Well–Posedness for a scalar conservation law with singular non–conservative source*

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Abstract

We consider the Cauchy problem for the 2×2 strictly hyperbolic system

$$\begin{cases} a_t = 0 \\ u_t + f(a,u)_x - g(a,u)a_x = 0 \end{cases} \qquad (a,u)(0,\cdot) = (a_o, u_o).$$

For possibly large, discontinuous data, the continuous dependence of the solution with respect to both a_o and u_o is shown. Moreover the solutions are characterized as unique limits of Kružkov's entropic solutions constructed with regularized initial data a_o^ε.

1 Introduction

The main concern of the present paper is with the Cauchy problem in $\mathbb{R}$ for the following 2×2 system of partial differential equations

$$\begin{cases} a_t = 0 \\ u_t + f(a,u)_x - g(a,u)a_x = 0, \end{cases} \tag{1.1}$$

completed with the initial data

$$a(0,\cdot) = a_o, \qquad u(0,\cdot) = u_o. \tag{1.2}$$

We assume that the functions f and g are $\mathbf{C}^2$ and that the system is strictly hyperbolic, i.e. we will always make the key assumption

$$f_u(a,u) > 0. \tag{1.3}$$

Like the authors of [3], we are interested in the dependence of the solution on both the initial data (a_o, u_o). In particular we want to generalize the results in [3] where the same system (1.1) was considered without the non–conservative term ($g = 0$). Without assumption (1.3), nonlinear resonance occurs (see [2, 15, 16, 17, 18, 19, 24]) and one can not hope to have continuous dependence with respect to the initial data a_o as can be seen by the counter example in [26].

*Support by the European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, is acknowledged.

Let us pause to state a few comments. System (1.1) can be viewed either as a 2×2 hyperbolic system, not in conservation form, or as a different way to write a general scalar balance law with a possibly singular source term,

$$u_t + f(a(x), u)_x = a'(x) g(a(x), u), \tag{1.4}$$

as advocated for instance in [7] to derive a simple model of shallow–water flow. Another case of special interest for (1.1) lies in the modeling of one–dimensional flow in a nozzle as pointed out in [19]; in this context, f, g do not depend on a:

$$u_t + f(u)_x = k(x) g(u), \qquad k(x) = a'(x). \tag{1.5}$$

Equation (1.5) can also be considered within a limiting process (see for instance [14, 12, 26]) where a family of smooth sources k^ε converges weakly to a singular (measure) source μ. Corollary 4.9 implies that the corresponding solutions u^ε converges in $\mathbf{L}^1{}_{loc}$. In particular we obtain the convergence result of [26] in a more general context and with weaker hypotheses.

Our main result, which can be obtained as a straightforward consequence of Theorems 4.1, 4.3 and 4.6 in Section 4, is the following.

Theorem 1.1 *Let K be the invariant region defined in (2.5). For any initial data $(a_o, u_o) \in \mathbf{L}^\infty(\mathbb{R}, K)$, with a_o having bounded total variation, it is possible to define a generalized solution $u(t,x)$ to the Cauchy problem (1.1), (1.2) such that, the map $(t, a_o, u_o) \mapsto u(t, \cdot)$ is continuous in the $\mathbf{L}^1{}_{loc}$ topology. Moreover, whenever the initial data a_o is continuously differentiable, the corresponding generalized solution $u(t,x)$ coincides with the Kružkov's entropic solution to (1.4) with initial data u_o and $a(x) = a_o(x)$.*

Remark 1.2 *Since every function a_o with bounded total variation can be obtained as the $\mathbf{L}^1{}_{loc}$ limit of regular approximations a_o^ε, Theorem 1.1 implies that all the generalized solutions to (1.1) can be obtained as unique limits of Kružkov's entropic solutions u^ε to (1.4) with initial data $u^\varepsilon(0, \cdot) = u_o$ and $a(x) = a_o^\varepsilon(x)$.*

All our analysis is based on considering (1.1) as a particular non–conservative 2×2 "Temple system" [25, 22]. One eigenvalues of the system is linearly degenerate while the integral curves of the other one are straight lines. For the analysis of general conservative Temple systems see for instance [20, 22, 5, 4, 6, 9]. For the non–conservative system (1.1), following the ideas in [10], we construct a functional equivalent to the $\mathbf{L}^1$ distance and which is almost decreasing in time along any pair of front tracking approximate solutions.

The paper is organized as follows. In Section 2 we give some preliminaries and construct the wave front tracking approximate solutions. In Section 3 we define the almost decreasing functional and derive the basic stability estimates. In Section 4 we construct a Lipschitz continuous semigroup of solutions to (1.1), we make some observations about the extension to $\mathbf{L}^\infty$ initial data and characterize the trajectories using the classical Kružkov's entropy inequalities.

2 Preliminaries and front tracking approximations

For smooth solutions, (1.1) corresponds to the quasilinear system

$$U_t + A(U) U_x = 0, \qquad U = (a, u), \qquad A(U) = \begin{pmatrix} 0 & 0 \\ f_a - g & f_u \end{pmatrix}. \tag{2.1}$$

The eigenvalues of A are $\lambda_1 = 0$, $\lambda_2 = f_u$; the corresponding eigenvectors are $\mathbf{r}_1 = (f_u\,, g - f_a)$, $\mathbf{r}_2 = (0, 1)$. Observe that one of the characteristic fields is linearly degenerate, while the integral curves of the other are parallel straight lines. The strict hyperbolicity is given by condition (1.3). In the case $a = \bar{a}$ constant, the system reduces to the scalar homogeneous conservation law with parameter $\bar{a}$

$$u_t + f(\bar{a}, u)_x = 0.$$

As in [16, 1, 2] let us introduce some special solutions of the system (1.1): the stationary ones, which correspond to the integral curves of the linearly degenerate field,

$$f(a, u)_x = a_x g(a, u).$$

Regular solutions satisfy $f_u u_x = (g - f_a) a_x$, which is equivalent to

$$\frac{du}{da} = \frac{g - f_a}{f_u}. \tag{2.2}$$

We denote by $\phi(a; a_o, u_o)$ the solution to (2.2) with initial data

$$u(a_o) = u_o\,. \tag{2.3}$$

Standard results in the o.d.e. theory ensure that ϕ is $\mathbf{C}^1$ with respect to all its arguments. Moreover if two states (a', u') and (a'', u'') are connected by a stationary solution, i.e. $u'' = \phi(a''; a', u')$ then (2.2) implies

$$f(a'', u'') - f(a', u') = \int_{a'}^{a''} g\left[\bar{a}, \phi(\bar{a}; a', u')\right] d\bar{a}. \tag{2.4}$$

A closed region K in the (a, u) plane will be called an invariant domain (see Figure 1) if it is bounded from above and below by two solutions to equation (2.2). More precisely K is determined by two functions, $\phi_1(a)$ and $\phi_2(a)$ solutions to (2.2) and defined in a common interval $[a_1, a_1]$, such that

$$K = \left\{(a, u) \in \mathbb{R}^2 \,:\, a_1 \le a \le a_2,\ \phi_1(a) \le u \le \phi_2(a)\right\}. \tag{2.5}$$

From now on we suppose the invariant region K be fixed. As shown in [2] it is very convenient

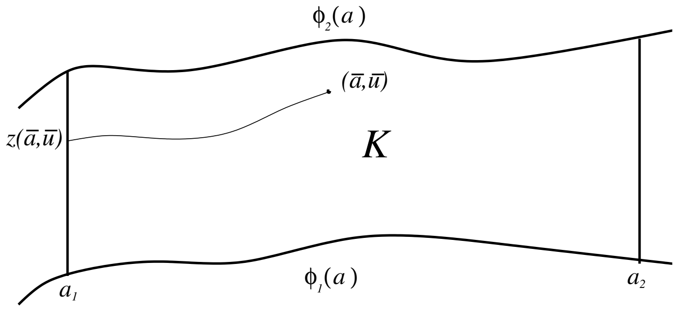

Figure 1: Example of an invariant domain

to define a pair of Riemann coordinates. Since in our case there is no resonance, the Riemann invariants are easily constructed: given a point $(\bar{a},\bar{u}) \in K$ we define the coordinate

$$z\,[\bar{a},\bar{u}] = \phi(a_1;\bar{a},\bar{u}), \tag{2.6}$$

i.e. z is given by (see Figure 1) the intersection of the vertical line $a = a_1$ with the solution to (2.2) with initial data $(\bar{a},\bar{u})$. It is now straightforward to see that z is constant along the integral curves of the first characteristic field while a is constant along the integral curves of the second one. Standard results in the o.d.e. theory imply that this is a regular change of variables.

2.1 Front tracking approximations

Let us now consider the Riemann problem for (1.1),

$$U(0,x) = \begin{cases} U_l = (a_l,u_l), & x < 0 \\ U_r = (a_r,u_r), & x > 0, \end{cases} \tag{2.7}$$

for any $U_1, U_2 \in K$. It will be solved, as usual, connecting the two states with waves of increasing speed. We will focus on the (a,z) coordinates since in these variables the characteristic curves of the 1^{st} (*standing waves*) and 2^{nd} family (*homogeneous waves*) are straight lines with the variable z and a constant, respectively. Therefore if we define

$$z_l = z\,[a_l,u_l] \ \text{ and } \ z_r = z\,[a_r,u_r]\,,$$

the solution of the Riemann problem (see Figure 2) is given by a single standing wave traveling with zero velocity connecting (a_l,z_l) to (a_r,z_l) followed by a set of homogeneous waves traveling with positive velocity connecting (a_r,z_l) to (a_r,z_r). This last set of homogeneous waves coincides with the entropic solution (see for instance [8] for a complete description of entropic solutions for scalar conservation laws) to the following homogeneous Riemann problem:

$$\begin{cases} u_t + f(a_r,u)_x = 0 \\ u(0,x) = \begin{cases} u^* = \phi(a_r;a_l,u_l) & \text{for } x < 0 \\ u_r & \text{for } x > 0. \end{cases} \end{cases}$$

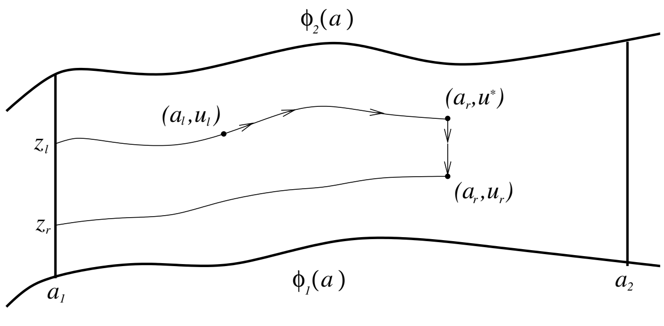

Figure 2: Solution of the Riemann problem in the space of the states

Since we want to construct approximate solutions using a front tracking algorithm, we need to approximate rarefaction waves with piecewise constant functions. Moreover we do not want

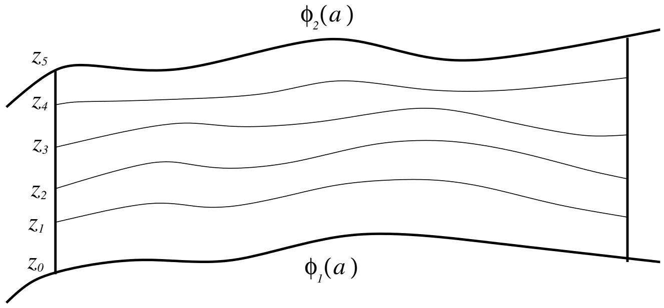

Figure 3: Partition of the z axis

the interactions between waves to accumulate in finite time. Therefore we fix a partition $\mathcal{P} = \{z_0, z_1, \ldots, z_n\}$ of the z axis with $|\mathcal{P}| \doteq \max_{i=1,\ldots,n} \{z_i - z_{i-1}\}$ (see Figure 3). Given a Riemann data $U(0,x)$ such that $z[U](0,x) \in \mathcal{P}$ we solve the Riemann problem in the following approximate way: first we go from (a_l, z_l) to (a_r, z_l) with a single standing wave traveling with zero velocity as in the exact solution; then we consider the function $\bar{f}(u)$ that linearly interpolate the curve $u \mapsto f(a_r, u)$ between the points $u_i = \phi(a_r; a_1, z_i)$, $i = 0, \ldots, n$ and connect (a_r, z_l) to (a_r, z_r) with the exact entropic solution to the Riemann problem

$$\begin{cases} u_t + \bar{f}(u)_x = 0 \\ u(0,x) = \begin{cases} u^* = \phi(a_r; a_l, u_l) & \text{for } x < 0 \\ u_r & \text{for } x > 0. \end{cases} \end{cases}$$

It is clear that (see [8]) in this way we found an approximate solution $U(t,x)$ to the Riemann problem (1.1), (2.7) which satisfies $z[U](t,x) \in \mathcal{P}$. This procedure, for the single conservation law, was introduced by Dafermos [11] and is now a standard tools for the analysis of scalar conservation laws. Our application of Dafermos discretization is similar to the one found for instance in [3, 18].

