ON THE APPROXIMATION OF THE NON-AUTONOMOUS NON-LINEAR RIEMANN PROBLEM *

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Abstract

In this paper we study conservation laws with spatially-varying flux functions. We give a survey of some schemes (based on finite volume methods) to solve non-autonomous conservation laws of the form $\mathbf{q}_t + [\mathbf{g}(\mathbf{q}, x)]_x = \mathbf{0}$. Numerical experiments are presented.

Keywords: finite volume methods, upwind schemes, central schemes, fluid flow through the male urethra

AMS subject classification: 65M06

1. Basic facts

In this section we give a short survey of basic facts for hyperbolic conservation laws.

1.1. Scalar conservation laws

For the non-linear conservation law in the standard (autonomous) form

$$\begin{aligned} u_t + [f(u)]_x &= 0,\ x \in \mathbb{R},\ t \in (0, T) \\ u(x, 0) &= u_0(x), x \in \mathbb{R}, \end{aligned} \tag{1}$$

where $f \in C^2(\mathbb{R})$ and $u_0 \in BV(\mathbb{R})$, we usually define weak, vanishing viscosity and entropy solutions.

*This work was supported by grant MSM 235200001 (CEZ: J23/98) of the Ministry of Education of the Czech Republic (S. Míka) and GA106/04/0201 of the Grant Agency of the Czech Republic (M. Brandner)

Definition. *Let $u_0(x) \in \mathcal{L}_{1,loc}(R)$ and the function $f = f(u)$ is Lipschitz continuous. Any function $u(x,t) \in \mathcal{L}^{\infty}_{loc}(\mathbb{R} \times (0,\infty))$ that fulfils*

$$\int_0^{\infty} \int_{-\infty}^{\infty} [\phi_t u + \phi_x f(u)] \; \mathrm{d}x \; \mathrm{d}t = - \int_{-\infty}^{\infty} \phi(x,0) u_0(x) \; \mathrm{d}x \tag{2}$$

for all $\phi \in \mathcal{C}^1_0(\mathbb{R} \times \mathbb{R})$ is called the weak solution of (1). $\mathcal{C}^1_0(\mathbb{R} \times \mathbb{R})$ is the set of continuously differentiable functions $\phi = \phi(x,t)$ where

$$supp \; \phi = \overline{\left\{(x,t) \in \mathbb{R}^2 : \; \phi(x,t) \neq 0\right\}}$$

is bounded.

Example.

$$u_t + (u^3)_x = 0, \quad u(x,0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

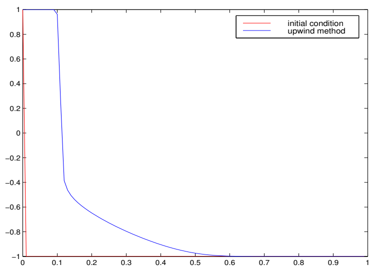

Figure 1: $u_L = 1$, $u_R = -1$ $T = 0.15$

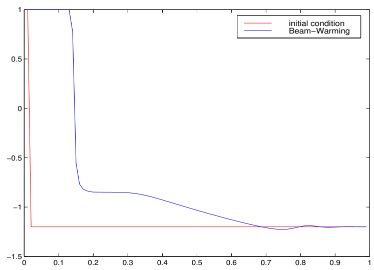

Figure 2: $u_L = 1$, $u_R = -1.2$ $T = 0.15$

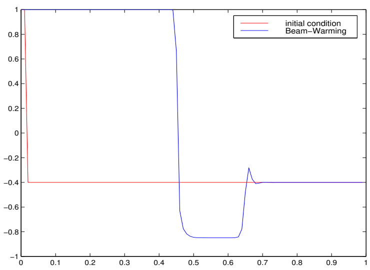

Figure 3: $u_L = 1$, $u_R = -0.4$ $T = 0.5$

We can obtain three types of weak solutions for the previous conservation law. The first one is the compound wave — the shock wave with the attached rarefaction[1], the second one is the slow non-classical shock followed by the rarefaction and third one is the slow non-classical shock followed by the fast classical shock[2].

[1]The solution is computed by the first-order upwind method ($h = 0.01$, $\tau = h/100$).
[2]These two solutions are computed by the second-order Beam-Warming scheme ($h = 0.01$, $\tau = h/100$).

Definition. *Let u^ε is the solution of the initial value problem*

$$\begin{aligned} u_t + [f(u)]_x &= \varepsilon u_{xx}, \quad x \in \mathbb{R},\ 0 < t < T,\ \epsilon > 0 \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{3}$$

If the limit

$$u^* = \lim_{\varepsilon \to 0} u^{(\varepsilon)}, \tag{4}$$

exists, it is called the vanishing viscosity solution of (1).

The solution in Fig. 1 is the vanishing viscosity solution, the other two are also limiting solutions. However, the equation with the artificial viscosity (or diffusion) (3) is replaced by a viscosity-dispersion equation (see [HL97]). To distinguish which weak solution is a vanishing-viscosity solution we introduce entropy conditions (see [HL97]).

1.2. Systems of conservation laws

We consider the system of conservation laws

$$\begin{aligned} \mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x &= \mathbf{0},\ x \in \mathbb{R},\ t \in (0,T) \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), x \in \mathbb{R}, \end{aligned} \tag{5}$$

where $\mathbf{u}_0 = \mathbf{u}_0(x) : \mathbb{R} \to \mathbb{R}^m$, $\mathbf{f} = \mathbf{f}(\mathbf{u}) : \mathbb{R}^m \to \mathbb{R}^m$ is sufficiently smooth. The weak and vanishing-viscosity solutions are defined by the same way as in the scalar case. This system can be rewritten to the form

$$\begin{aligned} \mathbf{u}_t + \mathbf{A}(\mathbf{u})\mathbf{u}_x &= \mathbf{0},\ x \in \mathbb{R},\ t \in (0,T) \\ \mathbf{u}(x,0) &= \mathbf{u}_0(x), x \in \mathbb{R}, \end{aligned} \tag{6}$$

where $\mathbf{A}(\mathbf{u}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}$ is the Jacobi matrix of the function $\mathbf{f} = \mathbf{f}(\mathbf{u})$. This system is said to be hyperbolic if $\mathbf{A}$ is diagonizable with real eigenvalues $\lambda_i(\mathbf{u})$, i. e., a regular matrix $\mathbf{R} = \mathbf{R}(\mathbf{u})$ exists for all $\mathbf{u}$ that

$$\mathbf{\Lambda}(\mathbf{u}) = [\mathbf{R}(\mathbf{u})]^{-1}\mathbf{A}(\mathbf{u})\mathbf{R}(\mathbf{u}),$$

where $\mathbf{\Lambda}(\mathbf{u})$ is a diagonal matrix. If the eigenvalues $\lambda_i(\mathbf{u})$ are distinct then (5) is called the strictly hyperbolic system. We distinguish two cases:

- $\nabla\lambda_i(\mathbf{u}) \cdot \mathbf{r}_i(\mathbf{u}) \equiv 0\ \forall \mathbf{u}$ (linear degeneracy, LD),
- $\nabla\lambda_i(\mathbf{u}) \cdot \mathbf{r}_i(\mathbf{u}) \neq 0\ \forall \mathbf{u}$ (genuine non-linearity, GNL).

