$\delta^{(n)}$-Shock wave as a new type solutions of hyperbolic systems of conservation laws

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Abstract. A concept of a new type of singular solutions to hyperbolic systems of conservation laws is introduced. It is so-called $\delta^{(n)}$-shock wave, where $\delta^{(n)}$ is n-th derivative of the delta function.

We introduce a definition of δ'-shock wave type solution for the system

$$u_t + \big(f(u)\big)_x = 0, \quad v_t + \big(f'(u)v\big)_x = 0, \quad w_t + \big(f''(u)v^2 + f'(u)w\big)_x = 0.$$

Within the framework of this definition, the Rankine–Hugoniot conditions for δ'-shock are derived and analyzed from geometrical point of view. We prove δ'-shock *balance relations* connected with *area transportation*. A solitary δ'-shock wave type solution to the Cauchy problem of the system of conservation laws

$$u_t + \big(u^2\big)_x = 0, \quad v_t + 2\big(uv\big)_x = 0, \quad w_t + 2\big(v^2 + uw\big)_x = 0$$

with piecewise continuous initial data is constructed.

These results show that solutions of hyperbolic systems of conservation laws can develop not only Dirac measures (as in the case of δ-shocks) but their derivatives as well.

1. Introduction

1.1. Singular solutions to systems of conservation laws. Let us consider the Cauchy problem for hyperbolic system of conservation laws

$$\left\{ \begin{array}{rcll} U_t + \big(F(U)\big)_x & = & 0, & \text{in } \mathbb{R} \times (0,\ \infty), \\ U & = & U^0, & \text{in } \mathbb{R} \times \{t = 0\}, \end{array} \right. \tag{1.1}$$

where $F : \mathbb{R}^m \to \mathbb{R}^m$ and $U^0 : \mathbb{R} \to \mathbb{R}^m$ are given smooth vector-functions, and $U = U(x,t) = (u_1(x,t), \dots, u_m(x,t))$ is the unknown function, $x \in \mathbb{R}$, $t \geq 0$.

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As is well known, even in the case of smooth (and, certainly, in the case of discontinuous) initial data $U^0(x)$, this system may have discontinuous solutions. In this case, it is said that $U \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^m\big)$ is a *generalized solution* of the Cauchy problem (1.1) if the integral identities

$$\int_0^\infty \int \Big(U\cdot\widetilde{\varphi}_t + F(U)\cdot\widetilde{\varphi}_x\Big)\,dx\,dt + \int U^0(x)\cdot\widetilde{\varphi}(x,0)\,dx = 0 \tag{1.2}$$

hold for all compactly supported test vector-functions $\widetilde{\varphi}:\mathbb{R}\times[0,\,\infty)\to\mathbb{R}^m$, where $\cdot$ is the scalar product of vectors, $\int f(x)\,dx$ denotes the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$.

1.2. **δ-shock wave type solutions.** Let us consider the hyperbolic systems of conservation laws

$$u_t + \big(F(u,v)\big)_x = 0, \qquad v_t + \big(G(u,v)\big)_x = 0, \tag{1.3}$$

and

$$v_t + \big(G(u,v)\big)_x = 0, \qquad (uv)_t + \big(H(u,v)\big)_x = 0, \tag{1.4}$$

where $F(u,v)$, $G(u,v)$, $H(u,v)$ are smooth functions, *linear* with respect to v; $u = u(x,t)$, $v = v(x,t)\in\mathbb{R}$; $x\in\mathbb{R}$. The well-known "zero-pressure gas dynamics" system is a particular case of system (1.4), where $G(u,v) = uv$, $H(u,v) = u^2v$. In this case $v(x,t)\geq 0$ is density, and $u(x,t)$ is velocity.