Take now a piecewise constant initial data $(a_o, u_o)(x)$ satisfying $z\,[a_o, u_o]\,(x) \in \mathcal{P}$. At $t = 0$ each Riemann problem is approximately solved as indicated above. The fronts are prolonged until the first collision occurs, then the new Riemann problem is solved and so on. We call $(a,u)(t,x)$ the approximate solution obtained in this way, observe that $a(t,x) = a_o(x)$. Obviously one has $z\,[a,u]\,(t,x) \in \mathcal{P}$ until the approximate solution is defined. Moreover, because we are dealing with Riemann coordinates, the total variation of z is non increasing, while the total variation of a is constant.

Since we do not assume genuine nonlinearity, an interaction may produce a number of outgoing fronts larger then the number of incoming fronts. Therefore to ensure that the approximate solution is defined for any time $t \geq 0$ we need the following lemma.

Lemma 2.1 *The possible interactions in the approximate solution $(a,u)(t,x)$ constructed above are finite.*

Proof. Let M be the maximum number of waves which can be created in an approximate solution of a generic Riemann problem: $M = n + 1$ where $n + 1$ is the number of points in the partition $\mathcal{P}$. Define $\delta' \doteq \min_{i=1,\ldots,n} \{z_i - z_{i-1}\} > 0$ and

$$R(x) \doteq 1 + \#\{\text{jumps in } a \text{ to the right of } x\}\,.$$

Let $N(t)$ and $x_1(t), \ldots, x_{N(t)}(t)$ be respectively the number and locations of the jumps in $z[a,u]$ at the time t and

$$\Delta z_\alpha(t) \doteq z\left[a,u\right]\left((t, x_\alpha(t)+\right) - z\left[a,u\right]\left((t, x_\alpha(t)-\right).$$

In every interaction, the number of waves does not strictly decrease only when there is an interaction between a standing wave and homogeneous waves or among homogeneous waves in which the consecutive states separated by the interacting waves are not ordered. In this last case there is a strict decrease (at least of δ') in the variation of z, hence it is easy to check that the functional

$$F(t) \doteq \delta' N(t) + M \sum_\alpha |\Delta z_\alpha(t)| R(x_\alpha(t))$$

decreases at least by the quantity δ' at every interaction, therefore there can not be infinite interactions. □

3 Stability for piecewise constant approximate solutions

Following the techniques in [10], in this section we introduce a functional which controls the $\mathbf{L}^1$ distance between two piecewise constant approximate solutions.

We fix two (possibly different) partitions of the z axis $\mathcal{P}_1$ and $\mathcal{P}_2$. Then we choose two initial data $U_1(0,x) = (b_o, v_o)(x)$ and $U_2(0,x) = (a_o, u_o)(x)$ satisfying

$$z\left[U_1\right](0,x) \in \mathcal{P}_1, \quad \text{and} \quad z\left[U_2\right](0,x) \in \mathcal{P}_2.$$

The corresponding approximate solutions, constructed as described in the previous section, will be denoted by $U_1(t,x) = (b,v)(t,x)$ and $U_2(t,x) = (a,u)(t,x)$. Observe that $z\left[U_1\right]$ will stay for ever in the partition $\mathcal{P}_1$ and $z\left[U_2\right]$ in the partition $\mathcal{P}_2$, moreover, since the first component is constant, one has $b(t,x) = b(x) = b_o(x)$ and $a(t,x) = a(x) = a_o(x)$.

To estimate the distance between U_1 and U_2, we solve the Riemann problem with left and right state respectively $U_1(t,x)$ and $U_2(t,x)$, then we introduce the strength of the zero wave and the strength of the set of the homogeneous waves in the solution of this Riemann problem respectively as

$$\begin{aligned} &p(x) = a(x) - b(x), \\ &q(t,x) = u(t,x) - \omega(t,x) \quad \text{with} \quad \omega(t,x) = \phi\left(a(x); b(x), v(t,x)\right). \end{aligned} \tag{3.1}$$

It is straightforward to check that the distance $\left|U_2(t,x) - U_1(t,x)\right|$ is equivalent to the sum of the strength of the waves: $|p(x)| + |q(t,x)|$. Indeed one has

$$\left|u(t,x) - v(t,x)\right| \leq |q(t,x)| + \left\|\frac{g - f_a}{f_u}\right\|_\infty |p(x)| \tag{3.2}$$

and

$$\left|q(t,x)\right| \leq \left|u(t,x) - v(t,x)\right| + \left\|\frac{g - f_a}{f_u}\right\|_\infty \left|a(x) - b(x)\right|, \tag{3.3}$$

where the $\mathbf{L}^\infty$ norms are taken over the invariant region K.

Now we define the maximum velocity L of the homogeneous waves

$$L = \max_{(a',u') \in K} f_u(a', u') \tag{3.4}$$

and concentrate on the triangle $D(x_1,x_2)$ (a domain of determinacy) in the (t,x) plane defined as

$$D(x_1,x_2)\doteq\left\{(t,x)\in\mathbb{R}^2\,:\,0\le t\le\frac{x_2-x_1}{L},\;x_1+Lt<x<x_2\right\},\qquad\text{for } x_1<x_2.$$

For $t\in\left[0,(x_2-x_1)/L\right]$ we define the functionals

$$\mathbf{\Phi}_1\left(U_1(t),U_2(t)\right)=\int_{x_1+Lt}^{x_2}W_2(x)\,\left[W_1(t,x)|p(x)|+|q(t,x)|\right]\,dx, \tag{3.5}$$

$$\mathbf{\Phi}_2\left(U_1(t),U_2(t)\right)=\int_{x_1+Lt}^{x_2}\left[W_1(t,x)|p(x)|+|q(t,x)|\right]\,dx. \tag{3.6}$$

The second functional will be used to obtain sharper results when the source satisfies dissipative conditions. The weight W_1 and W_2 are defined in the following way. Denote with $x_\alpha(t)$, $\alpha=1,\dots,N$ the locations at the time t of all the jumps that U_1 and U_2 have in the region $D(x_1,x_2)$, let $\mathcal{J}(w)$ be the collections of all the indexes α of the jumps in the piecewise constant function w, then

$$W_1(t,x)\;=\;\kappa_1\sum_{\substack{x_\alpha(t)<x\\ \alpha\in\mathcal{J}(U_1(t))}}\left|\Delta z\,[b,v]\left(t,x_\alpha(t)\right)\right|, \tag{3.7}$$

$$W_2(x)\;=\;\exp\left\{\kappa_2\sum_{\substack{x_\alpha(t)>x\\ \alpha\in\mathcal{J}(U_2(t))}}\left|\Delta a\left(x_\alpha(t)\right)\right|\right\}, \tag{3.8}$$

where we have used the obvious notations

$$\begin{aligned}\Delta z\,[b,v]\left(t,x_\alpha(t)\right)&=z\,[b,v]\left(t,x_\alpha(t)+\right)-z\,[b,v]\left(t,x_\alpha(t)-\right),\\ \Delta a\left(x_\alpha(t)\right)&=a\left(x_\alpha(t)+\right)-a\left(x_\alpha(t)-\right).\end{aligned} \tag{3.9}$$

The values of the constants κ_1, κ_2 in (3.7) and (3.8) will be specified later and will depend only on the system (i.e. on f, g and the invariant region K).

Our $\mathbf{L}^1$ stability estimate for front tracking approximate solutions can be stated as follows.

Theorem 3.1 *There exists a constant $C>0$ and a choice of the constants κ_1, κ_2 in (3.7) and (3.8), such that, for any two wave front tracking approximate solutions $U_1=(b,v)$ and $U_2=(a,u)$, with values in K and constructed using two possibly different partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, the functional $\mathbf{\Phi}_1$ in (3.5) satisfies for all $0\le s\le t\le(x_2-x_1)/L$*

$$\begin{aligned}&\mathbf{\Phi}_1\left(U_1(t),U_2(t)\right)-\mathbf{\Phi}_1\left(U_1(s),U_2(s)\right)\\ &\quad\le C\exp\left[\kappa_2\text{Tot.Var.}\left\{a,[x_1,x_2]\right\}\right]\cdot\Big[\delta_1\text{Tot.Var.}\left\{z[U_1](0,\cdot),[x_1,x_2]\right\}\\ &\qquad\qquad+\delta_2\text{Tot.Var.}\left\{z[U_2](0,\cdot),[x_1,x_2]\right\}\Big]\cdot(t-s),\end{aligned} \tag{3.10}$$

where $\delta_1\doteq|\mathcal{P}_1|$ and $\delta_2\doteq|\mathcal{P}_2|$.

If moreover one has $a'g_u \leq 0$ (i.e. whenever $g_u > 0$, the function a can not have upward jumps while whenever $g_u < 0$, the function a can not have downward jumps) a similar estimate holds also for the second functional (3.6) for all $0 \leq s \leq t \leq (x_2 - x_1)/L$

$$\begin{aligned}\mathbf{\Phi}_2\left(U_1(t), U_2(t)\right) - \mathbf{\Phi}_2\left(U_1(s), U_2(s)\right) \leq C\Big(&\delta_1 \text{Tot.Var.}\left\{z[U_1](0,\cdot), [x_1, x_2]\right\} \\ &+ \delta_2 \text{Tot.Var.}\left\{z[U_2](0,\cdot), [x_1, x_2]\right\}\Big) \cdot (t-s).\end{aligned} \tag{3.11}$$

From the above result, the existence of a Lipschitz semigroup generated by (1.1) can be easily proved as we will show in Section 4. Before beginning the proof of Theorem 3.1, we need two preliminary lemmas.

Lemma 3.2 *Let $f(u)$ be a $\mathbf{C}^2$ function and $\mathcal{P}^* = \{u_0, u_1, \ldots, u_n\}$ be a partition of the u axis with $\delta^* \doteq |\mathcal{P}^*|$. Denote by $\hat{f}(u)$ the function which linearly interpolate the curve $u \mapsto f(u)$ between the points u_i, $i = 0, \ldots, n$. If u^-, $u^+ \in \mathcal{P}^*$ are connected by an entropic discontinuity of the conservation law $u_t + \hat{f}(u)_x = 0$, then for any state v in the interval with extrema u^- and u^+ one has*

$$\begin{aligned}\left|u^+ - v\right| \left[\frac{f(u^+) - f(v)}{u^+ - v} - \frac{f(u^+) - f(u^-)}{u^+ - u^-}\right] &\leq \|f''\|_\infty \delta^* \left|u^+ - u^-\right|, \\ -\left|u^- - v\right| \left[\frac{f(u^-) - f(v)}{u^- - v} - \frac{f(u^+) - f(u^-)}{u^+ - u^-}\right] &\leq \|f''\|_\infty \delta^* \left|u^+ - u^-\right|.\end{aligned} \tag{3.12}$$

Proof. We will prove the first inequality with $u^- \leq v \leq u^+$, the other cases being similar. Denote by λ the velocity of the discontinuity, i.e. $\lambda = \frac{f(u^+) - f(u^-)}{u^+ - u^-}$, then consider the line connecting the points $\left(u^-, f(u^-)\right)$ and $\left(u^+, f(u^+)\right)$: $h(u) \doteq \lambda(u - u^+) + f(u^+)$. Now choose two contiguous points u', $u'' \in \mathcal{P}^*$ with $u^- \leq u' \leq v \leq u'' \leq u^+$, so that $|u'' - v| \leq \delta^*$. Since the discontinuity is entropic for the conservation law $u_t + \hat{f}(u)_x = 0$, the point $\left(u'', f(u'')\right)$ has to stay above the graph of h [8], i.e. $f(u'') - h(u'') \geq 0$. Therefore we can compute

$$\begin{aligned}\left(u^+ - v\right) \left[\frac{f(u^+) - f(v)}{u^+ - v} - \frac{f(u^+) - f(u^-)}{u^+ - u^-}\right] &= f(u^+) - f(v) - \lambda(u^+ - v) \\ &= h(v) - f(v) \\ &\leq f(u'') - h(u'') + h(v) - f(v).\end{aligned} \tag{3.13}$$

Finally, since $\lambda = f'(c)$ for a $c \in [u^-, u^+]$, we have

$$\begin{aligned}f(u'') - h(u'') + h(v) - f(v) &= \int_v^{u''} f'(\bar{u})\, d\bar{u} - \lambda(u'' - v) \\ &= \int_v^{u''} \left[f'(\bar{u}) - f'(c)\right]\, d\bar{u} \\ &\leq \|f''\|_\infty |u'' - v| \cdot |u^+ - u^-| \leq \|f''\|_\infty \delta^* |u^+ - u^-|,\end{aligned} \tag{3.14}$$

which proves the lemma. □

Lemma 3.3 *Let (a'', u'') and (a'', v'') be two states connected respectively to the states (a', u') and (a', v') by standing waves, more precisely*

$$u'' = \phi(a''; a', u'), \qquad v'' = \phi(a''; a', v').$$

Then one has the estimates

$$\left|(u'' - v'') - (u' - v')\right| \le C_1 |a'' - a'| \cdot \left|u' - v'\right|,$$
$$\left|\left[f(a'', u'') - f(a'', v'')\right] - \left[f(a', u') - f(a', v')\right]\right| \le C_2 |a'' - a'| \cdot \left|u' - v'\right|,$$

where the constants C_1 and C_2 depend only on the system.