The linearly degenerate case generates a contact discontinuity, the genuinely non-linear case generates either a shock or a rarefaction wave (for suitable initial conditions). The system is said to be weakly hyperbolic if $\mathbf{A}$ has real eigenvalues $\lambda_i(\mathbf{u})$ and it is not diagonizable.

1.3. Discretization

We discretize the space-time domain by choosing a mesh sizes h and a time step τ and define the discrete mesh points (x_j, t_n) by

$$x_j = jh,\ j = 0, \pm 1, \dots,\ t_n = n\tau,\ n = 0, 1, \dots, N.$$

For simplicity purposes we consider only the domain $\Omega = \mathbb{R} \times (0, T)$ in what follows. In this case we have $N = T/\tau$. Futhermore, we consider a uniform mesh, with h and τ constant, although the discussed methods can be extended to nonuniform meshes. The values of the true solution at the meshpoints are denoted by $\mathbf{u}_j^n = \mathbf{u}(x_j, t_n)$. Our numerical methods will produce approximations to $\mathbf{u}_j^n$ denoted by $\mathbf{U}_j^n$. Futhermore, we define $x_{j+1/2} = x_j + \frac{h}{2} = \left(j + \frac{1}{2}\right) h$ and put $\mathbf{U}_{j+1/2}^n \approx \mathbf{u}(x_{j+1/2}, t_n)$.

2. Autonomous case

We consider the difference schemes approximating the conservation law (5) in the conservative form

$$\mathbf{U}_j^{n+1} = \mathbf{U}_j^n - \frac{\tau}{h}\left(\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n\right). \tag{7}$$

The function $\mathbf{F}_{j+1/2}^n$ is called the numerical flux. We define the set of the Riemann problems for the linearized conservation laws

$$\begin{aligned} \mathbf{u}_t + \bar{\mathbf{A}}_{j+1/2}^n \mathbf{u}_x &= \mathbf{0}, \quad x \in \mathbb{R},\ t \in (t_n, t_{n+1}), \\ \mathbf{u}(x, t_n) &= \begin{cases} \mathbf{U}_j^n, & x < x_{j+1/2}, \\ \mathbf{U}_{j+1}^n, & x > x_{j+1/2}. \end{cases} \end{aligned} \tag{8}$$

where $\bar{\mathbf{A}}_{j+1/2}^n = \bar{\mathbf{A}}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n)$. For the autonomous case $\mathbf{f} = \mathbf{f}(\mathbf{u})$ the matrix $\bar{\mathbf{A}}(\widetilde{\mathbf{U}}, \widehat{\mathbf{U}})$ has to fulfil three conditions

- the system (8) is strictly hyperbolic,
- the matrix $\bar{\mathbf{A}}(\widetilde{\mathbf{U}}, \widehat{\mathbf{U}})$ is the linearization of the $\mathbf{f}$, i. e.,

$$\mathbf{f}(\widetilde{\mathbf{U}}) - \mathbf{f}(\widehat{\mathbf{U}}) = \bar{\mathbf{A}}(\widetilde{\mathbf{U}}, \widehat{\mathbf{U}})(\widetilde{\mathbf{U}} - \widehat{\mathbf{U}}) \quad \forall \widetilde{\mathbf{U}}, \widehat{\mathbf{U}}, \tag{9}$$

- $\bar{\mathbf{A}} \to \mathbf{A}(\mathbf{U})$, i e., $\bar{\mathbf{A}}(\widetilde{\mathbf{U}}, \widehat{\mathbf{U}}) \to \mathbf{A}(\mathbf{U})$ for $\widetilde{\mathbf{U}} \to \mathbf{U}$, $\widehat{\mathbf{U}} \to \mathbf{U}$.

Then we can write

$$\mathbf{f}(\mathbf{U}_{j+1}^n) - \mathbf{f}(\mathbf{U}_j^n) = \bar{\mathbf{A}}(\mathbf{U}_j^n, \mathbf{U}_{j+1}^n)(\mathbf{U}_{j+1}^n - \mathbf{U}_j^n) \tag{10}$$

and decompose the jump $\mathbf{U}^n_{j+1} - \mathbf{U}^n_j$ on the right hand side by the following way

$$\mathbf{U}^n_{j+1} - \mathbf{U}^n_j = \sum_{p=1}^{m} \mathit{\Upsilon}^{n,p}_{j+1/2} \bar{\mathbf{r}}^{n,p}_{j+1/2} = \sum_{p=1}^{m} \mathbf{w}^{n,p}_{j+1/2} \tag{11}$$

where $\mathbf{\Upsilon}^n_{j+1/2} = (\bar{\mathbf{R}}^n_{j+1/2})^{-1}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j)$, $\bar{\mathbf{r}}^{n,p}_{j+1/2}$ are the eigenvectors of $\bar{\mathbf{A}}(\mathbf{U}^n_j, \mathbf{U}^n_{j+1})$ (the columns of the matrix $\bar{\mathbf{R}}^n_{j+1/2}$). We obtain the decomposition

$$\begin{aligned} \mathbf{f}(\mathbf{U}^n_{j+1}) - \mathbf{f}(\mathbf{U}^n_j) = \bar{\mathbf{A}}(\mathbf{U}^n_j, \mathbf{U}^n_{j+1})(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j) = \\ = \bar{\mathbf{R}}^n_{j+1/2}\bar{\mathbf{\Lambda}}^n_{j+1/2}(\bar{\mathbf{R}}^n_{j+1/2})^{-1}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j) = \bar{\mathbf{R}}^n_{j+1/2}\bar{\mathbf{\Lambda}}^n_{j+1/2}\mathbf{\Upsilon}^n_{j+1/2} = \\ = \sum_{p=1}^{m} \mathit{\Upsilon}^{n,p}_{j+1/2}\bar{\lambda}^{n,p}_{j+1/2}\bar{\mathbf{r}}^{n,p}_{j+1/2}. \end{aligned} \tag{12}$$

