A Mathematical Model for Hurricanes

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Abstract

The source waves are some particular solutions to genuinely non linear hyperbolic systems with a source term, whose propagation velocity is a constant determined by the roots of this source term. We propose a system of 2 equations on a 2 dimension space, which modelizes the velocity field of the atmosphere near the ground. The source term is made of three parts: a given pressure gradient, a friction or suction effect and the Coriolis force. In the case where these parameters are constant, we build a solution which is a constant outside a circular crown. The internal circle represents the eye wall of the hurricane and corresponds to a share shock wave. The external circle is a set generating the bifurcation which actually modelizes the hurricane.

1 Introduction

A source wave is a wave whose velocity is completely determined by a root of the source term of a non linear hyperbolic system. The source term must be understood in a general sense, that is including the loss. For example for any $R > 0$, $u_* \in \mathbb{R}$ and $k \geq 0$, the function $u(x,t) = u_* + k \dfrac{\partial}{\partial x} \chi_R\left(|x - u_* t|\right)$, with $\chi_R(\xi) = \frac{1}{2}\left(\xi - R\right)^2$, is a solution to the non homoge$u_t + uu_x + k\left(u - u_*\right) = 0$ and also a solution to the linear homogeneous equation $u_t + u_* u_x = 0$. One can also build a solution shaped as a wave train, by concatening several such solutions. One notices that this solution contains a discontinuity which is compatible with the Rankine Hugoniot shock condition and with the usual entropy condition. Such source waves are already present in many applications, most of them in hydraulics, for the representation of the Roll Waves on a slope, the tidal bore waves in an estuary or the surf waves coming on the beach. See [1] , [2] , [3] . We propose here a new application of this notion in a two dimension space, concerning the hurricanes or cyclones. More physical descriptions of the phenomenon are to be found on the net, for example [7], [8], [9]. For source waves, see [5], [6] .

2 A hurricane model

We denote by u and v the two horizontal componants of the velocity, $\omega = \frac{\pi \; sin\lambda}{21600}$ is the Coriolis coefficient, where λ is the latitude of the place, $\lambda > 0$ in the northern hemisphere and $\lambda < 0$ in the southern hemisphere, $\mu \geq 0$ is a friction coefficient and $k \geq 0$ is a suction coefficient. This suction coefficient has no effect outside the hurricane and simulates the suction due to the vertical componant of the velocity which is not taken explicitely in account here. The gradient of pressure $(p_x,\ p_y)$ is supposed to be a datum here, given from the meteorological previsions or measures. The model is the following one

$$\begin{aligned} u_t \ + \ u\, u_x \ + \ v\, u_y \ + \ p_x \ + \ \mu\, u \ - \ \omega\, v \ &= \ k\ (u - u_*) \\ v_t \ + \ u\, v_x \ + \ v\, v_y \ + \ p_y \ + \ \mu\, v \ + \ \omega\, u \ &= \ k\ (v - v_*) \end{aligned} \tag{2.1}$$

where

$$u_* \ = \ -\frac{\mu\, p_x \ + \ \omega\, p_y}{\mu^2 \ + \ \omega^2} \ , \ v_* \ = \ \frac{\omega\, p_x \ - \ \mu\, p_y}{\mu^2 \ + \ \omega^2} \ . \tag{2.2}$$

The wind of velocity field (u_*, v_*) is known as the "trade wind".
By writing $p_x = -\mu u_* + \omega v_*$ and $p_y = -\omega u_* - \mu v_*$, the system becomes

$$\begin{aligned} u_t \ + \ u\, u_x \ + \ v\, u_y \ &= \ (k - \mu)\,(u - u_*) \ + \ \omega\,(v - v_*) \\ v_t \ + \ u\, v_x \ + \ v\, v_y \ &= \ (k - \mu)\,(v - v_*) \ - \ \omega\,(u - u_*) \end{aligned} \tag{2.3}$$

where the source term is zero when $u = u_*$ and $v = v_*$. By the way, a source wave is also a solution to the associated linear system:

$$u_t + u_* u_x + v_* u_y \ = \ 0 \ \ , \ \ v_t + u_* v_x + v_* v_y \ = \ 0. \tag{2.4}$$

We consider now the shock waves. Since the system has a non conservative form, a Rankine Hugoniot condition may be obtained by using the generalized functions theory (see [4] for details). A shock wave for the system (2.1) corresponds to a discontinuity curve of the form $S(x - \frac{u_1 + u_2}{2} t \ , \ y - \frac{v_1 + v_2}{2} t) \ = \ 0$ where (u_1, v_1) and (u_2, v_2) are the values on both sides of the discontinuity. In the particular case where the support of the discontinuity is a circle of radius R and center $(x_0 + u_* t,\ y_0 + v_* t)$ for a given point $(x_0,\ y_0)$ and taking $u_1 = u_*,\ v_1 = v_*$ on one side and $u_2 = \gamma u,\ v_2 = \gamma v$ on the other side, as the trace of the solution, that is $\left(x - \frac{u_* + \gamma u}{2} t - x_0\right)^2 + \left(y - \frac{v_* + \gamma v}{2} t - y_0\right)^2 - R^2 = 0$, the Rankine Hugoniot condition reads

$$(\gamma u - u_*)\ (x - u_* t - x_0) \ + \ (\gamma v - v_*)\ (y - v_* t - y_0) \ = \ 0. \tag{2.5}$$