Proof. Concerning the first inequality we compute

$$\begin{aligned}\left|(u'' - v'') - (u' - v')\right| &= \left|\phi(a''; a', u') - \phi(a''; a', v') - u' + v'\right| \\ &= \left|\int_{a'}^{a''} \left[\phi'(\bar a; a', u') - \phi'(\bar a; a', v')\right] d\bar a\right| \\ &= \left|\int_{a'}^{a''} \left[\left(\frac{g - f_a}{f_u}\right)\left[\bar a, \phi(\bar a; a', u')\right] - \left(\frac{g - f_a}{f_u}\right)\left[\bar a, \phi(\bar a; a', v')\right]\right] d\bar a\right| \\ &\le C_1 |a'' - a'| \cdot \left|u' - v'\right|,\end{aligned} \tag{3.15}$$

where we have used the Lipschitz continuity of $(g - f_a)/f_u$ and ϕ. Concerning the second inequality, (2.4) implies

$$\begin{aligned}&\left|f(a'', u'') - f(a'', v'') - f(a', u') + f(a', v')\right| \\ &\quad = \left|\int_{a'}^{a''} \left[g\left(\bar a, \phi(\bar a; a', u')\right) - g\left(\bar a, \phi(\bar a; a', v')\right)\right] d\bar a\right| \le C_2 |a'' - a'| \cdot \left|u' - v'\right|,\end{aligned} \tag{3.16}$$

where we have used the Lipschitz continuity of g and ϕ. □

Proof of Theorem 3.1 In the following, the Landau notation $\mathcal{O}(1)$ will be used to indicate a function whose absolute value remains uniformly bounded, the bound depending only on the system and the invariant region K. We consider first the functional $\mathbf{\Phi}_1$. We need to study how it evolves in time. The map $t \mapsto q(t, \cdot)$ is continuous in $\mathbf{L}^1{}_{loc}$ also when there are interactions in U_1 or in U_2. The weight W_2 is constant in time. The weight W_1 is $\mathbf{L}^1{}_{loc}$ continuous outside the interaction times and it can have only downward jumps when an interaction occur. Therefore $\mathbf{\Phi}_1$ can not have upward jumps at the interaction times.
Outside interaction times a direct computation yields

$$\begin{aligned}\frac{d}{dt}\mathbf{\Phi}_1\left(U_1(t), U_2(t)\right) = \sum_{\alpha=1}^{N} \dot x_\alpha &\left\{W_{1,\alpha}^- W_{2,\alpha}^- |p_\alpha^-| - W_{1,\alpha}^+ W_{2,\alpha}^+ |p_\alpha^+| + W_{2,\alpha}^- |q_\alpha^-| - W_{2,\alpha}^+ |q_\alpha^+|\right\} \\ &- L\left[W_{1,0}^+ W_{2,0}^+ |p_0^+| + W_{2,0}^+ |q_0^+|\right],\end{aligned} \tag{3.17}$$

where for notational convenience we have dropped the variable t and have defined

$$
\begin{aligned}
W_{i,\alpha}^{\pm} &= W_i(x_\alpha\pm), & W_{i,0}^{+} &= W_i\left((x_1+Lt)+\right), & &\text{for } i=1,2\\
p_\alpha^{\pm} &= p(x_\alpha\pm), & p_0^{+} &= p\left((x_1+Lt)+\right),\\
q_\alpha^{\pm} &= q(x_\alpha\pm), & q_0^{+} &= q\left((x_1+Lt)+\right).
\end{aligned}
$$

Recalling the definition of the wave strength (3.1), define

$$\omega_\alpha^{\pm} = \phi\left(a_\alpha^{\pm}; b_\alpha^{\pm}, v_\alpha^{\pm}\right), \tag{3.18}$$

with the following obvious notations

$$v_\alpha^{\pm} = v(x_\alpha\pm),\ u_\alpha^{\pm} = u(x_\alpha\pm),\ b_\alpha^{\pm} = b(x_\alpha\pm),\ a_\alpha^{\pm} = a(x_\alpha\pm), \tag{3.19}$$

so that we also have

$$p_\alpha^{\pm} = b_\alpha^{\pm} - a_\alpha^{\pm}, \qquad \text{and} \qquad q_\alpha^{\pm} = u_\alpha^{\pm} - \omega_\alpha^{\pm}.$$

Finally define

$$\lambda_\alpha^{-} = \frac{f(a_\alpha^{-}, u_\alpha^{-}) - f(a_\alpha^{-}, \omega_\alpha^{-})}{u_\alpha^{-} - \omega_\alpha^{-}}, \qquad \lambda_\alpha^{+} = \frac{f(a_\alpha^{+}, u_\alpha^{+}) - f(a_\alpha^{+}, \omega_\alpha^{+})}{u_\alpha^{+} - \omega_\alpha^{+}}. \tag{3.20}$$

Observe that hypothesis (1.3) implies

$$\lambda_\alpha^{\pm}|q_\alpha^{\pm}| = \left| f(a_\alpha^{\pm}, u_\alpha^{\pm}) - f(a_\alpha^{\pm}, \omega_\alpha^{\pm}) \right|. \tag{3.21}$$

Since U_1 and U_2 are piecewise constant, one clearly has

$$\lambda_\alpha^{+}|q_\alpha^{+}|W_{2,\alpha}^{+} = \lambda_{\alpha+1}^{-}|q_{\alpha+1}^{-}|W_{2,\alpha+1}^{-},$$

therefore, using the inequalities $\lambda_1^{-} \leq L$ and $\lambda_N^{+} > 0$, we can sum

$$
\begin{aligned}
\sum_{\alpha=1}^{N}\left[\lambda_\alpha^{+}|q_\alpha^{+}|W_{2,\alpha}^{+} - \lambda_\alpha^{-}|q_\alpha^{-}|W_{2,\alpha}^{-}\right] &= \lambda_N^{+}|q_N^{+}|W_{2,N}^{+} - \lambda_1^{-}|q_1^{-}|W_{2,1}^{-}\\
&\geq -L|q_1^{-}|W_{2,1}^{-} = -L|q_0^{+}|W_{2,0}^{+}.
\end{aligned} \tag{3.22}
$$

Putting together (3.17) and (3.22) we get

$$\frac{d}{dt}\mathbf{\Phi}_1\left(U_1(t), U_2(t)\right) \leq \sum_{\alpha=1}^{N}\left[E_\alpha' + E_\alpha''\right] \tag{3.23}$$

where we have defined

$$E_\alpha' = W_{2,\alpha}^{+}|q_\alpha^{+}|\left(\lambda_\alpha^{+} - \dot{x}_\alpha\right) - W_{2,\alpha}^{-}|q_\alpha^{-}|\left(\lambda_\alpha^{-} - \dot{x}_\alpha\right), \tag{3.24}$$

$$E_\alpha'' = \dot{x}_\alpha\left(W_{1,\alpha}^{-}W_{2,\alpha}^{-}|p_\alpha^{-}| - W_{1,\alpha}^{+}W_{2,\alpha}^{+}|p_\alpha^{+}|\right). \tag{3.25}$$

The next step is the estimate of (3.24) and (3.25). We consider separately three different cases

$$\sum_{\alpha}\left[E'_\alpha+E''_\alpha\right]=\left\{\sum_{\substack{\text{standing waves}\\ \alpha\in\mathcal{J}(U_1)\cup\mathcal{J}(U_2)}}+\sum_{\substack{\text{homogeneous waves}\\ \alpha\in\mathcal{J}(U_2)}}+\sum_{\substack{\text{homogeneous waves}\\ \alpha\in\mathcal{J}(U_1)}}\right\}\left[E'_\alpha+E''_\alpha\right]. \tag{3.26}$$

Case 1: The jump at x_α is a standing waves. In this case $\dot{x}_\alpha=0$ and consequently $E''_\alpha=0$. From definition (3.8), we have

$$W^-_{2,\alpha}=W^+_{2,\alpha}\cdot\exp\left[\kappa_2\left|\Delta a\left(x_\alpha\right)\right|\right],$$

hence from (3.24) we obtain

$$\begin{aligned}E'_\alpha&=W^+_{2,\alpha}\left[|q^+_\alpha|\lambda^+_\alpha-|q^-_\alpha|\lambda^-_\alpha-\left(e^{\kappa_2|\Delta a(x_\alpha)|}-1\right)|q^-_\alpha|\lambda^-_\alpha\right]\\&\le W^+_{2,\alpha}\left[|q^+_\alpha|\lambda^+_\alpha-|q^-_\alpha|\lambda^-_\alpha-\kappa_2\left|\Delta a\left(x_\alpha\right)\right||q^-_\alpha|\lambda^-_\alpha\right].\end{aligned}\tag{3.27}$$

The states $(a^-_\alpha,\omega^-_\alpha)$, (b^-_α,v^-_α), (b^+_α,v^+_α) and $(a^+_\alpha,\omega^+_\alpha)$ are all connected by standing waves as are the states (a^-_α,u^-_α) and (a^+_α,u^+_α). This fact implies

$$\operatorname{sgn}(u^-_\alpha-\omega^-_\alpha)=\operatorname{sgn}(u^+_\alpha-\omega^+_\alpha).$$

Observe that, in this computation, we do not exclude the case in which both U_1 and U_2 have a standing jump in the same point x_α. Lemma 3.3 together to (3.21) implies

$$\begin{aligned}|q^+_\alpha|\lambda^+_\alpha-|q^-_\alpha|\lambda^-_\alpha&=\operatorname{sgn}(u^+_\alpha-\omega^+_\alpha)\left[f(a^+_\alpha,u^+_\alpha)-f(a^+_\alpha,\omega^+_\alpha)-f(a^-_\alpha,u^-_\alpha)+f(a^-_\alpha,\omega^-_\alpha)\right]\\&\le\mathcal{O}(1)|a^-_\alpha-a^+_\alpha|\cdot|u^-_\alpha-\omega^-_\alpha|=\mathcal{O}(1)\left|\Delta a\left(x_\alpha\right)\right||q^-_\alpha|.\end{aligned}\tag{3.28}$$

Finally, since λ^-_α is uniformly bounded away from zero, for κ_2 sufficiently large (depending only on the system) we obtain

$$E'_\alpha\le W^+_{2,\alpha}\left[\mathcal{O}(1)-\kappa_2\lambda^-_\alpha\right]\left|\Delta a\left(x_\alpha\right)\right||q^-_\alpha|\le 0. \tag{3.29}$$

Therefore one has the estimate

$$\sum_{\substack{\text{standing waves}\\ \alpha\in\mathcal{J}(U_1)\cup\mathcal{J}(U_2)}}\left[E'_\alpha+E''_\alpha\right]\le 0. \tag{3.30}$$

Case 2: The jump at x_α is a homogeneous wave in U_2. In this case one has (see Figure 4)

$$a^+_\alpha=a^-_\alpha\doteq a_\alpha,\quad b^+_\alpha=b^-_\alpha\doteq b_\alpha,\quad,\quad |p^+_\alpha|=|p^-_\alpha|=|b_\alpha-a_\alpha|, \tag{3.31}$$

$$v^+_\alpha=v^-_\alpha\doteq v_\alpha,\quad \omega^+_\alpha=\omega^-_\alpha=\phi(a_\alpha;b_\alpha,v_\alpha)\doteq\omega_\alpha,\quad \dot{x}_\alpha=\frac{f(a_\alpha,u^+_\alpha)-f(a_\alpha,u^-_\alpha)}{u^+_\alpha-u^-_\alpha}, \tag{3.32}$$

$$W^+_{2,\alpha}=W^-_{2,\alpha},\quad W^+_{1,\alpha}=W^-_{1,\alpha}. \tag{3.33}$$

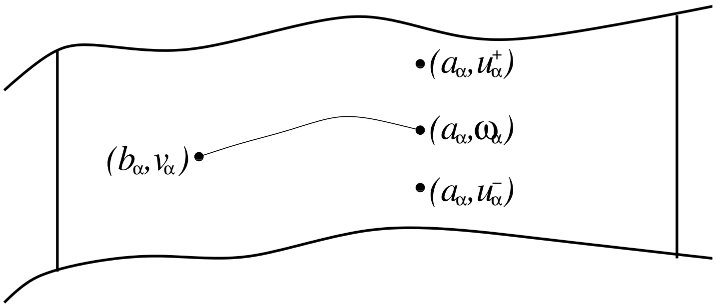

Figure 4: Homogeneous jump in U_2

Therefore

$$E''_\alpha = 0 \tag{3.34}$$

and

$$E'_\alpha = W^+_{2,\alpha}\left[|q^+_\alpha|\left(\lambda^+_\alpha - \dot{x}_\alpha\right) - |q^-_\alpha|\left(\lambda^-_\alpha - \dot{x}_\alpha\right)\right]. \tag{3.35}$$