The first-order upwind conservative scheme can be written in the form

$$\mathbf{U}^{n+1}_j = \mathbf{U}^n_j - \frac{\tau}{h}(\mathbf{F}^n_{j+1/2} - \mathbf{F}^n_{j-1/2}) \tag{13}$$

where

$$\mathbf{F}^n_{j+1/2} = \frac{1}{2}[\mathbf{f}(\mathbf{U}^n_j) + \mathbf{f}(\mathbf{U}^n_{j+1})] - \frac{1}{2}|\bar{\mathbf{A}}(\mathbf{U}^n_j, \mathbf{U}^n_{j+1})|(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j) \tag{14}$$

and

$$|\bar{\mathbf{A}}(\mathbf{U}^n_j, \mathbf{U}^n_{j+1})| = \bar{\mathbf{R}}^n_{j+1/2}|\mathbf{\Lambda}^n_{j+1/2}|(\bar{\mathbf{R}}^n_{j+1/2})^{-1}. \tag{15}$$

If we put (12) to (14) we obtain the following formula

$$\mathbf{F}^n_{j+1/2} = \frac{1}{2}[\mathbf{f}(\mathbf{U}^n_j) + \mathbf{f}(\mathbf{U}^n_{j+1})] - \frac{1}{2}\sum_{p=1}^{m} \mathit{\Upsilon}^{n,p}_{j+1/2} \left|\bar{\lambda}^{n,p}_{j+1/2}\right| \bar{\mathbf{r}}^{n,p}_{j+1/2}. \tag{16}$$

The wave-propagation algorithm is in the form

$$\mathbf{U}^{n+1}_j = \mathbf{U}^n_j - \frac{\tau}{h}[\mathcal{A}^+(\Delta\mathbf{U}^n_{j-1/2}) + \mathcal{A}^-(\Delta\mathbf{U}^n_{j+1/2})] \tag{17}$$

where

$$\mathcal{A}^+(\Delta\mathbf{U}^n_{j-1/2}) = \sum_{p=1}^{m} \max\{\bar{\lambda}^{n,p}_{j-1/2}, 0\}\mathbf{w}^{n,p}_{j-1/2},$$

$$\mathcal{A}^-(\Delta\mathbf{U}^n_{j+1/2}) = \sum_{p=1}^{m} \min\{\bar{\lambda}^{n,p}_{j+1/2}, 0\}\mathbf{w}^{n,p}_{j+1/2}.$$

The high-resolution wave-propagation algorithm is in the form

$$\mathbf{U}^{n+1}_j = \mathbf{U}^n_j - \frac{\tau}{h}[\mathcal{A}^+(\Delta\mathbf{U}^n_{j-1/2}) + \mathcal{A}^-(\Delta\mathbf{U}^n_{j+1/2})] - \frac{\tau}{h}(\mathbf{F}^{n,H}_{j+1/2} - \mathbf{F}^{n,H}_{j-1/2}), \tag{18}$$

where

$$\mathbf{F}^{n,H}_{j+1/2} = \frac{1}{2}\sum_{p=1}^{m} |\bar{\lambda}^{n,p}_{j+1/2}|(1 - \frac{\tau}{h}|\bar{\lambda}^{n,p}_{j+1/2}|)\,\widehat{\mathit{\Upsilon}}^{n,p}_{j+1/2}\bar{\mathbf{r}}^{n,p}_{j+1/2}, \tag{19}$$

$$\widehat{\Upsilon}^{n,p}_{j+1/2} = \Upsilon^{n,p}_{j+1/2}\Psi(\Theta^{n,p}_{j+1/2}), \tag{20}$$

$$\Theta^{n,p}_{j+1/2} = \begin{cases} \frac{\Upsilon^{n,p}_{j-1/2}}{\Upsilon^{n,p}_{j+1/2}} & \text{if } \bar{\lambda}^{n,p}_{j+1/2} \geq 0, \\ \frac{\Upsilon^{n,p}_{j+3/2}}{\Upsilon^{n,p}_{j+1/2}} & \text{otherwise.} \end{cases} \tag{21}$$

The function $\Psi = \Psi(\Theta)$ is called the limiter. It can be defined by several ways, for example

- $\Psi(\Theta) = \text{minmod}(1, \Theta)$ (minmod limiter),
- $\Psi(\Theta) = \max\{0, \min\{1, 2\Theta\}, \min\{2, \Theta\}\}$ (superbee limiter),
- $\Psi(\Theta) = \max\{0, \min\{\frac{1+\Theta}{2}, 2, 2\Theta\}\}$ (MC limiter),
- $\Psi(\Theta) = \frac{\Theta+|\Theta|}{1+|\Theta|}$ (van Leer limiter),

where

$$\text{minmod}(a, b) = \begin{cases} a & \text{if } |a| < |b| \text{ and } ab > 0, \\ b & \text{if } |b| < |a| \text{ and } ab > 0, \\ 0 & \text{if } ab \leq 0. \end{cases}$$

3. Strictly hyperbolic systems with spatially-varying flux functions

Herein, we describe upwind and central schemes to solve the strictly hyperbolic systems. We will study (GNL or LD) strictly hyperbolic non-autonomous system

$$\begin{aligned} \mathbf{q}_t + [\mathbf{g}(\mathbf{q}, x)]_x &= \mathbf{0}, \quad x \in \mathbb{R}, t > 0 \\ \mathbf{q}(x, 0) &= \mathbf{q}_0(x), \quad x \in \mathbb{R}. \end{aligned} \tag{22}$$

3.1. Scalar conservation law – augmented formulation

We will study the general initial value scalar problem in the form

$$\begin{aligned} q_t + [g(q, x)]_x &= 0, \; x \in \mathbb{R}, \; t > 0, \\ q(x, 0) &= q_0(x), \; x \in \mathbb{R}, \end{aligned} \tag{23}$$

rewritten to the initial value vector problem

$$\begin{aligned} \mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x &= \mathbf{0}, \; x \in \mathbb{R}, \; t > 0, \; \mathbf{u} = [q, w]^T, \; \mathbf{f} = [g(q, w), 0]^T, \\ [q(x, 0), w(x, 0)]^T &= [q_0(x), x]^T, \; x \in \mathbb{R}. \end{aligned} \tag{24}$$