In [**1**], [**6**]–[**9**], [**11**], [**12**], [**13**], [**19**]– [**22**], [**23**], [**25**] it is shown that for some cases of hyperbolic systems (1.3), (1.4) "nonclassical" situations may occur, when the Riemann problem does not possess a weak L^∞-solution except for some particular initial data. In contrast to the standard results of existence of weak solutions to strictly hyperbolic systems, here the *linear* component v of the solution may contain Dirac measures and must be sought in the space of measures, while the nonlinear component u of the solution has bounded variation. In order to solve the Cauchy problem in this nonclassical situation, it is necessary to introduce new singularities called δ-shocks, which are solutions of the hyperbolic system, such that the *linear* component of the solution can have the form $v(x,t) = V(x,t) + e(x,t)\delta(\Gamma)$, Γ is a graph in the upper half-plane $\{(x,t) : x\in\mathbb{R}, t\geq 0\}$, $V\in L^\infty$, $e\in C^1(\Gamma)$, and the *nonlinear* component $u\in L^\infty\big(\mathbb{R}\times(0,\,\infty);\mathbb{R}\big)$.

Several approaches to constructing *δ-shock type solutions* are known. An apparent difficulty in defining such solutions arises due to the fact that, to introduce a definition of the δ-shock type solution, we need to define a *singular superposition* of distributions (for example, a *product of the Heaviside function and the delta function*). We also need to define *in which sense* a distributional solution satisfies a nonlinear system.

We recall that in [**13**], to construct a δ-shock wave type solution of the system

$$u_t + \big(f(u)\big)_x = 0, \qquad v_t + \big(f'(u)v\big)_x = 0, \tag{1.5}$$

(here $F(u,v) = f(u)$, $G(u,v) = f'(u)v$) the problem of *multiplication of distributions* is solved by using the definition of Volpert's averaged superposition [**24**]. In [**17**], a general framework for *nonconservative product*

$$g(u)\frac{du}{dx} \tag{1.6}$$

was introduced, where $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally bounded Borel function, and $u : (a,b) \to \mathbb{R}^n$ is a discontinuous function of bounded variation. In the framework of the approach [**17**] the Cauchy problems for nonlinear hyperbolic systems in nonconservative form can be considered [**13**], [**14**], [**15**]. Note that in [**14**], [**15**], for nonconservative systems the notion of generalized solution *does depend on the specific family of paths*, which *can not be derived from the hyperbolic system only*.

One of the approaches to solving the problems related to singular solutions of quasilinear equations was developed in [**3**], [**4**]– [**9**], [**19**]– [**22**]. In these papers a new asymptotics method – the *weak asymptotics method* – for studying the *dynamics of propagation and interaction* of different singularities of quasilinear differential equations and hyperbolic systems of conservation laws was developed. Algebraic aspects of the *weak asymptotics method* are given in detail in [**2**], [**3**], [**18**].

In [**7**]– [**9**], in the framework of the *weak asymptotics method new definitions of a δ-shock wave type solution* by integral identities were introduced for two classes of hyperbolic systems of conservation laws (1.3), (1.4). These definitions give *natural* generalizations of the classical definition of the weak L^∞-solutions (1.1) relevant to the structure of δ-shocks. (For the case of system (1.3) see Definition 2.1 below.)

1.3. Main results and contents of the paper. In this paper we introduce a concept of a new type of singular solutions to hyperbolic systems of conservation laws. This type of solutions we call $\delta^{(n)}$*-shock waves*, where $\delta^{(n)}$ is n-th derivative of the Dirac delta function.

Our main interest is in the following hyperbolic system of conservation laws:

$$\begin{aligned} L_1[u] = u_t + \big(f(u)\big)_x &= 0, \\ L_2[u,v] = v_t + \big(f'(u)v\big)_x &= 0, \\ L_3[u,v,w] = w_t + \big(f''(u)v^2 + f'(u)w\big)_x &= 0, \end{aligned} \tag{1.7}$$

where $f(u)$ is a smooth function, $f''(u) > 0$, $u = u(x,t), v = v(x,t), w = w(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$. This system is extremely degenerate with repeated eigenvalues $\lambda = f'(u)$.

For system (1.7) a *definition of a δ'-shock wave type solution* (Definition 3.1) is introduced. Roughly speaking, it is such solution that its second component v may contain Dirac measures, and the third component w may contain the linear combination of Dirac measures and their derivatives, while the first component u of the solution has bounded variation.