If $\operatorname{sgn} q^+_\alpha = \operatorname{sgn} q^-_\alpha$ i.e. ω_α lies outside the interval with extrema u^+_α and u^-_α, then

$$\begin{aligned} E'_\alpha &= W^+_{2,\alpha} \operatorname{sgn} q^+_\alpha \left[q^+_\alpha\left(\lambda^+_\alpha - \dot{x}_\alpha\right) - q^-_\alpha\left(\lambda^-_\alpha - \dot{x}_\alpha\right)\right] \\ &= W^+_{2,\alpha} \operatorname{sgn} q^+_\alpha \left[f(a_\alpha, u^+_\alpha) - f(a_\alpha, \omega_\alpha) - q^+_\alpha \dot{x}_\alpha - f(a_\alpha, u^-_\alpha) + f(a_\alpha, \omega_\alpha) + q^-_\alpha \dot{x}_\alpha\right] \\ &= W^+_{2,\alpha} \operatorname{sgn} q^+_\alpha \left[f(a_\alpha, u^+_\alpha) - f(a_\alpha, u^-_\alpha) - (u^+_\alpha - u^-_\alpha)\dot{x}_\alpha\right] = 0. \end{aligned} \tag{3.36}$$

The last equality holds since the approximate solutions satisfy the exact Rankine Hugoniot relation (3.32). If instead ω_α is in between the interval with extrema u^+_α and u^-_α, Lemma 3.2 implies

$$\begin{aligned} |q^+_\alpha|\left(\lambda^+_\alpha - \dot{x}_\alpha\right) &\le \|f_{uu}\|_\infty \sup_{i=1,\dots n} \left|\phi(a_\alpha; a_1, z_i) - \phi(a_\alpha; a_1, z_{i-1})\right| \left|u^+_\alpha - u^-_\alpha\right| \\ &\le \mathcal{O}(1)\delta_2 \left|u^+_\alpha - u^-_\alpha\right| \end{aligned} \tag{3.37}$$

and analogously

$$-|q^-_\alpha|\left(\lambda^-_\alpha - \dot{x}_\alpha\right) \le \mathcal{O}(1)\delta_2 \left|u^+_\alpha - u^-_\alpha\right|. \tag{3.38}$$

Relations (3.34), (3.35), (3.36), (3.37) and (3.38) together imply for every homogeneous jump in U_2:

$$E'_\alpha + E''_\alpha \le \mathcal{O}(1) W^+_{2,\alpha}\delta_2 \left|u^+_\alpha - u^-_\alpha\right|. \tag{3.39}$$

Across homogeneous waves the variation in u is equivalent to the variation in the Riemann coordinate z. Since the total variation in the z variable is non increasing in time we can compute

$$\begin{aligned} \sum_{\substack{\text{homogeneous waves} \\ \alpha \in \mathcal{J}(U_2)}} \left[E'_\alpha + E''_\alpha\right] &\le \mathcal{O}(1)\|W_2\|_\infty \delta_2 \sum_{\substack{\text{homogeneous waves } \alpha \\ \alpha \in \mathcal{J}(U_2)}} \left|u^+_\alpha - u^-_\alpha\right| \\ &\le \mathcal{O}(1)\|W_2\|_\infty \delta_2 \text{Tot.Var.}\left\{z[a,u](t,\cdot), [x_1 + Lt, x_2]\right\} \\ &\le \mathcal{O}(1)\|W_2\|_\infty \delta_2 \text{Tot.Var.}\left\{z\left[U_2\right](0,\cdot), [x_1, x_2]\right\}. \end{aligned} \tag{3.40}$$

Case 3: The jump at x_α is a homogeneous wave of U_1 (see Figure 5). In this case the relations

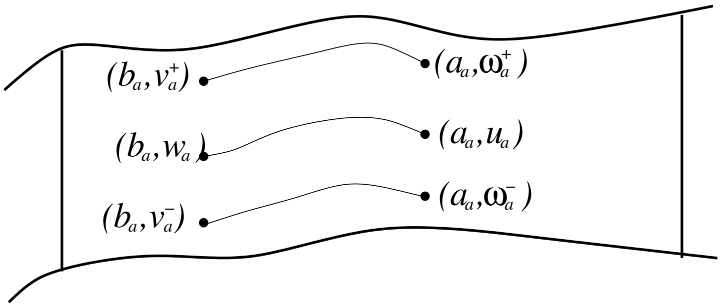

Figure 5: Homogeneous jump in U_1

(3.31) continue to hold while (3.32) (3.33) have to be substituted with

$$u_\alpha^+ = u_\alpha^- \doteq u_\alpha, \quad \dot{x}_\alpha = \frac{f(b_\alpha, v_\alpha^+) - f(b_\alpha, v_\alpha^-)}{v_\alpha^+ - v_\alpha^-}, \tag{3.41}$$

$$W_{2,\alpha}^+ = W_{2,\alpha}^-, \quad W_{1,\alpha}^+ = W_{1,\alpha}^- + \kappa_1 \left|\Delta z[b,v](x_\alpha)\right|. \tag{3.42}$$

Concerning E''_α, due to (3.42), one has

$$E''_\alpha \le -W_{2,\alpha}^+ \dot{x}_\alpha \kappa_1 |b_\alpha - a_\alpha| \left|\Delta z[b,v](x_\alpha)\right|. \tag{3.43}$$

Now we consider

$$E'_\alpha = W_{2,\alpha}^+ \left[|q_\alpha^+| \left(\lambda_\alpha^+ - \dot{x}_\alpha\right) - |q_\alpha^-| \left(\lambda_\alpha^- - \dot{x}_\alpha\right)\right]. \tag{3.44}$$

If $\operatorname{sgn} q_\alpha^+ = \operatorname{sgn} q_\alpha^-$, with the help of Lemma 3.3 and Rankine Hugoniot condition (3.41) we obtain

$$\begin{aligned} E'_\alpha &= W_{2,\alpha}^+ \operatorname{sgn} q_\alpha^+ \left[\dot{x}_\alpha(\omega_\alpha^+ - \omega_\alpha^-) - f(a_\alpha, \omega_\alpha^+) + f(a_\alpha, \omega_\alpha^-)\right] \\ &= W_{2,\alpha}^+ \operatorname{sgn} q_\alpha^+ \left[\dot{x}_\alpha(v_\alpha^+ - v_\alpha^-) - f(b_\alpha, v_\alpha^+) + f(b_\alpha, v_\alpha^-) + \mathcal{O}(1)|b_\alpha - a_\alpha| \cdot \left|v_\alpha^+ - v_\alpha^-\right|\right] \\ &\le \mathcal{O}(1) W_{2,\alpha}^+ |b_\alpha - a_\alpha| \cdot \left|v_\alpha^+ - v_\alpha^-\right|. \end{aligned} \tag{3.45}$$

If instead $\operatorname{sgn} q_\alpha^+ \neq \operatorname{sgn} q_\alpha^-$, then the state u_α is inside the interval with extrema ω_α^- and ω_α^+. In this case we have to introduce the new state $w_\alpha = \phi(b_\alpha; a_\alpha, u_\alpha)$, which is inside the interval with extrema v_α^+ and v_α^-, and the velocities

$$\widetilde{\lambda}_\alpha^+ = \frac{f(b_\alpha, v_\alpha^+) - f(b_\alpha, w_\alpha)}{v_\alpha^+ - w_\alpha}, \qquad \widetilde{\lambda}_\alpha^- = \frac{f(b_\alpha, v_\alpha^-) - f(b_\alpha, w_\alpha)}{v_\alpha^- - w_\alpha}. \tag{3.46}$$

Simple computations lead to

$$\begin{aligned} E'_\alpha &= W_{2,\alpha}^+ \left[|v_\alpha^+ - w_\alpha| \left(\widetilde{\lambda}_\alpha^+ - \dot{x}_\alpha\right) - |v_\alpha^- - w_\alpha| \left(\widetilde{\lambda}_\alpha^- - \dot{x}_\alpha\right)\right] \\ &\quad + W_{2,\alpha}^+ \left[|u_\alpha - \omega_\alpha^+| \lambda_\alpha^+ - |u_\alpha - \omega_\alpha^-| \lambda_\alpha^- - |w_\alpha - v_\alpha^+| \widetilde{\lambda}_\alpha^+ + |w_\alpha - v_\alpha^-| \widetilde{\lambda}_\alpha^-\right] \\ &\quad + W_{2,\alpha}^+ \dot{x}_\alpha \left[-|u_\alpha - \omega_\alpha^+| + |v_\alpha^+ - w_\alpha| + |u_\alpha - \omega_\alpha^-| - |v_\alpha^- - w_\alpha|\right] \end{aligned} \tag{3.47}$$

The first term in (3.47) can be estimated using Lemma 3.2

$$|v_\alpha^+ - w_\alpha| \left(\widetilde{\lambda}_\alpha^+ - \dot{x}_\alpha\right) - |v_\alpha^- - w_\alpha| \left(\widetilde{\lambda}_\alpha^- - \dot{x}_\alpha\right) \leq \mathcal{O}(1)\delta_1 \left|v_\alpha^+ - v_\alpha^-\right|. \tag{3.48}$$

The second term in (3.47) can be estimated using Lemma 3.3, indeed since $\mathrm{sgn}(u_\alpha - \omega_\alpha^+) = \mathrm{sgn}(w_\alpha - v_\alpha^+)$, we can compute

$$\begin{aligned} \left| |u_\alpha - \omega_\alpha^+| \lambda_\alpha^+ - |w_\alpha - v_\alpha^+| \widetilde{\lambda}_\alpha^+ \right| &= \left| f(a_\alpha, u_\alpha) - f(a_\alpha, \omega_\alpha^+) - f(b_\alpha, w_\alpha) + f(b_\alpha, v_\alpha^+) \right| \\ &\leq \mathcal{O}(1)|b_\alpha - a_\alpha| \cdot \left|v_\alpha^+ - w_\alpha\right| \leq \mathcal{O}(1)|b_\alpha - a_\alpha| \cdot \left|v_\alpha^+ - v_\alpha^-\right|, \end{aligned} \tag{3.49}$$

the same holds for $|u_\alpha - \omega_\alpha^-|\lambda_\alpha^- - |w_\alpha - v_\alpha^-|\widetilde{\lambda}_\alpha^-$.
Also the third term can be estimated with the help of lemma 3.3

$$\begin{aligned} \left| |v_\alpha^+ - w_\alpha| - |u_\alpha - \omega_\alpha^+| \right| &\leq \left| v_\alpha^+ - w_\alpha - \omega_\alpha^+ + u_\alpha \right| \leq \mathcal{O}(1)|b_\alpha - a_\alpha| \cdot \left| v_\alpha^+ - w_\alpha \right| \\ &\leq \mathcal{O}(1)|b_\alpha - a_\alpha| \cdot \left| v_\alpha^+ - v_\alpha^- \right| \end{aligned} \tag{3.50}$$

and the same holds for $|v_\alpha^- - w_\alpha| - |u_\alpha - \omega_\alpha^-|$.

The Lipschitz continuity of ϕ implies

$$\left|v_\alpha^+ - v_\alpha^-\right| \leq \mathcal{O}(1) \left|\Delta z[b,v](x_\alpha)\right|,$$

therefore, putting together (3.43), (3.45), (3.47), (3.48), (3.49) and (3.50), one gets for every homogeneous jump in U_1:

$$\begin{aligned} E'_\alpha + E''_\alpha &\leq W_{2,\alpha}^+ \left[\mathcal{O}(1) - \dot{x}_\alpha \kappa_1\right] |b_\alpha - a_\alpha| \cdot \left|\Delta z[b,v](x_\alpha)\right| + W_{2,\alpha}^+ \mathcal{O}(1)\delta_1 \left|v_\alpha^+ - v_\alpha^-\right| \\ &\leq W_{2,\alpha}^+ \mathcal{O}(1)\delta_1 \left|v_\alpha^+ - v_\alpha^-\right|, \end{aligned} \tag{3.51}$$

for κ_1 chosen sufficiently large depending only on the system. Computations similar to the previous case show that

$$\sum_{\substack{\text{homogeneous waves} \\ \alpha \in \mathcal{J}(U_1)}} \left[E'_\alpha + E''_\alpha\right] \leq \mathcal{O}(1)\|W_2\|_\infty \delta_1 \text{Tot.Var.} \left\{z\left[U_1\right](0,\cdot), [x_1, x_2]\right\}. \tag{3.52}$$

Finally, putting together (3.23), (3.26), (3.30), (3.40) and (3.52) one gets

$$\begin{aligned} \frac{d}{dt}\mathbf{\Phi}_1\left(U_1(t), U_2(t)\right) \leq \mathcal{O}(1)\|W_2\|_\infty \Big[& \delta_1 \text{Tot.Var.} \left\{z\left[U_1\right](0,\cdot), [x_1, x_2]\right\} \\ & + \delta_2 \text{Tot.Var.} \left\{z\left[U_2\right](0,\cdot), [x_1, x_2]\right\} \Big] \end{aligned} \tag{3.53}$$

which together the definition (3.8) of the weight $W_{2,\alpha}^+$ proves the first estimate (3.10).