We can define the following linearization (we have to find an appropriate matrix $\bar{\mathbf{A}}^n_{j+1/2}$ – an approximation of the Jacobi matrix that fulfils three conditions (9))

$$\mathbf{f}(\mathbf{U}^n_{j+1}) - \mathbf{f}(\mathbf{U}^n_j) = \bar{\mathbf{A}}^n_{j+1/2}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j),\ \bar{\mathbf{A}}^n_{j+1/2} = \begin{bmatrix} (\bar{a}_{11})^n_{j+1/2} & (\bar{a}_{12})^n_{j+1/2} \\ 0 & 0 \end{bmatrix} \quad (25)$$

where

$$(\bar{a}_{11})^n_{j+1/2} = \frac{1}{2}\left[\frac{g(Q^n_{j+1}, x_{j+1}) - g(Q^n_j, x_{j+1})}{Q^n_{j+1} - Q^n_j} + \frac{g(Q^n_{j+1}, x_j) - g(Q^n_j, x_j)}{Q^n_{j+1} - Q^n_j}\right],$$

$$(\bar{a}_{12})^n_{j+1/2} = \frac{1}{2}\left[\frac{g(Q^n_{j+1}, x_{j+1}) - g(Q^n_{j+1}, x_j)}{x_{j+1} - x_j} + \frac{g(Q^n_j, x_{j+1}) - g(Q^n_j, x_j)}{x_{j+1} - x_j}\right].$$

We can also formulate the upwind method in the form (13, 14) rewritten to the scalar form

$$\begin{aligned} Q^{n+1}_j &= Q^n_j - \tfrac{\tau}{h}[G^n_{j+1/2} - G^n_{j-1/2}], \\ G^n_{j+1/2} &= \tfrac{1}{2}[g(Q^n_j, x_j) + g(Q^n_{j+1}, x_{j+1})] - \tfrac{1}{2}\bar{b}^n_{j+1/2}(Q^n_{j+1} - Q^n_j), \\ \bar{b}^n_{j+1/2} &= \mathrm{sign}[(a_{11})^n_{j+1/2}]\bar{a}^n_{j+1/2},\ \bar{a}^n_{j+1/2} = \tfrac{g(Q^n_{j+1}, x_{j+1}) - g(Q^n_j, x_j)}{Q^n_{j+1} - Q^n_j}, \\ \bar{a}^n_{j+1/2} &= 0 \text{ if } Q^n_{j+1} = Q^n_j \end{aligned} \quad (26)$$

3.2. System of conservation laws – augmented formulation

We can apply the same approach to the initial value problem for the system of conservation laws in the form

$$\begin{aligned} \mathbf{q}_t + [\mathbf{g}(\mathbf{q}, x)]_x &= \mathbf{0},\ x \in \mathbb{R},\ t > 0, \\ \mathbf{q}(x, 0) &= \mathbf{q}_0(x),\ x \in \mathbb{R}. \end{aligned} \quad (27)$$

We rewrite this problem to

$$\begin{aligned} \mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x &= \mathbf{0},\ x \in \mathbb{R},\ t > 0,\ \mathbf{u} = [\mathbf{q}^T, w]^T,\ \mathbf{f} = [\mathbf{g}^T(\mathbf{q}, w), 0]^T, \\ [\mathbf{q}^T(x, 0), w(x, 0)]^T &= [\mathbf{q}^T_0(x), x]^T,\ x \in \mathbb{R}. \end{aligned} \quad (28)$$

To be able to use the method based on the linearization (10)

$$\mathbf{f}(\mathbf{U}^n_{j+1}) - \mathbf{f}(\mathbf{U}^n_j) = \bar{\mathbf{A}}^n_{j+1/2}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j)$$

where we have

$$\mathbf{f}(\mathbf{U}^n_{j+1}) - \mathbf{f}(\mathbf{U}^n_j) = \begin{bmatrix} \mathbf{g}(\mathbf{Q}^n_{j+1}, x_{j+1}) - \mathbf{g}(\mathbf{Q}^n_j, x_j) \\ 0. \end{bmatrix}$$

We propose the following linearization

$$\bar{\mathbf{A}}^n_{j+1/2} = \left[\begin{array}{cc} (\bar{\mathbf{A}}_{11})^n_{j+1/2} & (\bar{\mathbf{a}}_{12})^n_{j+1/2} \\ \mathbf{0} & 0 \end{array} \right] \tag{29}$$

where

$$[\bar{\mathbf{A}}_{11}(x_j)]^n_{j+1/2}(\mathbf{Q}^n_{j+1} - \mathbf{Q}^n_j) = \mathbf{g}(\mathbf{Q}^n_{j+1}, x_j) - \mathbf{g}(\mathbf{Q}^n_j, x_j),$$

$$(\bar{\mathbf{A}}_{11})^n_{j+1/2} = \frac{1}{2}[\bar{\mathbf{A}}_{11}(x_j)]^n_{j+1/2} + \frac{1}{2}[\bar{\mathbf{A}}_{11}(x_{j+1})]^n_{j+1/2}$$

$$(\bar{\mathbf{a}}_{12})^n_{j+1/2} = \frac{1}{2}\left[\frac{\mathbf{g}(\mathbf{Q}^n_{j+1}, x_{j+1}) - \mathbf{g}(\mathbf{Q}^n_{j+1}, x_j)}{x_{j+1} - x_j} + \frac{\mathbf{g}(\mathbf{Q}^n_j, x_{j+1}) - \mathbf{g}(\mathbf{Q}^n_j, x_j)}{x_{j+1} - x_j}\right].$$

Remark (we drop the subscripts and superscripts below) If we are able to construct a regular diagonizable matrix $\bar{\mathbf{A}}_{11}$ (i.e., $\bar{\mathbf{A}}_{11}\bar{\mathbf{R}}_{11} = \bar{\mathbf{R}}_{11}\bar{\mathbf{\Lambda}}_{11}$) then we are also able to construct the diagonizable matrix $\bar{\mathbf{A}}$ (i.e., $\bar{\mathbf{A}}\bar{\mathbf{R}} = \bar{\mathbf{R}}\bar{\mathbf{\Lambda}}$) in the form

$$\left[\begin{array}{cc} \bar{\mathbf{A}}_{11} & \bar{\mathbf{a}}_{12} \\ \mathbf{0} & 0 \end{array} \right] \left[\begin{array}{cc} \bar{\mathbf{R}}_{11} & -(\bar{\mathbf{A}}_{11})^{-1}\bar{\mathbf{a}}_{12} \\ \mathbf{0} & 1 \end{array} \right] = \left[\begin{array}{cc} \bar{\mathbf{R}}_{11} & -(\bar{\mathbf{A}}_{11})^{-1}\bar{\mathbf{a}}_{12} \\ \mathbf{0} & 1 \end{array} \right] \left[\begin{array}{cc} \bar{\mathbf{\Lambda}}_{11} & \mathbf{0} \\ \mathbf{0} & 0 \end{array} \right].$$