In this paper, as a step on the way to develop the theory of solutions of this type, we construct a δ'-shock wave type solution to the Cauchy problem of the system of conservation laws

$$
\begin{array}{rcl}
L_{11}[u]=u_t+\big(u^2\big)_x & = & 0, \\
L_{12}[u,v]=v_t+2\big(uv\big)_x & = & 0, \\
L_{13}[u,v,w]=w_t+2\big(v^2+uw\big)_x & = & 0
\end{array}
\tag{1.8}
$$

with the singular initial data

$$
\begin{array}{rcl}
u^0(x) & = & u_0^0(x)+u_1^0(x)H(-x), \\
v^0(x) & = & v_0^0(x)+v_1^0(x)H(-x)+e^0\delta(-x), \\
w^0(x) & = & w_0^0(x)+w_1^0(x)H(-x)+g^0\delta(-x)+h^0\delta'(-x),
\end{array}
\tag{1.9}
$$

where $u_k^0(x)$, $v_k^0(x)$, $w_k^0(x)$ are given smooth functions, $k=0,1$, e^0, g^0, h^0 are given constants, $H(x)$ is the Heaviside function, $\delta(x)$ is the delta function, $\delta'(x)$ is the derivative of the delta function. System (1.8) is the simplest case of system (1.7).

Taking into account that system (1.7) has repeated eigenvalues $\lambda=f'(u)$, we shall use the following admissibility condition for the δ'-shocks:

$$
f'(u_+)\le\dot\phi(t)\le f'(u_-),
\tag{1.10}
$$

where $\dot\phi(t)$ is the velocity of motion of the δ'-shock wave, and u_-, u_+ are the respective left- and right-hand values of u on the discontinuity curve. Condition (1.10) means that all characteristics on both sides of the discontinuity are in-coming. For system (1.8) condition (1.10) has the form

$$
2u_+\le\dot\phi(t)\le 2u_-.
\tag{1.11}
$$

The construction of a δ'-shock type solution for the simplest system (1.8) points out an entirely new perspective in the theory of singular solutions to hyperbolic systems of conservation laws. This result shows that hyperbolic systems can develop not only Dirac measures (as in the case of δ-shocks) but also their derivatives, i.e., they admit solutions of "unlimited" degree of singularity.

In Sec. 2, we recall some facts on *δ-shock wave type solutions* for system (1.3) obtained in [**7**]– [**9**], [**20**]– [**22**], to compare them with our results on δ'-shocks. Namely, we repeat definition for δ-shock type solutions (Definition 2.1), derive the Rankine–Hugoniot conditions for δ-shocks by Theorem 2.1, and prove δ-shock *balance relations* by Theorem 2.2.

In Subsec. 3.1 we explain why system (1.7) can admit δ'-shock wave type solutions. A *δ'-shock wave type solution* for system (1.7) is introduced by Definition 3.1 in Subsec. 3.2. This definition is of the same type as Definition 2.1 for *δ-shock wave type solutions* mentioned above. The integral identities used in these definitions are introduced after analyzing *asymptotic solutions* of hyperbolic systems. In this subsection, within the framework of Definition 3.1, the Rankine–Hugoniot conditions for δ'-shocks (3.16)–(3.19) are derived by Theorem 3.2. In Subsec. 3.3, the Rankine–Hugoniot conditions for δ'-shocks are analyzed from geometrical point of view. δ'-Shock *balance relations* (3.22), (3.23) connected with *area transportation* are proved.

We construct a *δ'-shock wave type solution* of the Cauchy problem as the weak limit of a *weak asymptotic solution*. A definition of a *weak asymptotic solution* for system (1.7) is given in Subsec. 3.4.

In Sec. 4, we study the problem of propagation of a δ'-shock in system (1.8), i.e., we solve the Cauchy problem (1.8), (1.9). In Subsec. 4.1, by Theorem 4.1, we construct a *weak asymptotic solution* of the Cauchy problem (1.8), (1.9). In Subsec. 4.2, using the *weak asymptotic solution* of the Cauchy problem, a *δ'-shock wave type solution* of the Cauchy problem is constructed by Theorem 4.2. By Corollary 4.1 we solve the Cauchy problem (1.8), (1.9) for the case of piecewise constant initial data. To solve these problems, we use the *weak asymptotics method* mentioned above, extended to the case of this type of singular solutions.