This imposes the discontinuity jump $([u], [v]) = (\gamma u - u_*, \gamma v - v_*)$ to be orthogonal to the radial direction. The shock velocity is still equal to (u_*, v_*) which means that the shock wave is a contact discontinuity. We shall see in the following that the edge of the hurricane may be such a discontinuity, with (u_*, v_*) as the external velocity value. We have also, and that is the most important, the eye wall described by such a shock wave, with (u_*, v_*) as the velocity inside the eye, and a strong value $(\gamma u, \gamma v)$ on the other side of the eye wall. The way the velocity behaves between the external edge and the eye wall corresponds to a bifurcation solution of (2.1).

3 The caracteristics solution

Using the time t as a parameter, the characteristics system associated with (2.1) reads

$$\begin{array}{rclrcl} x'(t) &=& u(t)\ , & u'(t) &=& (k-\mu)\,(u(t)-u_*)\ +\ \omega\,(v(t)-v_*) \\ y'(t) &=& v(t)\ , & v'(t) &=& (k-\mu)\,(v(t)-v_*)\ -\ \omega\,(u(t)-u_*)\ . \end{array} \tag{3.6}$$

We consider as a reference trajectory the one which starts from $x = R,\ y = 0$ where R is the radius of the hurricane. We notice that the integration of the characteristics system leads to an explicit expression of the solution
$u = (k-\mu)\,(x-u_*t) + \omega\,(y-v_*t) + u_*,\ v = -\omega\,(x-u_*t) + (k-\mu)\,(y-v_*t) + v_*$.
We use the Lagrangian variables $X = x - u_*t,\ Y = y - v_*t,\ U = u - u_*,\ V = v - v_*$ and we look for a solution of the form

$$\begin{pmatrix} U \\ V \end{pmatrix} = a(\xi)\begin{pmatrix} X \\ Y \end{pmatrix} + b(\xi)\begin{pmatrix} -Y \\ X \end{pmatrix} \tag{3.7}$$

where $\xi = \sqrt{X^2+Y^2}$. This way $a(\xi)\xi$ stands for the radial componant of the velocity and $b(\xi)\xi$ for the transverse componant. The Rankine Hugoniot condition reduces to impose the quantity $a(\xi)\xi^2$ to be continuous, which means that the only possible discontinuity for $a(\xi)$ will stay at $\xi = 0$ and that any discontinuity of the transverse componant $b(\xi)\xi$ is allowed.
After introducing (3.7) into (2.1) and some computations, we get the following differential system as a compatibility condition

$$a\,a'\,\xi\ +\ a^2\ -\ b^2\ -\ (k-\mu)\,a\ +\ \omega b\ =\ 0,\ a\,b'\,\xi\ +\ 2\,a\,b\ -\ (k-\mu)\,b\ -\ \omega a\ . \tag{3.8}$$

We look for the non zero solutions of this system, which are the bifurcation solutions. In the very particular case where $\mu = 0,\ \omega = 0$, a bifurcation solution reads $a(\xi) = k\left(1-\frac{R}{\xi}\right), b(\xi) = 0$. This an immediate generalization of the solution of one dimension example given in the introduction, and it can be generalized easily to the three dimension case. This can be seen as a kind of "black hole".