Concerning the estimate for the second functional $\mathbf{\Phi}_2$, we observe that the weight W_2 is needed only to obtain estimate (3.29). If the source is dissipative (in the sense precised in the statement of the theorem), W_2 is no more needed for (3.29). Indeed since in the Case 1 the states $(a_\alpha^-, \omega_\alpha^-)$, (b_α^-, v_α^-), (b_α^+, v_α^+) and $(a_\alpha^+, \omega_\alpha^+)$ are all connected by standing waves as are the states (a_α^-, u_α^-) and (a_α^+, u_α^+), we have

$$\operatorname{sgn}(u_\alpha^- - \omega_\alpha^-) = \operatorname{sgn}(u_\alpha^+ - \omega_\alpha^+) = \operatorname{sgn}\left[\phi(\bar a; a_\alpha^-, u_\alpha^-) - \phi(\bar a; a_\alpha^-, \omega_\alpha^-)\right], \qquad \forall \bar a \in [a_1, a_2].$$

Therefore, with the help of (2.4), we compute

$$\begin{aligned} E'_\alpha &= |q_\alpha^+|\lambda_\alpha^+ - |q_\alpha^-|\lambda_\alpha^- \\ &= \operatorname{sgn}(u_\alpha^+ - \omega_\alpha^+)\left[f(a_\alpha^+, u_\alpha^+) - f(a_\alpha^+, \omega_\alpha^+) - f(a_\alpha^-, u_\alpha^-) + f(a_\alpha^-, \omega_\alpha^-)\right] \\ &= \int_{a_\alpha^-}^{a_\alpha^+}\int_0^1 g_u\left[\bar a, s\phi(\bar a; a_\alpha^-, u_\alpha^-) + (1-s)\phi(\bar a; a_\alpha^-, \omega_\alpha^-)\right]\left|\phi(\bar a; a_\alpha^-, u_\alpha^-) - \phi(\bar a; a_\alpha^-, \omega_\alpha^-)\right| ds d\bar a \\ &\le 0. \end{aligned} \tag{3.54}$$

□

The following $\mathbf{L}^1$ estimate for wave front tracking approximations is a direct consequence of Theorem 3.1 and will be used in Section 4.

Corollary 3.4 *In the hypotheses of Theorem 3.1, for any $x_1 < x_2$ and $t \in \left[0, (x_2 - x_1)/L\right]$ the following $\mathbf{L}^1$ estimate holds*

$$\begin{aligned} \int_{x_1+Lt}^{x_2} \left|u(t,x) - v(t,x)\right| dx \le & e^{\kappa_2\rho_2}\int_{x_1}^{x_2}\left|u(0,x) - v(0,x)\right| dx \\ & + C'\left(1 + r_1\right) e^{\kappa_2\rho_2}\int_{x_1}^{x_2}\left|a(x) - b(x)\right| dx \\ & + Ce^{\kappa_2\rho_2}\left[\delta_1 r_1 + \delta_2 r_2\right] t, \end{aligned} \tag{3.55}$$

where C' is a constant which depends only on the system and we have set for notational convenience

$$\begin{aligned} &\rho_2 \doteq \text{Tot.Var.}\left\{a, [x_1, x_2]\right\}, \ r_1 \doteq \text{Tot.Var.}\left\{z\left[b, v\right](0, \cdot), [x_1, x_2]\right\}, \\ &r_2 \doteq \text{Tot.Var.}\left\{z\left[a, u\right](0, \cdot), [x_1, x_2]\right\}. \end{aligned}$$

If moreover $a' g_u \le 0$ then the following sharper estimate holds

$$\begin{aligned} \int_{x_1+Lt}^{x_2}\left|u(t,x) - v(t,x)\right| dx \le & \int_{x_1}^{x_2}\left|u(0,x) - v(0,x)\right| dx + C'\left(1 + r_1\right)\int_{x_1}^{x_2}\left|a(x) - b(x)\right| dx \\ & + C\left[\delta_1 r_1 + \delta_2 r_2\right] t. \end{aligned} \tag{3.56}$$

Proof. From inequalities (3.2), (3.3), definitions (3.5), (3.7), (3.8) and Theorem 3.1 we obtain:

$$\begin{aligned}
\int_{x_1+Lt}^{x_2} \left|u(t,x)-v(t,x)\right| dx &\leq \int_{x_1+Lt}^{x_2} \Big[\left|q(t,x)\right| + \mathcal{O}(1)|p(x)|\Big]\, dx \\
&\leq \mathbf{\Phi}_1\left(U_1(t),U_2(t)\right) + \mathcal{O}(1)\int_{x_1+Lt}^{x_2} |p(x)|dx \\
&\leq \mathbf{\Phi}_1\left(U_1(0),U_2(0)\right) + Ce^{\kappa_2\rho_2}\left(\delta_1 r_1 + \delta_2 r_2\right)t + \mathcal{O}(1)\int_{x_1+Lt}^{x_2} |p(x)|dx \\
&\leq e^{\kappa_2\rho_2}\int_{x_1}^{x_2} \left|u(0,x)-v(0,x)\right| dx \\
&\quad + \mathcal{O}(1)e^{\kappa_2\rho_2}\left[1+r_1\right]\int_{x_1}^{x_2} |p(x)|dx + Ce^{\kappa_2\rho_2}\left(\delta_1 r_1 + \delta_2 r_2\right)t,
\end{aligned}$$

which proves (3.55) with $C' \geq |\mathcal{O}(1)|$. Estimate (3.56) can be proved in the same way using $\mathbf{\Phi}_2$ instead of $\mathbf{\Phi}_1$. □

4 Construction and characterization of the limit semigroup

4.1 Construction and continuity properties

In this section we will construct a semigroup of solutions to equation (1.1) as limit of front tracking approximate solutions.

Given the invariant domain K defined in (2.5), we consider the following domains of functions with bounded total variation:

$$\begin{aligned}
\mathcal{B}_\rho &= \left\{a \in \mathbf{BV}\left(\mathbb{R},\mathbb{R}\right) :\ a_1 \leq a(x) \leq a_2\ \forall x \in \mathbb{R},\ \text{Tot.Var.}a \leq \rho\right\}, \qquad \forall \rho > 0, \\
\mathcal{B} &= \left\{a \in \mathbf{BV}\left(\mathbb{R},\mathbb{R}\right) :\ a_1 \leq a(x) \leq a_2\ \forall x \in \mathbb{R}\right\}, \\
\mathcal{D}_r^a &= \left\{u \in \mathbf{BV}\left(\mathbb{R},\mathbb{R}\right) :\ (a,u)(x) \in K\ \forall x \in \mathbb{R},\ \text{Tot.Var.}z[a,u] \leq r\right\}, \quad \forall a \in \mathcal{B},\ \forall r > 0.
\end{aligned}$$

We measure the total variation of u using the Riemann coordinate z because in this way $\mathcal{D}_r^a$ turns out to be an invariant domain for the semigroup we are going to construct.

Theorem 4.1 *Given $a_o \in \mathcal{B}$, $u_o \in \mathcal{D}_r^{a_o}$, consider a sequence of partitions of the z axis $\mathcal{P}_k$ with $\delta_k \doteq |\mathcal{P}_k| \xrightarrow{k\to+\infty} 0$ and a sequence of approximate initial data $(a_{o,k}, u_{o,k})$ satisfying*

$$\text{Tot.Var.}(a_{o,k},u_{o,k}) \leq R, \quad (a_{o,k},u_{o,k}) \xrightarrow[k\to+\infty]{\mathbf{L}^1{}_{loc}} (a_o,u_o), \quad z[a_{o,k},u_{o,k}] \in \mathcal{P}_k, \tag{4.1}$$

where $R > 0$ is a constant independent of k. Then, for any $t \geq 0$, the corresponding sequence of wave front tracking approximate solutions $u_k(t,\cdot)$ to (1.1), constructed as described in Section 2, converges, in the $\mathbf{L}^1{}_{loc}$ topology to a unique limit solution $u(t,\cdot) \in \mathcal{D}_r^{a_o}$. The map $(t,a_o,u_o) \mapsto u(t,\cdot) \doteq \boldsymbol{S}_t^{a_o}u_o$ is a uniformly Lipschitz continuous semigroup in the following sense. There exist positive constants $L'(\rho,r)$, C', C'' and L such that for any $a_o, b_o \in \mathcal{B}_\rho$, $u_o \in \mathcal{D}_r^{a_o}$, $v_o \in \mathcal{D}_r^{b_o}$ one has

$$\boldsymbol{S}_0^{a_o}u_o = u_o, \qquad \boldsymbol{S}_s^{a_o}\left(\boldsymbol{S}_t^{a_o}u_o\right) = \boldsymbol{S}_{s+t}^{a_o}u_o, \qquad \forall s,t \geq 0, \tag{4.2}$$

$$\left\|\boldsymbol{S}_s^{a_o}u_o - \boldsymbol{S}_t^{a_o}u_o\right\|_{\mathbf{L}^1} \leq L'(\rho,r)|t-s|, \qquad \forall s,t \geq 0. \tag{4.3}$$

Moreover, for all $x_1 < x_2$, $0 \leq t \leq (x_2 - x_1)/L$, it holds

$$\begin{aligned}\int_{x_1+Lt}^{x_2} \left| \left(\boldsymbol{S}_t^{b_o} v_o\right)(x) - \left(\boldsymbol{S}_t^{a_o} u_o\right)(x) \right| dx \leq & e^{\kappa_2 \rho} \int_{x_1}^{x_2} \left| v_o(x) - u_o(x) \right| dx \\ & + C'(1+r) e^{\kappa_2 \rho} \int_{x_1}^{x_2} \left| b_o(x) - a_o(x) \right| dx.\end{aligned} \tag{4.4}$$

Proof. Fix L as in the definition (3.4). From Corollary 3.4 we obtain for any two elements $U_k = (a_k, u_k)$, $U_h = (a_h, u_h)$ of the sequence of front tracking approximate solutions and for any $x_1 < x_2$, $t \in [0, (x_2 - x_1)/L]$

$$\begin{aligned}\int_{x_1+Lt}^{x_2} \left| u_k(t,x) - u_h(t,x) \right| dx \leq e^{\kappa_2 R} \Bigg\{ & \int_{x_1}^{x_2} \left| u_{o,k}(x) - u_{o,h}(x) \right| dx + \mathcal{O}(1) R t \left(\delta_k + \delta_h\right) \\ & + C' \left[1 + \mathcal{O}(1) R\right] \int_{x_1}^{x_2} \left| a_{o,k}(x) - a_{o,h}(x) \right| dx \Bigg\}.\end{aligned}$$

Therefore $u_k(t,\cdot)$ is a Cauchy sequence in $\mathbf{L}^1\left((x_1 + Lt, x_2)\right)$. The arbitrariness of x_1 and x_2 implies that for any $t \geq 0$, the sequence $u_k(t,\cdot)$ converges in $\mathbf{L}^1{}_{loc}(\mathbb{R})$ to a function $u(t,\cdot) \doteq \boldsymbol{S}_t^{a_o} u_o$.