System (4.28), which determines the trajectory $x = \phi(t)$ of a δ'-shock wave and the coefficients $e(t)$, $g(t)$, $h(t)$ of the singularities, constitutes the *Rankine–Hugoniot conditions for δ'-shock*. Here the first equation in (4.28) is the "standard" Rankine–Hugoniot condition for the *shock*, while the first and second equations are the "standard" *Rankine–Hugoniot conditions for δ-shock* (see Theorem 2.1).

The problem of defining *δ'-shock wave type solutions* for the Cauchy problems (1.8), (1.9) in connection with the construction of *singular superposition* (*products*) of distributions (4.35)–(4.37) is discussed in Subsec. 4.3. If we knew the *"right" singular superpositions* (4.35)–(4.37) in advance then Theorem 4.2 and Corollary 4.1 could be proved explicitly by substituting these superpositions into system (1.8).

It remains to note that, since system (1.7) has no terms of the type (1.6), it is *impossible* to construct a *δ'-shock wave type solution* for it by using the *nonconservative product* [**13**], [**14**], [**15**], [**17**].

The problem of constructing $\delta^{(n)}$-shock waves is discussed in Sec. 5.

According to the *weak asymptotics method* mentioned above, we shall seek a δ'-shock wave type solution of the Cauchy problem (1.7), (1.9) in the form of the singular ansatz

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t)H(-x+\phi(t)), \\ v(x,t) &= v_0(x,t) + v_1(x,t)H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \\ w(x,t) &= w_0(x,t) + w_1(x,t)H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) \\ &\qquad + h(t)\delta'(-x+\phi(t)), \end{aligned} \tag{1.12}$$

where $u_k(x,t)$, $v_k(x,t)$, $w_k(x,t)$, $k = 0,1$, $\phi(t)$, $e(t)$, $g(t)$, $h(t)$ are desired functions. The singular ansatzs (1.12) *correspond* to the structure of the initial data (1.9).

Within the framework of the *weak asymptotics method*, we find a *δ'-shock wave type solution* of the Cauchy problem (1.7), (1.9) as the weak limit of a *weak asymptotic solution* given by Definition 3.2.

Let $\alpha \in \mathbb{R}$. Denote by $O_{\mathcal{D}'}(\varepsilon^\alpha)$, $\varepsilon \to +0$ the collection of distributions $f(x,t,\varepsilon) \in \mathcal{D}'(\mathbb{R}_x)$ such that

$$\langle f(x,t,\varepsilon),\ \psi(x)\rangle = O(\varepsilon^\alpha), \quad \varepsilon \to +0,$$

for any test function $\psi(x) \in \mathcal{D}(\mathbb{R}_x)$. Moreover, $\langle f(x,t,\varepsilon),\ \psi(x)\rangle$ is a continuous function in t, and the estimate $O(\varepsilon^\alpha)$ is understood in the standard sense being uniform with respect to t. The relation $o_{\mathcal{D}'}(\varepsilon^{\alpha})$, $\varepsilon \to +0$ is understood in a corresponding way.

We will construct a *weak asymptotic solution* to the Cauchy problem (1.7), (1.9) in the form of the sum of the singular ansatz regularized *with respect to singularities* and *corrections*:

$$\begin{aligned} u(x,t,\varepsilon) &= \widetilde{u}(x,t,\varepsilon) + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= \widetilde{v}(x,t,\varepsilon) + R_v(x,t,\varepsilon), \\ w(x,t,\varepsilon) &= \widetilde{w}(x,t,\varepsilon) + R_w(x,t,\varepsilon). \end{aligned}$$

Here, a triple of functions $\big(\widetilde{u}(x,t,\varepsilon), \widetilde{v}(x,t,\varepsilon), \widetilde{w}(x,t,\varepsilon)\big)$ is a regularization of the singular ansatz (1.12) with respect to singularities $H(-x+\phi(t))$, $\delta(-x+\phi(t))$, $\delta'(-x+\phi(t))$, and the *corrections* $R_u(x,t,\varepsilon)$, $R_v(x,t,\varepsilon)$, $R_w(x,t,\varepsilon)$ are the desired functions, which must admit the estimates:

$$R_j(x,t,\varepsilon) = o_{\mathcal{D}'}(1), \quad \frac{\partial R_j(x,t,\varepsilon)}{\partial t} = o_{\mathcal{D}'}(1), \quad \varepsilon \to +0, \qquad j = u, v, w. \tag{1.13}$$

Let us note that choosing the corrections is an essential part of the "right" construction of the *weak asymptotic solution* [**3**], [**4**]– [**9**], [**19**], [**20**].

