the renormalized inequality*

$$\int_{\mathbb{R}_\lambda \times Q} \varphi(f)\{\partial_t \zeta + 1_{\lambda > l}\, \boldsymbol{v} \cdot \nabla_x \zeta + b'(\lambda) \Delta_x \zeta)\} d\boldsymbol{x} dt d\lambda$$
$$+ \int_{\mathbb{R}_\lambda \times \Omega} \varphi(f_0) \zeta(\boldsymbol{x}, 0, \lambda) d\boldsymbol{x} d\lambda - \int_{\mathbb{R}_\lambda \times Q} \zeta_\lambda dH_\varphi(\boldsymbol{x}, t, \lambda) \le 0 \tag{17}$$

holds for any nonnegative smooth 1-periodic in $\boldsymbol{x}$ *function* $\zeta(\boldsymbol{x}, t, \lambda)$ *vanishing in a neighborhood of the plane* $t = T$ *and for sufficiently large* $|\lambda|$.

This theorem gives a key for establishing the principle result of the work.

Theorem 3. *For any initial data* e_0 *satisfying* (1f) *there is at least one entropy solution of Problem D-S.*

Proof. Let us take the test functions in (17) in the forms

$$\varphi(f) = f(f-1) \text{ and } \zeta(\boldsymbol{x}, t, \lambda) = \zeta_1(\lambda) \zeta_2(t), \tag{18}$$

where

$$\zeta_1 \quad \text{is nonnegative and} \quad \zeta_1 = 1 \quad \text{on} \quad [-c_0, c_0], \tag{19}$$
$$\zeta_2 \quad \text{is nonnegative,} \quad \zeta_2(T) = 0, \quad \text{and} \quad \zeta_2' < 0 \quad \text{for} \quad t < T. \tag{20}$$

Clearly such a choice of φ and ζ is legitime. Observing that $\nabla_x \zeta = 0$, $\varphi(f_0) = 0$, and

$$\int_{Q \times \mathbb{R}_\lambda} \zeta_\lambda dH_\varphi(\boldsymbol{x}, t, \lambda) = 0,$$

on the strength of formulas (19) and $\operatorname{spt} H \subset (Q \times [-c_0, c_0]_\lambda)$, we conclude that (17) takes the form

$$\int_{Q \times [0,1]_\lambda} \zeta_1 \varphi(f) \partial_t \zeta_2 d\boldsymbol{x} dt d\lambda \le 0.$$

From this inequality, formulas (18)–(20), and item (a) in formulation of Problem K it follows that $\varphi(f) \equiv 0$. This implies that $f(\boldsymbol{x}, t, \lambda)$ attains only values 0 and 1 for a.e. $(\boldsymbol{x}, t, \lambda) \in \mathbb{R}_x^d \times [0,T] \times \mathbb{R}_\lambda$ and therefore has a structure

$$f(\boldsymbol{x}, t, \lambda) = 1_{\lambda \ge e(x,t)}, \tag{21}$$

with some measurable 1-periodic in $\boldsymbol{x}$ function e that satisfies the bound $-c_0 \le e \le c_0$. Substituting this representation in (16i) and (16j) and integrating in λ, we conclude that this function e along with the limiting vector-function $\nabla_x p$ (see limiting relation (11)) are an entropy solution of Problem D-S. Theorem is proved.

Remark 2. *It follows from* (15) *and* (21) *that the Young measure* $\nu_{x,t}$ *is the parametrized Dirac measure on* $\mathbb{R}_\lambda$ *concentrated at the point* $\lambda = e(\boldsymbol{x}, t)$. *By the theory of Young measures [14, 18] this means that the extracted subsequence* $\{e_\varepsilon\}$, *which generates the Young measure* $\nu_{x,t}$, *converges strongly to* e *in* $L^r_{loc}(\mathbb{R}^d_x \times [0,T]) \quad \forall r < +\infty$.

Remark 3. *The considerations in this work can be naturally adapted with just few easy modifications to the Cauchy problem without imposing periodic conditions* (1i)–(1j)*. In this case conditions* (2a) *and* (2b) *in Definition 1 should be substituted by the conditions*

$$e \in L^\infty((0,T)\times\mathbb{R}^d), \quad B(e) \in L^2(0,T;H^1_{loc}(\mathbb{R}^d)),$$

$$\nabla_x p \in L^r(0,T;H^1_{loc}(\mathbb{R}^d)), \quad \forall\, r < +\infty,$$

and all the 1-periodic in $\boldsymbol{x}$ test functions should be replaced by the finite in $\mathbb{R}^d_x$ functions.

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