The matrix $\bar{\mathbf{R}}^{-1}$ has the form

$$\bar{\mathbf{R}}^{-1} = \left[\begin{array}{cc} (\bar{\mathbf{R}}_{11})^{-1} & (\bar{\mathbf{R}}_{11})^{-1}(\bar{\mathbf{A}}_{11})^{-1}\bar{\mathbf{a}}_{12} \\ \mathbf{0} & 1 \end{array} \right].$$

■

We use the fact that the matrix $\bar{\mathbf{A}}^n_{j+1/2}$ has the special form (29) and construct the first-order upwind method (13, 16). We have

$$\sum_{p=1}^{m+1} |\bar{\lambda}^{n,p}_{j+1/2}|\,\Upsilon^{n,p}_{j+1/2}\bar{\mathbf{r}}^{n,p}_{j+1/2} = \left[\begin{array}{c} \sum_{p=1}^{m} |\bar{\lambda}^{n,p}_{j+1/2}|\,\Upsilon^{n,p}_{j+1/2}(\bar{\mathbf{r}}_{11})^{n,p}_{j+1/2} \\ 0 \end{array} \right] \tag{30}$$

where

$$\mathbf{\Upsilon}^n_{j+1/2} = (\bar{\mathbf{R}}^n_{j+1/2})^{-1}(\mathbf{U}^n_{j+1} - \mathbf{U}^n_j). \tag{31}$$

If we set $\Xi^{n,p}_{j+1/2} = \Upsilon^{n,p}_{j+1/2}\bar{\lambda}^{n,p}_{j+1/2}$ we obtain

$$\sum_{p=1}^{m} |\bar{\lambda}^{n,p}_{j+1/2}|\,\Upsilon^{n,p}_{j+1/2}(\bar{\mathbf{r}}_{11})^{n,p}_{j+1/2} = \sum_{p=1}^{m} \mathrm{sign}(\bar{\lambda}^{n,p}_{j+1/2})\,\Xi^{n,p}_{j+1/2}(\bar{\mathbf{r}}_{11})^{n,p}_{j+1/2}. \tag{32}$$

We formulate the upwind method in the form (13, 14)

$$\begin{aligned} \mathbf{Q}^{n+1}_j &= \mathbf{Q}^n_j - \tfrac{\tau}{h}[\mathbf{G}^n_{j+1/2} - \mathbf{G}^n_{j-1/2}], \\ \mathbf{G}^n_{j+1/2} &= \tfrac{1}{2}[\mathbf{g}(\mathbf{Q}^n_j, x_j) + \mathbf{g}(\mathbf{Q}^n_{j+1}, x_{j+1})] - \\ &\quad -\tfrac{1}{2}\sum_{p=1}^{m} \mathrm{sign}(\bar{\lambda}^{n,p}_{j+1/2})\,\Xi^{n,p}_{j+1/2}(\bar{\mathbf{r}}_{11})^{n,p}_{j+1/2}. \end{aligned} \tag{33}$$

In the wave-propagation form it can be formulated as

$$\begin{aligned}
\mathbf{Q}_j^{n+1} &= \mathbf{Q}_j^n - \tfrac{\tau}{h}[\mathcal{A}^+(\Delta \mathbf{Q}_{j-1/2}^n) + \mathcal{A}^-(\Delta \mathbf{Q}_{j+1/2}^n)],\\
\mathcal{A}^+(\Delta \mathbf{Q}_{j-1/2}^n) &= \sum_{\bar{\lambda}_{j-1/2}^{n,p}>0} \Xi_{j-1/2}^{n,p} (\bar{\mathbf{r}}_{11})_{j-1/2}^{n,p} = \sum_{\bar{\lambda}_{j-1/2}^{n,p}>0} \bar{\mathbf{z}}_{j-1/2}^{n,p},\\
\mathcal{A}^-(\Delta \mathbf{Q}_{j+1/2}^n) &= \sum_{\bar{\lambda}_{j+1/2}^{n,p}<0} \Xi_{j+1/2}^{n,p} (\mathbf{r}_{11})_{j+1/2}^{n,p} = \sum_{\bar{\lambda}_{j+1/2}^{n,p}<0} \bar{\mathbf{z}}_{j+1/2}^{n,p}.
\end{aligned} \tag{34}$$

The high-resolution wave-propagation algorithm is in the form

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\tau}{h}[\mathcal{A}^+(\Delta \mathbf{Q}_{j-1/2}^n) + \mathcal{A}^-(\Delta \mathbf{Q}_{j+1/2}^n)] - \frac{\tau}{h}(\mathbf{G}_{j+1/2}^{n,H} - \mathbf{G}_{j-1/2}^{n,H}), \tag{35}$$

where

$$\mathbf{G}_{j+1/2}^{n,H} = \frac{1}{2}\sum_{p=1}^{m} |\bar{\lambda}_{j+1/2}^{n,p}|(1 - \frac{\tau}{h}|\bar{\lambda}_{j+1/2}^{n,p}|)\,\widehat{\Upsilon}_{j+1/2}^{n,p} (\bar{\mathbf{r}}_{11})_{j+1/2}^{n,p}, \tag{36}$$

$$\widehat{\Upsilon}_{j+1/2}^{n,p} = \Upsilon_{j+1/2}^{n,p} \Psi(\Theta_{j+1/2}^{n,p}), \tag{37}$$

$$\Theta_{j+1/2}^{n,p} = \begin{cases} \frac{\Upsilon_{j-1/2}^{n,p}}{\Upsilon_{j+1/2}^{n,p}} & \text{if } \bar{\lambda}_{j+1/2}^{n,p} \geq 0,\\ \frac{\Upsilon_{j+3/2}^{n,p}}{\Upsilon_{j+1/2}^{n,p}} & \text{otherwise.} \end{cases} \tag{38}$$

4. General (non-strictly and weakly) hyperbolic systems with spatially-varying flux functions

In this section we study schemes for non-strictly weakly hyperbolic systems.

4.1. Upwind schemes

There are two possibilities how to apply the upwind schemes to the weakly hyperbolic systems. The first approach is based on the direct application of the conservative upwind scheme to the system (28). If we apply Roe's linearization we suppose that Riemann problems at cell interfaces (typically) have a large jump at most one wave family (i.e., the solution consists of a single wave and $||\,\Upsilon_{j+1/2}^{n,p}\mathbf{r}_{j+1/2}^{n,p}|| = O(h)$ for all other waves). In our case we obtain two wave families but the second one (for the function w) with $O(h)$.

The system (28) fulfils the simplified version of the jump condition (disscussed in [FMO00]) in the form

$$[\mathbf{g}] = s[\mathbf{q}],\ \ [w](0 - s) = 0.$$

This observation is used in the second approach. We solve the non-augmented system (27) but in order to obtain the correct projection into the characteristics field, we have to use the full left eigenvectors. In the case we have a weakly hyperbolic system we use the approach based on the complementary projection method [FMO98].