In order to construct a regularization $f(x,\varepsilon)$ of a distribution $f(x) \in \mathcal{D}'(\mathbb{R})$ we use the representation

$$f(x,\varepsilon) = f(x) * \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right), \quad \varepsilon > 0, \tag{1.14}$$

where $*$ is a convolution, and the mollifier $\omega(\eta)$ has the following properties: (a) $\omega(\eta) \in C^{\infty}(\mathbb{R})$, (b) $\omega(\eta)$ has a compact support or decreases sufficiently rapidly, as $|\eta| \to \infty$, (c) $\int \omega(\eta)\,d\eta = 1$, (d) $\omega(\eta) \ge 0$, (e) $\omega(-\eta) = \omega(\eta)$. It is known that $\lim\limits_{\varepsilon\to+0} \langle f(x,\varepsilon), \varphi(x)\rangle = \langle f(x), \varphi(x)\rangle$ for all $\varphi(x) \in \mathcal{D}(\mathbb{R})$.

Thus we will seek a weak asymptotic solution of the Cauchy problem (1.7), (1.9) in the form:

$$\begin{aligned} u(x,t,\varepsilon) &= u_0(x,t) + u_1(x,t)H_u(-x+\phi(t),\varepsilon) + R_u(x,t,\varepsilon), \\ v(x,t,\varepsilon) &= v_0(x,t) + v_1(x,t)H_v(-x+\phi(t),\varepsilon) \\ &\qquad\qquad + e(t)\delta_v(-x+\phi(t),\varepsilon) + R_v(x,t,\varepsilon), \\ w(x,t,\varepsilon) &= w_0(x,t) + w_1(x,t)H_w(-x+\phi(t),\varepsilon) \\ &\quad + g(t)\delta_w(-x+\phi(t),\varepsilon) + h(t)\delta'_w(-x+\phi(t),\varepsilon) \\ &\qquad\qquad + R_w(x,t,\varepsilon), \end{aligned} \tag{1.15}$$

where, according to (1.14),

$$\delta_v(\xi,\varepsilon) = \frac{1}{\varepsilon}\omega_e\Big(\frac{\xi}{\varepsilon}\Big), \qquad \delta_w(\xi,\varepsilon) = \frac{1}{\varepsilon}\omega_g\Big(\frac{\xi}{\varepsilon}\Big) \tag{1.16}$$

are regularizations of the δ-function,

$$\delta'_w(\xi,\varepsilon) = \frac{1}{\varepsilon^2}\omega'_h\Big(\frac{\xi}{\varepsilon}\Big) \tag{1.17}$$

is a regularization of the distribution δ', and

$$H_j(\xi,\varepsilon)=\omega_{0j}\Big(\frac{\xi}{\varepsilon}\Big)=\int_{-\infty}^{\xi/\varepsilon}\omega_j(\eta)\,d\eta \tag{1.18}$$

are regularizations of the Heaviside function $H(\xi)$, where $\omega_{0j}(z)\in C^\infty(\mathbb{R})$, and $\lim_{z\to+\infty}\omega_{0j}(z)=1$, $\lim_{z\to-\infty}\omega_{0j}(z)=0$, $j=u,v,w$. Here mollifiers ω_e, ω_g, ω_h, ω_j, $j=u,v,w$ have the properties (a)–(e).

2. δ-Shock wave type solutions.

2.1. Generalized solution and the Rankine–Hugoniot conditions. Suppose that $\Gamma=\{\gamma_i : i\in I\}$ is a graph in the upper half-plane $\{(x,t): x\in\mathbb{R},\ t\in[0,\infty)\}\in\mathbb{R}^2$ containing smooth arcs γ_i, $i\in I$, and I is a finite set. By I_0 we denote a subset of I such that an arc γ_k for $k\in I_0$ starts from the points of the x-axis. Denote by $\Gamma_0=\{x_k^0 : k\in I_0\}$ the set of initial points of arcs γ_k, $k\in I_0$.