Concerning the uniqueness, suppose we have another sequence of partitions $\overline{\mathcal{P}}_k$ with $\overline{\delta}_k \doteq |\overline{\mathcal{P}}_k| \xrightarrow{k \to +\infty} 0$ and another sequence of wave front tracking approximations $(\overline{a}_k, \overline{u}_k)$ with initial data $(\overline{a}_{o,k}, \overline{u}_{o,k})$ satisfying the analogous of (4.1)

$$\text{Tot.Var.}(\overline{a}_{o,k}, \overline{u}_{o,k}) \leq R, \quad (\overline{a}_{o,k}, \overline{u}_{o,k}) \xrightarrow[k \to +\infty]{\mathbf{L}^1{}_{loc}} (a_o, u_o), \quad z[\overline{a}_{o,k}, \overline{u}_{o,k}] \in \overline{\mathcal{P}}_k. \tag{4.5}$$

Now we apply Corollary 3.4 to $U_k = (a_k, u_k)$ and $\overline{U}_k = (\overline{a}_k, \overline{u}_k)$ to obtain

$$\begin{aligned}\int_{x_1+Lt}^{x_2} \left| u_k(t,x) - \overline{u}_k(t,x) \right| dx \leq e^{\kappa_2 R} \Bigg\{ & \int_{x_1}^{x_2} \left| u_{o,k}(x) - \overline{u}_{o,k}(x) \right| dx + \mathcal{O}(1) R t \left(\delta_k + \overline{\delta}_k\right) \\ & + C' \left[1 + \mathcal{O}(1) R\right] \int_{x_1}^{x_2} \left| a_{o,k}(x) - \overline{a}_{o,k}(x) \right| dx \Bigg\}.\end{aligned}$$

This last inequality shows that $u_k(t,\cdot)$ and $\overline{u}_k(t,\cdot)$ converge to the same limit $\boldsymbol{S}_t^{a_o} u_o$ as $k \to +\infty$. The initial approximating sequence $(a_{o,k}, u_{o,k})$ can be chosen in such a way that in addition to (4.1), it satisfies

$$\text{Tot.Var.} a_{o,k} \leq \text{Tot.Var.} a_o, \qquad \limsup_{k \to +\infty} \text{Tot.Var.} z[a_{o,k}, u_{o,k}] \leq r. \tag{4.6}$$

Observe that since the z coordinate has to stay on a grid, we can not choose $u_{o,k}$ such that $\text{Tot.Var.} z[a_{o,k}, u_{o,k}] \leq r$. The lower semicontinuity of the total variation, together to the fact that the total variation of the coordinate z, does not increase with time, implies that $\boldsymbol{S}_t^{a_o} u_o \in \mathcal{D}_r^{a_o}$ for all $t \geq 0$. The semigroup property (4.2) is a consequence of the definition of approximate solutions and the uniqueness of their limit. Properties (4.3) is a direct consequence of the finite speed propagation of waves and of the uniform bound (depending on ρ and r) on the total variation of $\boldsymbol{S}_t^{a_o} u_o$.

To conclude the proof we have to show (4.4). To this purpose, take the usual sequence of partitions $\mathcal{P}_k$ with $\delta_k \doteq |\mathcal{P}_k| \xrightarrow{k\to+\infty} 0$ and two sequences of approximate initial data $(a_{o,k}, u_{o,k})$ and $(b_{o,k}, v_{o,k})$ satisfying:

$$\text{Tot.Var.} a_{o,k} \le \rho,\ \text{Tot.Var.} z[a_{o,k}, u_{o,k}] \le r_k,\ (a_{o,k}, u_{o,k}) \xrightarrow[k\to+\infty]{\mathbf{L}^1{}_{loc}} (a_o, u_o),\ z[a_{o,k}, u_{o,k}] \in \mathcal{P}_k,$$

$$\text{Tot.Var.} b_{o,k} \le \rho,\ \text{Tot.Var.} z[b_{o,k}, v_{o,k}] \le r_k,\ (b_{o,k}, v_{o,k}) \xrightarrow[k\to+\infty]{\mathbf{L}^1{}_{loc}} (b_o, v_o),\ z[b_{o,k}, v_{o,k}] \in \mathcal{P}_k,$$

with $r_k \to r$. If we apply again Corollary 3.4 to the corresponding front tracking approximations $\big(a_k(x), u_k(t,x)\big)$ and $\big(b_k(x), v_k(t,x)\big)$ we obtain

$$\begin{aligned}\int_{x_1+Lt}^{x_2} \left|u_k(t,x) - v_k(t,x)\right| dx \le & e^{\kappa_2\rho} \int_{x_1}^{x_2} \left|u_{o,k}(x) - v_{o,k}(x)\right| dx \\ & + C'(1+r_k)e^{\kappa_2\rho} \int_{x_1}^{x_2} \left|a_{o,k}(x) - b_{o,k}(x)\right| dx + 2Ctr_k e^{\kappa_2\rho}\delta_k\end{aligned} \tag{4.7}$$

which, in the limit as $k \to +\infty$ concludes the proof. □

In the presence of dissipative sources, we have sharper estimates.

Theorem 4.2 *In the hypotheses of Theorem 4.1, suppose that the source g and the initial data $a_o,\ b_o \in \mathcal{B}_\rho$, $u_o \in \mathcal{D}_r^{a_o}$, $v_o \in \mathcal{D}_r^{b_o}$ satisfy one of the following additional dissipative conditions:*

i. $g_u(a,u) = 0$ *(conservative case);*

ii. $g_u(a,u) \le 0$ *and* a_o *non decreasing;*

iii. $g_u(a,u) \ge 0$ *and* a_o *non increasing.*

Then the semigroup trajectories satisfy the sharper estimate

$$\begin{aligned}\int_{x_1+Lt}^{x_2} \left|\left(\boldsymbol{S}_t^{b_o} v_o\right)(x) - \left(\boldsymbol{S}_t^{a_o} u_o\right)(x)\right| dx \le & \int_{x_1}^{x_2} \left|v_o(x) - u_o(x)\right| dx \\ & + C'\left[1+r\right] \int_{x_1}^{x_2} \left|b_o(x) - a_o(x)\right| dx.\end{aligned} \tag{4.8}$$

Proof. In all the three cases, one can choose an approximating initial data $a_{o,k}$ such that $a'_{o,k} g_u \le 0$ in the sense of Theorem 3.1, therefore one can use (3.56) instead of (3.55) in the estimate (4.7) to obtain the desired result. □

4.2 Extension to $\mathbf{L}^\infty$ initial data

It is well known that, for a scalar conservation law, there is stability with respect to the initial data within a domain of $\mathbf{L}^\infty$ functions. On the other hand, in the case of $n \times n$ systems, the usual stability results hold for domains of functions with uniformly bounded total variation. However there are also some stability results in $\mathbf{L}^\infty$ for classes of Temple systems in conservation form [3, 9, 6].

Here we want only to show that the particular dependence of the Lipschitz constants in (4.4) on the total variation of the initial data, allows us to extend the semigroup to a domain of

$\mathbf{L}^\infty$ function for the variable u, maintaining the continuity with respect to the variable a with bounded total variation.

First of all we define the following domains of $\mathbf{L}^\infty$ functions for any $a \in \mathcal{B}$

$$\mathcal{D}^a = \left\{u \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R}) : \ (a,u)(x) \in K \text{ for all } x \in \mathbb{R}\right\}. \tag{4.9}$$

It is clear that, since the Lipschitz constant $e^{\kappa_2 \rho}$ in (4.4) does not depend on r, for a fixed $a \in \mathcal{B}$ the semigroup $\boldsymbol{S}^a_t$ can be extended to a Lipschitz continuous semigroup defined on $\mathcal{D}^a$. In this extension, we maintain the Lipschitz continuity with respect to u, i.e. for $a \in \mathcal{B}_\rho$ and any $u, v \in \mathcal{D}^a$ one has

$$\int_{x_1+Lt}^{x_2} \left|\left(\boldsymbol{S}^a_t v\right)(x) - \left(\boldsymbol{S}^a_t u\right)(x)\right| dx \le e^{\kappa_2 \rho} \int_{x_1}^{x_2} \left|v(x) - u(x)\right| dx. \tag{4.10}$$

On the other hand we loose the Lipschitz continuity with respect to the variable a. Indeed it is easy to show (with a counter example, similar to the one found in [9]) that the map $a \mapsto \boldsymbol{S}^a_t u$, in general, does not satisfy Lipschitz conditions similar to (4.4) if u has unbounded total variation. However one can prove the following continuity result for the extended semigroup.

Theorem 4.3 *The map $(t, a, u) \mapsto \boldsymbol{S}^a_t u$ is continuous in the $\mathbf{L}^1{}_{loc}$ topology when $t \ge 0$, $a \in \mathcal{B}_\rho$ and $u \in \mathcal{D}^a$.*

Proof. Take a converging sequence $(t_k, a_k, u_k) \to (t, a, u)$ in the $\mathbf{L}^1{}_{loc}$ topology with $a_k \in \mathcal{B}_\rho$, $u_k \in \mathcal{D}^{a_k}$ and $t_k \ge 0$. We have to show that also $\boldsymbol{S}^{a_k}_{t_k} u_k$ converges to $\boldsymbol{S}^a_t u$ in $\mathbf{L}^1{}_{loc}$. Observe that $I \doteq z[K]$ is a closed interval. Fix two arbitrary points $x_1 < x_2$, $0 \le t < (x_2 - x_1)/L$ and $\varepsilon > 0$. Define the functions $\widetilde{z}_k(x) = z[a_k, u_k](x)$, $\widetilde{z}(x) = z[a, u](x)$ and choose an approximation $\widetilde{z}^\varepsilon(x) \in I$ with bounded total variation

$$r_\varepsilon \doteq \text{Tot.Var.}\widetilde{z}^\varepsilon, \qquad \int_{x_1}^{x_2} \left|\widetilde{z}(x) - \widetilde{z}^\varepsilon(x)\right| dx \le \varepsilon. \tag{4.11}$$

Since $\widetilde{z}_k$ converges to $\widetilde{z}$ in $\mathbf{L}^1{}_{loc}$, for k sufficiently large we also have

$$\int_{x_1}^{x_2} \left|\widetilde{z}_k(x) - \widetilde{z}^\varepsilon(x)\right| dx \le 2\varepsilon. \tag{4.12}$$

Now define two functions

$$\widetilde{u}^\varepsilon_k(x) = \phi\left(a_k(x); a_1, \widetilde{z}^\varepsilon(x)\right) \qquad \text{and} \qquad \widetilde{u}^\varepsilon(x) = \phi\left(a(x); a_1, \widetilde{z}^\varepsilon(x)\right)$$

so that $\widetilde{u}^\varepsilon_k \in \mathcal{D}^{a_k}_{r_\varepsilon}$ and $\widetilde{u}^\varepsilon \in \mathcal{D}^a_{r_\varepsilon}$. Since

$$u(x) = \phi\left(a(x); a_1, \widetilde{z}(x)\right), \qquad \text{and} \qquad u_k(x) = \phi\left(a_k(x); a_1, \widetilde{z}_k(x)\right),$$

the Lipschitz continuity of ϕ implies

$$\int_{x_1}^{x_2} \left|u_k(x) - \widetilde{u}^\varepsilon_k(x)\right| dx \le \mathcal{O}(1)\varepsilon, \quad \int_{x_1}^{x_2} \left|u(x) - \widetilde{u}^\varepsilon(x)\right| dx \le \mathcal{O}(1)\varepsilon,$$
$$\int_{x_1}^{x_2} \left|\widetilde{u}^\varepsilon_k(x) - \widetilde{u}^\varepsilon(x)\right| dx \le \mathcal{O}(1) \int_{x_1}^{x_2} \left|a_k(x) - a(x)\right| dx.$$

Therefore from (4.3), (4.4) and (4.10) we can compute for $t_k, t \leq \bar{t} \leq (x_2 - x_1)/L$

$$
\begin{aligned}
\int_{x_1+L\bar{t}}^{x_2} & \left| \left(\boldsymbol{S}_{t_k}^{a_k} u_k\right)(x) - \left(\boldsymbol{S}_t^a u\right)(x) \right| dx \leq \int_{x_1+Lt_k}^{x_2} \left| \left(\boldsymbol{S}_{t_k}^{a_k} u_k\right)(x) - \left(\boldsymbol{S}_{t_k}^{a_k} \widetilde{u}_k^{\varepsilon}\right)(x) \right| dx \\
& + \int_{x_1+L\bar{t}}^{x_2} \left| \left(\boldsymbol{S}_{t_k}^{a_k} \widetilde{u}_k^{\varepsilon}\right)(x) - \left(\boldsymbol{S}_t^a \widetilde{u}^{\varepsilon}\right)(x) \right| dx + \int_{x_1+Lt}^{x_2} \left| \left(\boldsymbol{S}_t^a \widetilde{u}^{\varepsilon}\right)(x) - \left(\boldsymbol{S}_t^a u\right)(x) \right| dx \\
\leq e^{\kappa_2 \rho} & \int_{x_1}^{x_2} \left| u_k(x) - \widetilde{u}_k^{\varepsilon}(x) \right| dx + e^{\kappa_2 \rho} \int_{x_1}^{x_2} \left| \widetilde{u}_k^{\varepsilon}(x) - \widetilde{u}^{\varepsilon}(x) \right| dx + L'(\rho, r_\varepsilon) \cdot |t_k - t| \\
& + C'\left(1 + r_\varepsilon\right) e^{\kappa_2 \rho} \int_{x_1}^{x_2} \left| a_k(x) - a(x) \right| dx + e^{\kappa_2 \rho} \int_{x_1}^{x_2} \left| \widetilde{u}^{\varepsilon}(x) - u(x) \right| dx \\
\leq \mathcal{O}(1) & e^{\kappa_2 \rho} \varepsilon + \left[C'\left(1 + r_\varepsilon\right) e^{\kappa_2 \rho} + \mathcal{O}(1) e^{\kappa_2 \rho} \right] \int_{x_1}^{x_2} \left| a_k(x) - a(x) \right| dx + L'(\rho, r_\varepsilon) \cdot |t_k - t|.
\end{aligned}
\tag{4.13}
$$

The constants r_ε and ρ do not depend on k, hence we can take the limit as $k \to +\infty$ in (4.13) to obtain

$$
\limsup_{k \to +\infty} \int_{x_1+L\bar{t}}^{x_2} \left| \left(\boldsymbol{S}_{t_k}^{a_k} u_k\right)(x) - \left(\boldsymbol{S}_t^a u\right)(x) \right| dx \leq \mathcal{O}(1) e^{\kappa_2 \rho} \varepsilon, \tag{4.14}
$$

which proves the theorem because of the arbitrariness of $\varepsilon > 0$ and $x_1 < x_2$. □

Remark 4.4 *In the conservative case $g_u = 0$, it can be proved (see [3]) that the semigroup can be continuously extended to $\mathbf{L}^\infty$ data also in the variable a. Since in this case the Lipschitz constant in (4.8) depends only on the total variation of u, one can obtain the same extension result changing a little bit the proof of the previous theorem.*

4.3 Characterization of the semigroup trajectories

In this section we want to characterize the trajectories of the semigroup. The following lemma shows that the non–conservative jump relation $z = const.$ continues to hold also for the exact trajectories.