The Godunov type method can also be used but the more general structure of waves for weakly hyperbolic systems has to be respected. If the function $\mathbf{g}(\mathbf{q}, x)$ is sufficiently smooth in $\mathbf{q}$ and x it is possible to use any standard entropy fix procedure. In other cases a special entropy fix procedure has to be applied (and it is better to use the formulation in the form $\mathbf{q}_t + [\mathbf{g}(\mathbf{q}, \mathbf{a}(x))]_x = \mathbf{0}$ where $\mathbf{a} = \mathbf{a}(x)$ is a non-smooth function).

4.2. Central schemes

The other possibility is to use a componentwise scheme – for example, the central schemes. By this way we circumvent the problem of the construction of the Riemann solver. We present here the second order central scheme (Tadmor, Nessyahu). This is the two-step predictor-corrector-type procedure:

$$\begin{aligned}
\mathbf{Q}_j^{n+1/2} &= \mathbf{Q}_j^n - \tfrac{1}{2}\tfrac{\tau}{h}(\mathbf{g}')_j^n, \\
(\mathbf{Q}')_j^n &= \mathbf{MM}\left(2\Delta\mathbf{Q}_{j+1/2}^n, \tfrac{1}{2}(\mathbf{Q}_{j+1}^n - \mathbf{Q}_{j-1}^n), 2\Delta\mathbf{Q}_{j-1/2}^n\right), \\
(\mathbf{g}')_j^n &= \mathbf{MM}\left(2\Delta\mathbf{g}_{j+1/2}^n, \tfrac{1}{2}\left[\mathbf{g}(\mathbf{Q}_{j+1}^n, x_{j+1}) - \mathbf{g}(\mathbf{Q}_{j-1}^n, x_{j-1})\right], 2\Delta\mathbf{g}_{j-1/2}^n\right), \\
\Delta\mathbf{Q}_{j+1/2}^n &= \mathbf{Q}_{j+1}^n - \mathbf{Q}_j^n, \quad \Delta\mathbf{g}_{j+1/2}^n = \mathbf{g}(\mathbf{Q}_{j+1}^n, x_{j+1}) - \mathbf{g}(\mathbf{Q}_j^n, x_j),
\end{aligned} \tag{39}$$

$$\begin{aligned}
\mathbf{MM}(\mathbf{b}_1, \mathbf{b}_2, \ldots) &= \begin{bmatrix} \mathrm{MM}\left((b_1)_1, (b_2)_1, \ldots\right) \\ \mathrm{MM}\left((b_1)_2, (b_2)_2, \ldots\right) \\ \cdots \\ \mathrm{MM}\left((b_1)_m, (b_2)_m, \ldots\right) \end{bmatrix} \\
\mathrm{MM}(b_1, b_2, \ldots) &= \begin{cases} \min\limits_k\{b_k\} & b_k > 0 \qquad \forall k, \\ \max\limits_k\{b_k\} & b_k < 0 \qquad \forall k, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned} \tag{40}$$

$$\mathbf{Q}_{j+1/2}^{n+1} = \frac{1}{2}(\mathbf{Q}_j^n + \mathbf{Q}_{j+1}^n) + \frac{1}{8}[(\mathbf{Q}')_j^n - (\mathbf{Q}')_{j+1}^n] - \frac{\tau}{h}[\mathbf{g}(\mathbf{Q}_{j+1}^{n+1/2}) - \mathbf{g}(\mathbf{Q}_j^{n+1/2})] \tag{41}$$

Instead of the basic scheme described above we propose to use the semidiscrete version of these algorithms where the degenerate conservation law $w_t = 0$ is solved exactly.

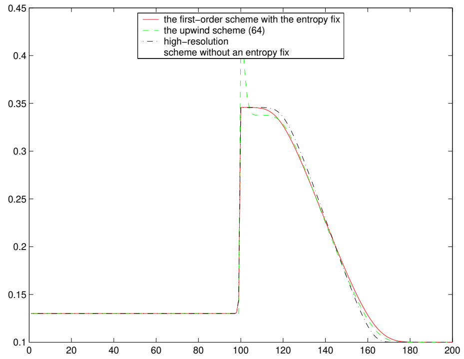

Figure 1: Traffic flow: $q_L = 0.13$, $q_R = 0.1$, $v_L = 2$, $v_R = 1$, $h = 0.01$, $\tau = h/2$, $t_{max} = 0.75$

5. Applications

5.1. Traffic flow

We consider non-autonomous scalar conservation law in the form

$$q_t + [v_{max}(x)(1-q)q]_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{42}$$

with the initial condition

$$q(x,0) = q_0(x), \quad x \in \mathbb{R}, \tag{43}$$

where $g(q,x) = vq = v_{max}(x)(1-q)q$, $q = q(x,t)$ is the density of cars and $u_{max} = u_{max}(x)$ is the limit speed. The upwind method (26) rewritten to the scalar form

$$\begin{aligned}
Q_j^{n+1} &= Q_j^n - \tfrac{\tau}{h}[G_{j+1/2}^n - G_{j-1/2}^n],\\
G_{j+1/2}^n &= \tfrac{1}{2}[g(Q_j^n, x_j) + g(Q_{j+1}^n, x_{j+1})] - \tfrac{1}{2}\bar{b}_{j+1/2}^n (Q_{j+1}^n - Q_j^n),\\
\bar{b}_{j+1/2}^n &= \operatorname{sign}\left[\tfrac{1}{2}\left(V_{max,j} + V_{max,j+1}\right)\left(1 - Q_j^n - Q_{j+1}^n\right)\right]\bar{a}_{j+1/2}^n,\\
\bar{a}_{j+1/2}^n &= \tfrac{g(Q_{j+1}^n, x_{j+1}) - g(Q_j^n, x_j)}{Q_{j+1}^n - Q_j^n},\\
\bar{a}_{j+1/2}^n &= 0 \text{ if } Q_{j+1}^n = Q_j^n
\end{aligned} \tag{44}$$