Consider *δ-shock wave type initial data* $(u^0(x),v^0(x))$, where

$$v^0(x)=V^0(x)+e^0\delta(\Gamma_0),$$

$u^0,V^0\in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, $e^0\delta(\Gamma_0)=\sum_{k\in I_0}e_k^0\delta(x-x_k^0)$, e_k^0 are constants, $k\in I_0$.

Definition 2.1. ([**7**]– [**9**]) A pair of distributions $(u(x,t),v(x,t))$ and a graph Γ, where $v(x,t)$ is represented in the form of the sum

$$v(x,t)=V(x,t)+e(x,t)\delta(\Gamma),$$

$u,V\in L^\infty\big(\mathbb{R}\times(0,\,\infty);\mathbb{R}\big)$, $e(x,t)\delta(\Gamma)=\sum_{i\in I}e_i(x,t)\delta(\gamma_i)$, $e_i(x,t)\in C(\Gamma)$, $i\in I$, is called a *generalized δ-shock wave type solution* of system (1.3) with the *δ-shock wave type initial data* $(u^0(x),v^0(x))$ if the integral identities

$$\begin{aligned}
\int_0^\infty\!\!\int\Big(u\varphi_t+F(u,V)\varphi_x\Big)\,dx\,dt+\int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty\!\!\int\Big(V\varphi_t+G(u,V)\varphi_x\Big)\,dx\,dt \qquad\qquad&\\
+\sum_{i\in I}\int_{\gamma_i}e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl \qquad&\\
+\int V^0(x)\varphi(x,0)\,dx+\sum_{k\in I_0}e_k^0\varphi(x_k^0,0) &= 0,
\end{aligned} \tag{2.1}$$

hold for all test functions $\varphi(x,t)\in\mathcal{D}(\mathbb{R}\times[0,\,\infty))$, where $\dfrac{\partial\varphi(x,t)}{\partial\mathbf{l}}$ is the tangential derivative on the graph Γ, $\int_{\gamma_i}\cdot\,dl$ is the line integral over the arc γ_i.

Suppose that arcs of the graph $\Gamma=\{\gamma_i : i\in I\}$ have the form $\gamma_i=\{(x,t): x=\phi_i(t)\}$, $i\in I$ and $\mathbf{n}=(\nu_1,\nu_2)$ is the unit oriented normal to the curve γ_i. In this case

$$\mathbf{n}=(\nu_1,\nu_2)=\frac{1}{\sqrt{1+(\dot\phi_i(t))^2}}\big(1,-\dot\phi_i(t)\big), \qquad \mathbf{l}=(-\nu_2,\nu_1), \tag{2.2}$$

and

$$\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\Big|_{\gamma_i}=\frac{\varphi_t(\phi_i(t),t)+\dot\phi_i(t)\varphi_x(\phi_i(t),t)}{\sqrt{1+(\dot\phi_i(t))^2}}$$

$$=\frac{1}{\sqrt{1+(\dot\phi_i(t))^2}}\frac{d\varphi(\phi_i(t),t)}{dt}. \tag{2.3}$$

Theorem 2.1. ([**20**]– [**22**]) *Let us assume that $\Omega\subset\mathbb{R}\times(0,\ \infty)$ is some region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_{\mp}$, $(u(x,t),v(x,t))$ and Γ is a generalized δ-shock wave type solution of system* (1.3)*, and $(u(x,t),v(x,t))$ is smooth in $\Omega_{\pm}$. Then the Rankine–Hugoniot conditions for δ-shocks*

$$\begin{array}{rcl}\big[F(u,v)\big]_{\Gamma}\nu_1+\big[u\big]_{\Gamma}\nu_2&=&0,\\ \big[G(u,v)\big]_{\Gamma}\nu_1+\big[v\big]_{\Gamma}\nu_2&=&\frac{\partial e(x,t)|_{\Gamma}}{\partial\mathbf{l}},\end{array} \tag{2.4}$$

hold along Γ, where $\mathbf{n}=(\nu_1,\nu_2)$ is the unit normal to the curve Γ pointing from Ω_- into Ω_+, $\mathbf{l}=(-\nu_2,\nu_1)$ is the tangential vector to Γ,

$$\big[h(u,v)\big]=h(u_-,v_-)-h(u_+,v_+)$$

is, as usual, a jump in function $h(u(x,t),v(x,t))$ across the discontinuity curve Γ, $(u_{\mp},v_{\mp})$ are respective left- and right-hand values of (u,v) on the discontinuity curve.