Lemma 4.5 *Let $u(t,x) = \left(\boldsymbol{S}_t^a u_o\right)(x)$ be a semigroup trajectory with u_o having bounded total variation, then for every $\bar{x} \in \mathbb{R}$ it holds:*

$$
z\left[a,u\right](t,\bar{x}-) = z\left[a,u\right](t,\bar{x}+), \qquad \textit{for almost every } t \geq 0. \tag{4.15}
$$

Proof. Take a converging sequence of front tracking approximate solutions (a_k, u_k). Across standing waves $z[a_k, u_k]$ is constant. Moreover since the velocity of the homogeneous waves is bounded away from zero, for every $0 \leq t_1 < t_2$ the map $x \mapsto z[a_k, u_k](\cdot, x)$ is Lipschitz continuous in $\mathbf{L}^1(t_1, t_2)$. The Lipschitz constant $\widetilde{C}$ depends only on a upper bound for the initial total variation of $z[a_k, u_k]$ and on a lower bound for the velocities. Therefore for any $x' \geq 0$ we can write

$$
\int_{t_1}^{t_2} \left| z[a_k, u_k](t, \bar{x} + x') - z[a_k, u_k](t, \bar{x} - x') \right| dt \leq 2\widetilde{C} x'. \tag{4.16}
$$

Now we fix $\varepsilon > 0$, integrate (4.16) on the interval $[0, \varepsilon]$ and apply Fubini Theorem to obtain

$$
\int_{t_1}^{t_2} \int_0^{\varepsilon} \left| z[a_k, u_k](t, \bar{x} + x') - z[a_k, u_k](t, \bar{x} - x') \right| dx' dt \leq \widetilde{C} \varepsilon^2. \tag{4.17}
$$

The $\mathbf{L}^1{}_{loc}$ convergence of the sequence $z[a_k, u_k](t, \cdot)$ allows us to pass to the limit in (4.17) and obtain for the exact semigroup trajectory

$$\int_{t_1}^{t_2} \int_0^{\varepsilon} \left| z[a,u](t, \bar{x} + x') - z[a,u](t, \bar{x} - x') \right| dx' dt \leq \widetilde{C} \varepsilon^2. \tag{4.18}$$

Finally, since the function $x \mapsto z[a,u](t,x)$ has bounded total variation, it has right and left limit at the point $\bar{x}$. Therefore we can divide (4.18) by ε and take the limit as $\varepsilon \to 0$ to obtain

$$\int_{t_1}^{t_2} \left| z[a,u](t, \bar{x}+) - z[a,u](t, \bar{x}-) \right| dt \leq 0, \tag{4.19}$$

which proves the lemma. □

The following theorem shows that whenever a is sufficiently regular, the semigroup trajectories coincide with the entropic Kružkov's solutions to the scalar equation

$$u_t + f\left[a(x), u\right]_x = a'(x) g\left[a(x), u\right]. \tag{4.20}$$

Theorem 4.6 *If $a \in W^{1,1}(\mathbb{R})$ and $u_o \in \mathcal{D}^a$, then the semigroup trajectory $u(t,x) = \left(\boldsymbol{S}_t^a u_o\right)(x)$ satisfies the Kružkov's entropy inequality*

$$\begin{aligned} \int_{\mathbb{R}^+ \times \mathbb{R}} \Big\{ & \left| u(t,x) - \kappa \right| \varphi_t(t,x) \\ & + \operatorname{sgn}\left[u(t,x) - \kappa\right] \Big[f\big(a(x), u(t,x)\big) - f\big(a(x), \kappa\big) \Big] \varphi_x(t,x) \\ & + \operatorname{sgn}\left[u(t,x) - \kappa\right] \Big[g\big(a(x), u(t,x)\big) - f_a\big(a(x), \kappa\big) \Big] a'(x) \varphi(t,x) \Big\} dt dx \geq 0 \end{aligned} \tag{4.21}$$

for any nonnegative $\mathbf{C}^1$ function φ with compact support in $(0, +\infty) \times \mathbb{R}$ and any real constant κ (for definiteness we assume $\operatorname{sgn}(0) = 0$*).*

Recall that the validity of (4.21) for $|\kappa|$ sufficiently large implies that u is a distributional solution of (4.20).

Proof. First suppose that the initial data u_o has bounded total variation, hence fix $r > 0$ and suppose $u_o \in \mathcal{D}_r^a$. We adapt the proof of Theorem 6.1 in [2] to our case. Fix a constant κ and a nonnegative $\mathbf{C}^1$ function φ with compact support in $(0, +\infty) \times \mathbb{R}$. For notational purpose define

$$\bar{\eta}(v) = |v - \kappa|, \qquad \bar{q}(b,v) = \operatorname{sgn}(v - \kappa) \left[f(b,v) - f(b,\kappa) \right].$$

Let $E \in \mathbf{C}^2(\mathbb{R})$ be such that: $E'' \geq 0$, $E(v) = |v|$ for $|v| \geq 1$ and $E'(0) = 0$. Then the sequence $\eta^\varepsilon(v) = \varepsilon E\left(\frac{v - \kappa}{\varepsilon}\right)$ converges to $\bar{\eta}(v)$ in $\mathbf{C}^0(\mathbb{R})$, as $\varepsilon \to 0$.

If we fix

$$\eta = \eta^\varepsilon, \qquad q = q^\varepsilon(b,v) = \int_\kappa^v \eta'(\bar{v}) f_u(b, \bar{v})\, d\bar{v}, \tag{4.22}$$

then, for any constant b, $\big(\eta, q(b, \cdot)\big)$ is an entropy–entropy flux pair for the homogeneous scalar law $u_t + f(b,u)_x = 0$. An easy computation (see Remark 3.1 in [13] or in a more general contest

Proposition 3.2 in [2]) shows that if two states (a^-, u^-) and (a^+, u^+) are connected through a standing wave it holds

$$q(a^+, u^+) - q(a^-, u^-) = \int_{a^-}^{a^+} \left[q_a + \eta'(g - f_a)\right] \left(\bar{a}, \phi(\bar{a}; a^-, u^-)\right) d\bar{a}. \tag{4.23}$$

Let us fix a sequence of spatial mesh lengths $\Delta x_\nu > 0$ with $\Delta x_\nu \to 0$ as $\nu \to +\infty$ and define for all $j \in \mathbb{Z}$

$$x_j^\nu \doteq j\Delta x_\nu, \quad a_j^\nu \doteq a(x_j^\nu), \qquad a^\nu(x) \doteq a_j^\nu, \text{ for } x_j^\nu \le x < x_{j+1}^\nu, \qquad u^\nu(t, \cdot) \doteq \boldsymbol{S}_t^{a^\nu} u_o. \tag{4.24}$$

Observe that u^ν is the exact semigroup trajectory corresponding to the initial data (a^ν, u_o). Theorem 4.1 implies that $(a^\nu, u^\nu)(t, \cdot)$ converges to $(a, u)(t, \cdot)$ in $\mathbf{L}^1{}_{loc}$ as $\nu \to +\infty$. Consider now the quantity

$$I_{\varepsilon,\nu} = \int_{\mathbb{R}^+ \times \mathbb{R}} \left[\eta\left(u^\nu\right) \varphi_t + q\left(a^\nu, u^\nu\right) \varphi_x + \eta'(u) \left[g(a, u) - f_a(a, \kappa)\right] a' \varphi\right] dt dx. \tag{4.25}$$

Clearly, as $\nu \to +\infty$, the Lipschitz continuity of η and q implies

$$I_{\varepsilon,\nu} \to I_\varepsilon = \int_{\mathbb{R}^+ \times \mathbb{R}} \left[\eta\left(u\right) \varphi_t + q\left(a, u\right) \varphi_x + \eta'(u) \left[g(a, u) - f_a(a, \kappa)\right] a' \varphi\right] dt dx. \tag{4.26}$$

Since a^ν is constant over the strip $x_j^\nu < x < x_{j+1}^\nu$, the couple $(\eta, q(a^\nu, \cdot))$ is, in that set, an entropy–entropy flux pair for the scalar law $u_t + f(a^\nu, u)_x = 0$, moreover, by construction, u^ν is an exact entropy solution to that scalar law in the same strip. Therefore, observing that $a^\nu(x_j^\nu+) = a^\nu(x_{j+1}^\nu-) = a(x_j^\nu) = a_j^\nu$, integrating by parts we can compute

$$\begin{aligned}
&\int_{\mathbb{R}^+ \times \mathbb{R}} \left[\eta\left(u^\nu\right) \varphi_t + q\left(a^\nu, u^\nu\right) \varphi_x\right] dt dx \\
&\quad \ge \sum_{j \in \mathbb{Z}} \int_0^{+\infty} \left[q\left(a_j^\nu, u^\nu(t, x_{j+1}^\nu-)\right) \varphi(t, x_{j+1}^\nu) - q\left(a_j^\nu, u^\nu(t, x_j^\nu+)\right) \varphi(t, x_j^\nu)\right] dt \\
&\quad \ge -\sum_{j \in \mathbb{Z}} \int_0^{+\infty} \left[q\left(a_{j+1}^\nu, u^\nu(t, x_{j+1}^\nu+)\right) - q\left(a_j^\nu, u^\nu(t, x_{j+1}^\nu-)\right)\right] \varphi(t, x_{j+1}^\nu) dt \\
&\quad \doteq J_{\varepsilon,\nu}.
\end{aligned} \tag{4.27}$$

Lemma 4.5 implies that the two states

$$\left(a_j^\nu, u^\nu(t, x_{j+1}^\nu-)\right) \quad \text{and} \quad \left(a_{j+1}^\nu, u^\nu(t, x_{j+1}^\nu+)\right)$$

are connected by a standing wave for almost all $t \ge 0$. Hence applying (4.23) and the (absolutely continuous) change of variable $\bar{a} = a(x)$ (see [23]), we obtain, for almost all $t \ge 0$

$$\begin{aligned}
&q\left(a_{j+1}^\nu, u^\nu(t, x_{j+1}^\nu+)\right) - q\left(a_j^\nu, u^\nu(t, x_{j+1}^\nu-)\right) \\
&\quad = \int_{x_j^\nu}^{x_{j+1}^\nu} \left[q_a + \eta'(g - f_a)\right] \left(a(x), \phi\left(a(x); a_j^\nu, u^\nu(t, x_{j+1}^\nu-)\right)\right) \cdot a'(x)\, dx.
\end{aligned} \tag{4.28}$$

Therefore it holds

$$
\begin{aligned}
J_{\varepsilon,\nu} &= -\sum_{j\in\mathbb{Z}} \int_0^\infty \int_{x_j^\nu}^{x_{j+1}^\nu} \Big[q_a + \eta'(g - f_a) \Big] \Big(a(x), \phi\left(a(x); a_j^\nu, u^\nu(t, x_{j+1}^\nu -) \right) \Big) a'(x) \varphi(t, x_{j+1}^\nu) dx dt \\
&= -\int_{\mathbb{R}^+\times\mathbb{R}} \Big[q_a + \eta'(g - f_a) \Big] \Big(a(x), \widehat{u}^\nu(t,x) \Big) \cdot a'(x) \widehat{\varphi}^\nu(t,x) dt dx,
\end{aligned}
$$

where we have defined

$$
\widehat{u}^\nu(t,x) = \phi\left(a(x); a_j^\nu, u^\nu(t, x_{j+1}^\nu -) \right), \quad \widehat{\varphi}^\nu(t,x) = \varphi(t, x_{j+1}), \quad \text{for } (t,x) \in \mathbb{R}^+ \times [x_j^\nu, x_{j+1}^\nu).
$$