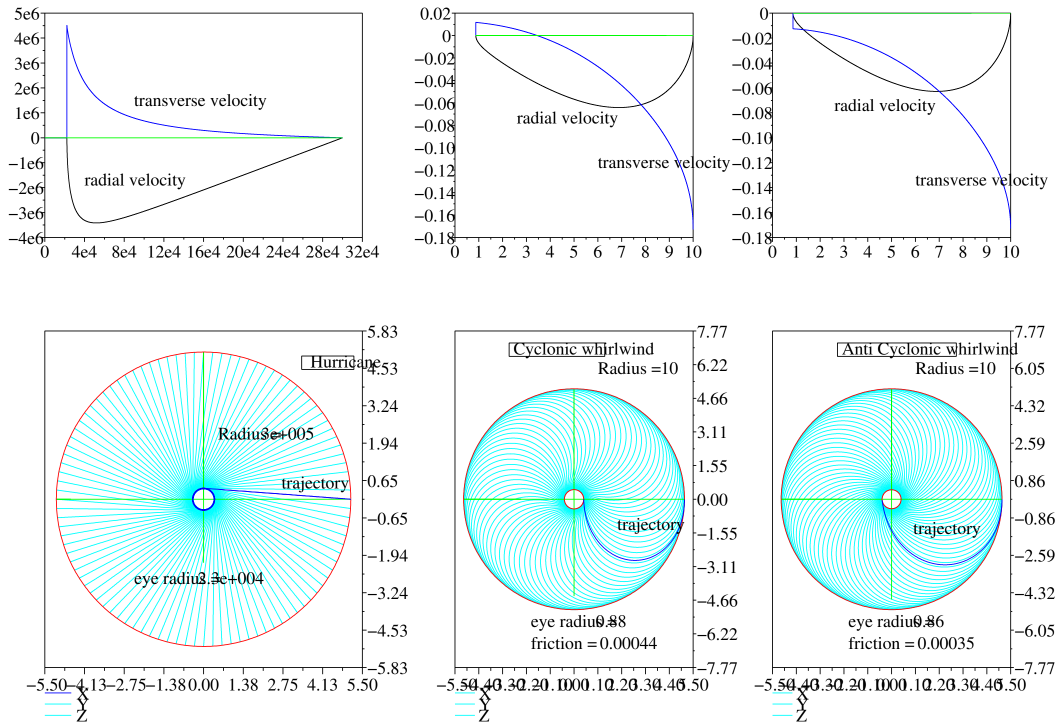
In the case of a hurricane, bifurcation solutions are obtained by writing $z(\xi) = -\frac{b(\xi)}{a(\xi)}$ which leads to the system

$$\frac{d}{d\xi}\left(a(\xi)\ \xi\right)\ =\ z(\xi)^2 a(\xi)\ +\ (k-\mu)\ +\ \omega z(\xi)\ ,\ \frac{d}{d\xi}\left(b(\xi)\ \xi\right)\ =\ -b(\xi)\ +\ \omega\ -(k-\mu)\,z(\xi) \tag{3.9}$$

that we solve by integrating by starting from $\xi = R$ with $a(R) = 0,\ b(R) = 0$ and $z(R) = z_0$, an incidence value to be given. It is interesting to notice that the sign of $a'(\xi)$ will change and that $a(\xi)$ will be zero for some value $\xi_0 < R$. When $\xi_0 > 0$ it corresponds to the eyewall. At this point, since $b >> \omega$ in practice, we must have from (3.8) an infinite slope, that is $\lim\limits_{\xi\longrightarrow\xi_0} a'(\xi) = \infty$ and $\lim\limits_{\xi\longrightarrow\xi_0} a(\xi)a'(\xi)$ bounded. With the same argument, $\lim\limits_{\xi\longrightarrow\xi_0} a(\xi)b'(\xi)$ is also bounded and $\lim\limits_{\xi\longrightarrow\xi_0} b'(\xi) = \infty$.

4 Numerical Tests

The computation of the caracteristics trajectories may be performed either by using the characteristics equations (3.6) or by using the non linear system (3.8) or (3.9). The first case is more adapted to the case when a shock is present on the edge of the hurricane and the second one is to be used when there is no external shock, in order to have a bifurcation solution.



We present three results. The first one in the first column corresponds to a hurricane with a small incidence entry and no external shock. We have taken $R = 300km$, and $z_0 = 0.075$, $\mu = 0$, $k = 15$, $\omega = 0.0001$. The radial and transverse velocity componants are given above the picture of the hurricane with a selection of characteristics trajectories.

We see that an eye wall is found and that these trajectories come to the eyewall and turn around in the counter clockwise sense since omega is positive (northern hemisphere). The computations are done by solving (3.9) with the Euler method, which explains a large value of the velocity near the eyewall. A more accurate method will reduce this, and this will be also reduced by the natural viscosity which is not taken in account here. On the other hand this experiment proves that even with an order one scheme the eye is captured.

The two next columns corresponds two smallest whirlwinds, like tornadoes or waterspout, starting with an external shock on the edge. Here $R = 10m$, $k = 0$, $\omega = 0.002$. We have tested two values of the friction, and found that for $\mu = 0.00044$ the flow around the eye is a counter clockwise one and for $\mu = 0.00035$ it is a clock wise one. This underline that the friction term plays some role in the way the flow reaches the eye. There is a value between these two ones for which the eye collapses. Here an explicit calculus of the differential system (3.6) has been performed.

5 Conclusion

We have proposed a rather simple model since it is made of two equations only in a two dimension space. And yet, it is able to compute the hurricane tracks using (2.2) to be updated at any time from meteorogical and local data , and to give an idea of the shape and behaviour of the hurricane, mainly when an eye appears. However, we have introduced some parameters whose physical meaning needs to be settled, this is the case of the so called aspiration term which may also be a nonlinear term which vanishes when$u = u_*, v = v_*$. The model is also very sensitive to the values of the parameters, as seen in the picture
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