If $\Gamma=\{(x,t):x=\phi(t)\}$, $\Omega_{\pm}=\{(x,t):\pm(x-\phi(t))>0\}$ then relations (2.4) *can be rewritten as*

$$\begin{array}{rcl}\dot\phi(t)&=&\frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)},\\ \dot e(t)&=&\Big([G(u,v)]-[v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)},\end{array} \tag{2.5}$$

where $e(t)\stackrel{def}{=}e(\phi(t),t)$ and $\dot{(\cdot)}=\frac{d}{dt}(\cdot)$.

Proof. Selecting the test function $\varphi(x,t)$ with compact support in $\Omega_{\pm}$, we deduce from (2.1) that (1.3) hold in $\Omega_{\pm}$, respectively. Now, choosing a test function $\varphi(x,t)$ with support in Ω, we deduce from the first identity (2.1) that

$$0=\int_0^\infty\int\Big(u\varphi_t+F(u,V)\varphi_x\Big)\,dx\,dt$$

$$=\int\int_{\Omega_-}\Big(u\varphi_t+F(u,V)\varphi_x\Big)\,dx\,dt+\int\int_{\Omega_+}\Big(u\varphi_t+F(u,V)\varphi_x\Big)\,dx\,dt.$$

Next, integrating by parts, we obtain

$$\begin{aligned}
&\int\int_{\Omega_\pm} \Big(u\varphi_t + F(u,V)\varphi_x\Big)\,dx\,dt \\
&= -\int\int_{\Omega_\pm} \Big(u_t + \big(F(u,V)\big)_x\Big)\varphi\,dx\,dt \mp \int_\Gamma \Big(\nu_2 u_\pm + \nu_1 F(u_\pm, v_\pm)\Big)\varphi\,dl \\
&= \mp \int_\Gamma \Big(\nu_2 u_\pm + \nu_1 F(u_\pm, v_\pm)\Big)\varphi\,dl,
\end{aligned}$$

owing to (1.3). Adding the last relations, we have

$$0 = \int_0^\infty \int \Big(u\varphi_t + F(u,V)\varphi_x\Big)\,dx\,dt = \int_\Gamma \Big(\big[F(u,v)\big]\nu_1 + \big[u\big]\nu_2\Big)\varphi(x,t)\,dl$$

for all $\varphi(x,t) \in \mathcal{D}(\Omega)$. This implies the first relation (2.4).

In the same way as above, we obtain the equality

$$\int_0^\infty \int \Big(V\varphi_t + G(u,V)\varphi_x\Big)\,dx\,dt = \int_\Gamma \Big(\big[G(u,v)\big]\nu_1 + \big[v\big]\nu_2\Big)\varphi(x,t)\,dl. \tag{2.6}$$

Now integrating by parts we can easily see that

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = -\int_\Gamma \frac{\partial e(x,t)}{\partial \mathbf{l}}\varphi(x,t)\,dl, \tag{2.7}$$

where $\frac{\partial e(x,t)}{\partial \mathbf{l}}\big|_\Gamma = \big(\frac{\partial e(x,t)}{\partial t}\nu_1 - \frac{\partial e(x,t)}{\partial x}\nu_2\big)\big|_\Gamma$.

Adding (2.6) and (2.7), we deduce

$$\int_\Gamma \left(\big[G(u,v)\big]\nu_1 + \big[v\big]\nu_2 - \frac{\partial e(x,t)}{\partial \mathbf{l}}\right)\varphi(x,t)\,dl = 0$$

for all $\varphi(x,t) \in \mathcal{D}(\Omega)$. Thus the second relation (2.4) holds.

If $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$ and $\Omega_\pm = \{(x,t) \in \Omega : \pm(x-\phi(t)) > 0\}$, taking into account that $\mathbf{n}$ and $\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\Big|_\Gamma$ are given by (2.2) and (2.3), conditions (2.4) can be rewritten in the form (2.5). □

The first equation (2.4) (or (2.5)) is the *standard* Rankine–Hugoniot condition. The left-hand side of the second equation in (2.4) (or the right-hand side of the second equation in (2.5)) is called the *Rankine–Hugoniot deficit*.