5.2. River flow

We consider non-autonomous scalar conservation law in the form

$$\phi_t + q_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{45}$$

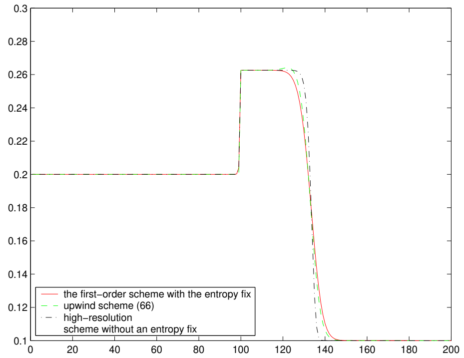

Figure 2: River flow: $a_L = 0.2$, $a_R = 0.1$, $\alpha_L = \frac{\pi}{10}$, $\alpha_R = \frac{\pi}{20}$, $h = 0.01$, $\tau = h/6$, $t_{max} = 0.33$

with the initial condition

$$\phi(x,0) = \phi_0(x), \quad x \in \mathbb{R}, \tag{46}$$

where $q = q(\phi, x) = \left(\frac{g}{\psi}\right)^{\frac{1}{2}} [\sin(\alpha(x))]^{1/2} \phi^{5/4}$ is the flow rate, $\phi = \phi(x,t)$ is the cross-section of the stream bed, g is the gravity acceleration, ψ is the friction factor and $\alpha = \alpha(x)$ is the downstream slope. The upwind method (26) rewritten to the scalar form

$$\begin{aligned}
A_j^{n+1} &= A_j^n - \tfrac{\tau}{h}[G_{j+1/2}^n - G_{j-1/2}^n], \\
G_{j+1/2}^n &= \tfrac{1}{2}[g(A_j^n, x_j) + g(A_{j+1}^n, x_{j+1})] - \tfrac{1}{2}\bar{b}_{j+1/2}^n (A_{j+1}^n - A_j^n), \\
\bar{b}_{j+1/2}^n &= \operatorname{sign}\left[\bar{\lambda}_{j+1/2}^n\right] \bar{a}_{j+1/2}^n, \\
\bar{a}_{j+1/2}^n &= \tfrac{g(A_{j+1}^n, x_{j+1}) - g(A_j^n, x_j)}{A_{j+1}^n - A_j^n}, \\
\bar{a}_{j+1/2}^n &= 0 \text{ if } A_{j+1}^n = A_j^n, \\
\bar{\lambda}_{j+1/2}^n &= \tfrac{1}{2}\left(\tfrac{g}{\psi}\right)^{1/2} \left[\sqrt{\sin(x_j)} + \sqrt{\sin(x_{j+1})}\right] \cdot \\
&\cdot \tfrac{A_j^n + (A_j^n)^{3/4}(A_{j+1}^n)^{1/4} + (A_j^n)^{1/2}(A_{j+1}^n)^{1/2} + (A_j^n)^{1/4}(A_{j+1}^n)^{3/4} + A_{j+1}^n}{(A_j^n)^{3/4} + (A_j^n)^{1/2}(A_{j+1}^n)^{1/4} + (A_j^n)^{1/4}(A_{j+1}^n)^{1/2} + (A_{j+1}^n)^{3/4}}
\end{aligned} \tag{47}$$

5.3. Fluid flow through the male urethra

In one space dimension the fluid flow through the male urethra is governed by the following system of equations

$$\begin{aligned}
\phi_t + (v\phi)_x &= 0, \ x \in (0,L), \ t \in (0,T), \\
v_t + (\tfrac{1}{2}v^2)_x + \tfrac{1}{\varrho}p_x &= 0, \ x \in (0,L), \ t \in (0,T), \\
p &= \tfrac{\phi - \psi(x)}{\beta(x)},
\end{aligned} \tag{48}$$

where we denote the cross-section of the urethra by $\phi = \phi(x,t)$, the fluid velocity by $v = v(x,t)$, the fluid pressure by $p = p(x,t)$. The fluid density is denoted by ϱ and the material parameters (generally dependent on the space variable x) by $\psi = \psi(x)$, $\beta = \beta(x)$. This system can be written as the non-autonomous non-linear system in the form

$$\begin{gathered}\mathbf{q}_t + [\mathbf{g}(\mathbf{q},x)]_x = \mathbf{0},\\ \mathbf{q} = [\phi, v]^T, \quad \mathbf{g} = \left[v\phi, \tfrac{1}{2}v^2 + \tfrac{\phi-\psi(x)}{\beta(x)\varrho}\right]^T,\end{gathered} \tag{49}$$

or as the autonomous non-linear system in the form

$$\begin{gathered}\mathbf{u}_t + [\mathbf{f}(\mathbf{u})]_x = \mathbf{0},\\ \mathbf{u} = [\phi, v, w]^T, \quad \mathbf{f} = \left[v\phi, \tfrac{1}{2}v^2 + \tfrac{\phi-\psi(w)}{\beta(w)\varrho}, 0\right]^T.\end{gathered} \tag{50}$$

This system has to be supplemented by the initial condition

$$w(x,0) \;=\; x,\ x \in (0,L). \tag{51}$$

The Jacobi matrix has the form

$$\mathbf{A}(\mathbf{u}) = \mathbf{f}_{\mathbf{u}} = \begin{bmatrix} v & \phi & 0 \\ \frac{1}{\beta(w)\varrho} & v & \frac{-\beta(w)\psi_w + [\psi(w)-\phi]\beta_w}{\varrho\beta^2(w)} \\ 0 & 0 & 0 \end{bmatrix}. \tag{52}$$

The eigenvalues of this system are

$$\lambda_1 = v + \sqrt{\frac{\phi}{\beta(x)\varrho}},\ \lambda_2 = v - \sqrt{\frac{\phi}{\beta(x)\varrho}},\ \lambda_3 = 0. \tag{53}$$

The matrix $(\bar{\mathbf{A}}_{11})^n_{j+1/2}$ has the form

$$(\bar{\mathbf{A}}_{11})^n_{j+1/2} = \begin{bmatrix} \bar{V}^n_{j+1/2} & \bar{\Phi}^n_{j+1/2} \\ \bar{K}_{j+1/2} & \bar{V}^n_{j+1/2} \end{bmatrix} \tag{54}$$

$$\left[(\bar{\mathbf{A}}_{11})^n_{j+1/2}\right]^{-1} = \begin{bmatrix} \frac{-\bar{V}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}\bar{K}^n_{j+1/2} - (\bar{V}^n_{j+1/2})^2} & \frac{\bar{\Phi}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}\bar{K}^n_{j+1/2} - (\bar{V}^n_{j+1/2})^2} \\ \frac{\bar{K}_{j+1/2}}{\bar{\Phi}^n_{j+1/2}\bar{K}^n_{j+1/2} - (\bar{V}^n_{j+1/2})^2} & \frac{-\bar{V}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}\bar{K}^n_{j+1/2} - (\bar{V}^n_{j+1/2})^2} \end{bmatrix} \tag{55}$$

where

$$\begin{aligned} \bar{\Phi}^n_{j+1/2} &= \tfrac{1}{2}\left(\Phi^n_j + \Phi^n_{j+1}\right),\\ \bar{V}^n_{j+1/2} &= \tfrac{1}{2}\left(V^n_j + V^n_{j+1}\right),\\ \bar{K}_{j+1/2} &= \tfrac{1}{2}\left(\tfrac{1}{\varrho\beta_j} + \tfrac{1}{\varrho\beta_{j+1}}\right). \end{aligned} \tag{56}$$