The Lipschitz continuity of ϕ implies for every $x \in [x_j^\nu, x_{j+1}^\nu)$:

$$
\begin{aligned}
\left| \widehat{u}^\nu(t,x) - u^\nu(t,x) \right| &\le \mathcal{O}(1) \left[\left| a(x) - a_j^\nu \right| + \left| u^\nu(t, x_{j+1}^\nu -) - u^\nu(t,x) \right| \right] \\
&\le \mathcal{O}(1) \left[\text{Tot.Var.} \left\{ a, (x_{j-1}^\nu, x_{j+1}^\nu) \right\} + \text{Tot.Var.} \left\{ u^\nu(t), (x_{j-1}^\nu, x_{j+1}^\nu) \right\} \right].
\end{aligned} \tag{4.29}
$$

Hence we compute

$$
\begin{aligned}
\int_{\mathbb{R}} \left| \widehat{u}^\nu(t,x) - u^\nu(t,x) \right| dx &= \sum_{j\in\mathbb{Z}} \int_{x_j^\nu}^{x_{j+1}^\nu} \left| \widehat{u}^\nu(t,x) - u^\nu(t,x) \right| dx \\
&\le \mathcal{O}(1) \Delta x_\nu \sum_{j\in\mathbb{Z}} \Big[\text{Tot.Var.} \left\{ a, (x_{j-1}^\nu, x_{j+1}^\nu) \right\} \\
&\qquad + \text{Tot.Var.} \left\{ u^\nu(t,\cdot), (x_{j-1}^\nu, x_{j+1}^\nu) \right\} \Big] \\
&\le \mathcal{O}(1) \Delta x_\nu \left[\text{Tot.Var.} a + \text{Tot.Var.} u^\nu(t,\cdot) \right]
\end{aligned} \tag{4.30}
$$

which, together the uniform bound on the total variation of u^ν and the $\mathbf{L}^1{}_{loc}$ convergence of u^ν to u, shows that $\widehat{u}^\nu(t,\cdot)$ converges to $u(t,\cdot)$ in $\mathbf{L}^1{}_{loc}$. Therefore Proposition B.1 in [2] and the uniform convergence $\widehat{\varphi} \to \varphi$ allows us to obtain

$$
\begin{aligned}
I_\varepsilon = \lim_{\nu\to+\infty} I_{\varepsilon,\nu} &\ge \lim_{\nu\to+\infty} J_{\varepsilon,\nu} + \int_{\mathbb{R}\times\mathbb{R}^+} \eta'(u) \left[g(a,u) - f_a(a,\kappa) \right] a' \varphi dx dt \\
&\ge \int_{\mathbb{R}\times\mathbb{R}^+} \Big\{ \eta'(u) \left[f_a(a,u) - f_a(a,\kappa) \right] - q_a(a,u) \Big\} a' \varphi \, dx dt.
\end{aligned} \tag{4.31}
$$

We have derived inequality (4.31) for $u(t,x) = \left(\boldsymbol{S}_t^a u_o \right)(x)$ when u_o has bounded total variation. However, since η, f_a, q_a and g are Lipschitz continuous functions, inequality (4.31) can be extended by density in $\mathbf{L}^1{}_{loc}$ to a generic semigroup trajectory $u(t,x) = \left(\boldsymbol{S}_t^a u_o \right)(x)$ with $u_o \in \mathcal{D}^a$. Now observe that the inequality

$$
\begin{aligned}
\left| \eta'(u) \left[f_a(a,u) - f_a(a,\kappa) \right] - q_a(a,u) \right| &= \left| \int_\kappa^u \left[\eta'(u) - \eta'(\xi) \right] f_{au}(a,\xi) \, d\xi \right| \\
&\le 2\varepsilon ||E||_{\mathbf{C}^2} ||f||_{\mathbf{C}^2}
\end{aligned}
$$

holds because $\eta'(u) = \eta'(\xi)$ for $u, \xi \notin [\kappa - \varepsilon, \kappa + \varepsilon]$. Therefore we have

$$I_\varepsilon \geq -2\varepsilon ||E||_{\mathbf{C}^2} ||f||_{\mathbf{C}^2} \int_{\mathbb{R}\times\mathbb{R}^+} |a'|\varphi \, dxdt,$$

which implies

$$\liminf_{\varepsilon \to 0} I_\varepsilon \geq 0.$$

Finally, the integrand in (4.26) converges pointwise as $\varepsilon \to 0$, hence by the dominated convergence theorem we can pass to the limit and complete the proof. □

Remark 4.7 *In [2] it is proved that when the the source satisfy also the supplementary condition*

$$\sup a' g_u < +\infty$$

then the entropic Kružkov's solution is unique, therefore in this case the trajectory of the semigroup coincide with the solution obtained in [2] with a Godunov–type algorithm.

Remark 4.8 *Given a right continuous initial data $a \in \mathcal{B}$, it is possible to choose an approximating sequence a_k such that for any $x \in \mathbb{R}$ one has*

$$\text{Tot.Var.}\left\{a_k, (x, +\infty)\right\} \to \text{Tot.Var.}\left\{a, (x, +\infty)\right\}.$$

Therefore, with such a choice of the converging subsequence, the corresponding weight defined in (3.8) $W_2(x) = \exp\left[\kappa_2 \text{Tot.Var.}\left\{a_k, (x, +\infty)\right\}\right]$ converges pointwise to the weight

$$W(x) = \exp\left[\kappa_2 \text{Tot.Var.}\left\{a, (x, +\infty)\right\}\right]. \tag{4.32}$$

If the initial data are chosen such that $p(x) = 0$, then from definitions (3.1), (3.5) the functional $\mathbf{\Phi}_1$, computed on two front tracking approximations (a_k, u_k) and (a_k, v_k) becomes

$$\mathbf{\Phi}_1\left(U_1(t), U_2(t)\right) = \int_{x_1+Lt}^{x_2} \exp\left[\kappa_2 \text{Tot.Var.}\left\{a_k, (x, +\infty)\right\}\right] |u_k(t,x) - v_k(t,x)| dx.$$

Therefore we have for the exact semigroup trajectories

$$\int_{x_1+Lt}^{x_2} W(x) \left| \left(\boldsymbol{S}_t^a u\right)(x) - \left(\boldsymbol{S}_t^a v\right)(x) \right| dx \leq \int_{x_1}^{x_2} W(x) |u(x) - v(x)| dx,$$

where $W(x)$ is defined in (4.32). In this way we have obtained a generalization, to an arbitrary function a with bounded total variation, of the decreasing functional introduced in Theorem 2.1 of [26]. Observe that, in the particular framework of [26], our jump conditions (see Lemma 4.5) coincide with the Layer and Gap problem described there.

When a is discontinuous, the semigroup trajectories can not be characterized by (4.21) since not all integrals are well defined. However as observed in the introduction (Remark 1.2), the continuity of the semigroup with respect to the variable a allows us to characterize the trajectories as unique limits of usual Kružkov's solutions. More explicitly, fix $a \in \mathcal{B}$ and take a smooth approximation a^ε in $\mathbf{L}^1{}_{loc}$. Then $\left(\boldsymbol{S}_t^{a^\varepsilon} u_o\right)(x)$ coincide with the unique entropic solution $u^\varepsilon(t,x)$ to

$$\begin{cases} u_t^\varepsilon + f(a^\varepsilon, u^\varepsilon)_x = (a^\varepsilon)' g(a^\varepsilon, u^\varepsilon) \\ u^\varepsilon(0,x) = u_o(x). \end{cases}$$

Therefore, the continuity of the semigroup implies that $u^\varepsilon(t,\cdot)$ converges to $\boldsymbol{S}_t^a u_o$ as $\varepsilon \to 0$, and this limit does not depend on the particular approximating sequence a^ε chosen.

If the functions $f(a,u)$ and $g(a,u)$ do not depend explicitly on a then the semigroup trajectory does not change if one adds a constant to the initial data a, i.e. we have

$$\boldsymbol{S}_t^{a+c} u_o = \boldsymbol{S}_t^a u_o. \tag{4.33}$$

Therefore we can prove the following corollary which generalizes the convergence results of [12, 26].

Corollary 4.9 *Consider the following family of scalar balance laws*

$$\begin{cases} u_t^\varepsilon + f(u^\varepsilon)_x = k^\varepsilon(x) g(u^\varepsilon) \\ u^\varepsilon(0,x) = u_o^\varepsilon(x), \quad u_o^\varepsilon(x) \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R}), \end{cases} \tag{4.34}$$

with f and g being $\mathbf{C}^2$ functions and satisfying, for some positive constants A, B

$$f'(u) > 0, \qquad \left|\frac{g(u)}{f'(u)}\right| \le A + B|u|. \tag{4.35}$$

Suppose that k^ε, $\varepsilon > 0$ is a family of smooth integrable functions, uniformly bounded in $\mathbf{L}^1(\mathbb{R})$ and converging weakly to some bounded measure μ as $\varepsilon \to 0$. Suppose also that, as $\varepsilon \to 0$, $u_o^\varepsilon(x)$ is uniformly bounded in $\mathbf{L}^\infty$ and converges to some function u_o in the $\mathbf{L}^1{}_{loc}$ topology. Then the Kružkov's entropic solution u^ε to (4.34) converges in the $\mathbf{L}^1{}_{loc}$ topology to a unique function $u \in \mathbf{L}^\infty$ that depends only on the limit measure μ and not on the particular converging family k^ε. We can consider this unique limit function as a generalized solution to the following scalar balance law with singular non–conservative source term:

$$\begin{cases} u_t + f(u)_x = \mu g(u) \\ u(0,x) = u_o(x), \quad u_o(x) \in \mathbf{L}^\infty(\mathbb{R},\mathbb{R}). \end{cases} \tag{4.36}$$

In particular if $\bar{a}$ is any function with bounded total variation such that $\bar{a}' = \mu$, then the generalized solution to (4.36) is given by the semigroup trajectory $\boldsymbol{S}_t^{\bar{a}} u_o$.

Proof. Define

$$a^\varepsilon(x) = \int_{-\infty}^x k^\varepsilon(\bar{x}) d\bar{x}.$$

The hypotheses on k^ε imply that there exist two constants $a_1 < a_2$ independent of ε such that $a_1 \le a^\varepsilon(x) \le a_2$. The first condition in (4.35) guarantees the strict hyperbolicity, the second one implies that the family of the initial data $(a^\varepsilon, u_o^\varepsilon)$ stays on an invariant domain K independent of ε (see definition (2.5)). By Theorem 4.6 the entropic solution to (4.34) coincides with the semigroup trajectory with initial data a^ε and u_o^ε: $u^\varepsilon(t,x) = \left(\boldsymbol{S}_t^{a^\varepsilon} u_o^\varepsilon\right)(x)$. Since k^ε is uniformly bounded in $\mathbf{L}^1$, also the total variation of a^ε is uniformly bounded. Therefore we can extract a converging subsequence a^{ε_ν} in $\mathbf{L}^1{}_{loc}$. Denote by $\bar{a}$ its limit which has bounded total variation. Theorem 4.3 implies the convergence of $u^{\varepsilon_\nu}(t,\cdot) = \boldsymbol{S}_t^{a^{\varepsilon_\nu}} u_o^{\varepsilon_\nu}$ to the function $\bar{u}(t,\cdot) = \boldsymbol{S}_t^{\bar{a}} u_o$ in $\mathbf{L}^1{}_{loc}$. Take now another converging subsequence $a^{\tilde{\varepsilon}_\nu}$ and denote by $\tilde{a}$ its limit. As before $u^{\tilde{\varepsilon}_\nu}(t,\cdot) = \boldsymbol{S}_t^{a^{\tilde{\varepsilon}_\nu}} u_o^{\tilde{\varepsilon}_\nu}$ converges in $\mathbf{L}^1{}_{loc}$ to the function $\tilde{u}(t,\cdot) = \boldsymbol{S}_t^{\tilde{a}} u_o$. Since the strong convergence in $\mathbf{L}^1{}_{loc}$ implies the weak convergence of the derivatives, from $(a^\varepsilon)' = k^\varepsilon$, $k^\varepsilon \rightharpoonup \mu$ and the uniqueness of the weak limit we get

$$\bar{a}' = \tilde{a}' = \mu. \tag{4.37}$$

Therefore $\bar{a}$ and $\tilde{a}$ differ by an additive constant c, $\tilde{a} = \bar{a} + c$. Finally, by (4.33), we have

$$\tilde{u}(t,\cdot) = \boldsymbol{S}_t^{\tilde{a}} u_o = \boldsymbol{S}_t^{\bar{a}+c} u_o = \boldsymbol{S}_t^{\bar{a}} u_o = \bar{u}(t,\cdot).$$

which concludes the proof. □

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