The eigenvalues are

$$\begin{aligned} \bar{\lambda}^{n,1}_{j+1/2} &= \bar{V}^n_{j+1/2} + \sqrt{\bar{\Phi}^n_{j+1/2}\bar{K}_{j+1/2}},\\ \bar{\lambda}^{n,2}_{j+1/2} &= \bar{V}^n_{j+1/2} - \sqrt{\bar{\Phi}^n_{j+1/2}\bar{K}_{j+1/2}}. \end{aligned} \tag{57}$$

The matrix of the right eigenvectors is

$$(\bar{\mathbf{R}}_{11})^n_{j+1/2} = \begin{bmatrix} \sqrt{\frac{\bar{\Phi}^n_{j+1/2}}{\bar{K}_{j+1/2}}} & -\sqrt{\frac{\bar{\Phi}^n_{j+1/2}}{\bar{K}_{j+1/2}}} \\ 1 & 1 \end{bmatrix} = \left[(\bar{\mathbf{r}}_{11})^{n,1}_{j+1/2}, (\bar{\mathbf{r}}_{11})^{n,2}_{j+1/2}\right], \tag{58}$$

and the inverse matrix has the form

$$\left[(\bar{\mathbf{R}}_{11})^n_{j+1/2}\right]^{-1} = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{\bar{K}_{j+1/2}}{\bar{\Phi}^n_{j+1/2}}} & \frac{1}{2} \\ -\frac{1}{2}\sqrt{\frac{\bar{K}_{j+1/2}}{\bar{\Phi}^n_{j+1/2}}} & \frac{1}{2} \end{bmatrix}. \tag{59}$$

Furthermore, we have

$$\bar{\mathbf{A}}^n_{j+1/2} = \begin{bmatrix} \bar{V}^n_{j+1/2} & \bar{\Phi}^n_{j+1/2} & 0 \\ \bar{K}_{j+1/2} & \bar{V}^n_{j+1/2} & \frac{\bar{\Phi}^n_{j+1/2}\widetilde{K}_{j+1/2}+\widetilde{L}^n_{j+1/2}}{h} \\ 0 & 0 & 0 \end{bmatrix} \tag{60}$$

and

$$\widetilde{L}^n_{j+1/2} = \frac{\Psi^n_j}{\varrho\beta_j} - \frac{\Psi^n_{j+1}}{\varrho\beta_{j+1}}, \quad \widetilde{K}^n_{j+1/2} = \frac{1}{\varrho\beta_{j+1}} - \frac{1}{\varrho\beta_j}, \tag{61}$$

$$\bar{\mathbf{R}}^n_{j+1/2} = \begin{bmatrix} \sqrt{\frac{\bar{\Phi}^n_{j+1/2}}{\bar{K}^n_{j+1/2}}} & -\sqrt{\frac{\bar{\Phi}^n_{j+1/2}}{\bar{K}^n_{j+1/2}}} & \widetilde{M}^n_{j+1/2} \\ 1 & 1 & \widetilde{N}^n_{j+1/2} \\ 0 & 0 & 1 \end{bmatrix}, \tag{62}$$

$$\left[\mathbf{R}^n_{j+1/2}\right]^{-1} = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{\bar{K}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}}} & \frac{1}{2} & -\frac{1}{2}\widetilde{N}^n_{j+1/2} - \frac{1}{2}\widetilde{M}^n_{j+1/2}\sqrt{\frac{\bar{K}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}}} \\ -\frac{1}{2}\sqrt{\frac{\bar{K}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}}} & \frac{1}{2} & -\frac{1}{2}\widetilde{N}^n_{j+1/2} + \frac{1}{2}\widetilde{M}^n_{j+1/2}\sqrt{\frac{\bar{K}^n_{j+1/2}}{\bar{\Phi}^n_{j+1/2}}} \\ 0 & 0 & 1 \end{bmatrix}, \tag{63}$$

$$\begin{aligned} \widetilde{M}^n_{j+1/2} &= \frac{-\bar{\Phi}^n_{j+1/2}\left(\bar{\Phi}^n_{j+1/2}\widetilde{K}^n_{j+1/2}+\widetilde{L}^n_{j+1/2}\right)}{h\left[\bar{\Phi}^n_{j+1/2}\bar{K}^n_{j+1/2}-(\bar{V}^n_{j+1/2})^2\right]}, \\ \widetilde{N}^n_{j+1/2} &= \frac{\bar{V}^n_{j+1/2}\left(\bar{\Phi}^n_{j+1/2}\widetilde{K}^n_{j+1/2}+\widetilde{L}^n_{j+1/2}\right)}{h\left[\bar{\Phi}^n_{j+1/2}\bar{K}^n_{j+1/2}-(\bar{V}^n_{j+1/2})^2\right]}. \end{aligned} \tag{64}$$

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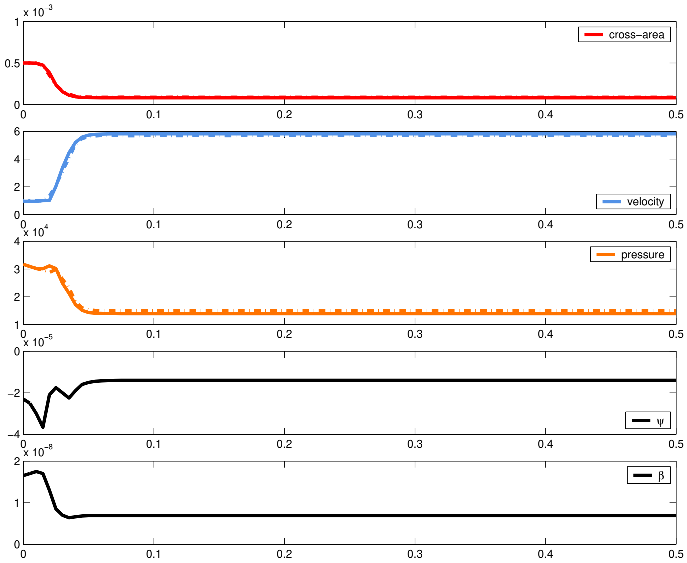

Figure 3: Fluid flow through the (simplified) mail urethra: $h = 0.005$, $\tau = h/100$, $t_{max} = 0.5$

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