The system of δ-*shocks* integral identities (2.1) is a *natural generalization* of the *usual* system of integral identities (1.2) (in the case of $m = 2$). The integral identities (2.1) differ from the integral identities (1.2) (in the case of $m = 2$) by the additional term

$$\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl = \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial \mathbf{l}}\,dl$$

in the second identity. This term appears due to the so-called *Rankine–Hugoniot deficit*.

2.2. Geometrical sense of δ-shock Rankine–Hugoniot conditions. It is well known that if a pair of compactly supported functions $\big(u(x,t), v(x,t)\big) \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}^2\big)$ with respect to x is a generalized solution of system (1.3) then integrals of the solution on the whole space

$$\int u(x,t)\,dx = \int u^0(x)\,dx, \qquad \int v(x,t)\,dx = \int v^0(x)\,dx, \qquad t \ge 0 \tag{2.8}$$

(that is, the total area, mass, momentum, energy, etc.) are independent of time, where $(u^0(x), v^0(x))$ is initial data.

For a δ-shock wave type solution this fact *does not hold*. However, there is a "generalized" analog of conservation laws (2.8).

Denote by

$$\begin{aligned} S_u(t) &= \textstyle\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx, \\ S_v(t) &= \textstyle\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx, \\ S_u(0) &= \textstyle\int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ S_v(0) &= \textstyle\int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx, \end{aligned} \tag{2.9}$$

the areas under the graphs $y = u(x,t)$, $y = V(x,t)$, and $y = u^0(x)$, $y = V^0(x)$, respectively, where $x = \phi(t)$ is a line in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\}$ issued from $\phi(0) = 0$.

THEOREM 2.2. ([**20**], [**22**]) *Let the pair of distributions $\big(u(x,t), v(x,t)\big)$ be a generalized δ-shock wave type solution of the Cauchy problem* (1.3) *with δ-shock wave type initial data, where $v(x,t) = V(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ is the discontinuity line, and $u(x,t)$, $V(x,t)$ are compactly supported functions with respect to x. Then the following balance relations hold:*

$$\dot S_u(t) = 0, \qquad \dot S_v(t) = -\dot e(t), \tag{2.10}$$

where $S_u(t)$, $S_v(t)$ given by (2.9), *and*

$$\dot e(t) = \Big([G(u,v)] - [v]\frac{[F(u,v)]}{[u]}\Big)\Big|_{x=\phi(t)}$$

is the Rankine–Hugoniot deficit. Thus,

$$\begin{aligned} &\int_{-\infty}^{\phi(t)} u(x,t)\,dx + \int_{\phi(t)}^{+\infty} u(x,t)\,dx \\ &\qquad\qquad = \int_{-\infty}^{0} u^0(x)\,dx + \int_{0}^{+\infty} u^0(x)\,dx, \\ &\int_{-\infty}^{\phi(t)} v(x,t)\,dx + \int_{\phi(t)}^{+\infty} v(x,t)\,dx + e(t) \\ &\qquad\qquad = \int_{-\infty}^{0} v^0(x)\,dx + \int_{0}^{+\infty} v^0(x)\,dx + e^0, \end{aligned} \tag{2.11}$$

where e^0 is the initial amplitude of the δ-function.

Proof. Let us prove the second relation (2.10). Let $v_{\pm} = \lim_{x\to\phi(t)\pm 0} v(x,t)$ denote the right- and left-hand sides value of $v(x,t)$ on the curve Γ. Differentiating the second relation (2.9) and using the second equation of system (1.3), we obtain

$$\dot{S}_v(t) = v_-\dot{\phi}(t) - v_+\dot{\phi}(t) + \int_{-\infty}^{\phi(t)} v_t(x,t)\,dx + \int_{\phi(t)}^{+\infty} v_t(x,t)\,dx$$

$$= [v]\Big|_{x=\phi(t)}\dot{\phi}(t) - \int_{-\infty}^{\phi(t)} \big(G(u,v)\big)_x\,dx - \int_{\phi(t)}^{+\infty} \big(G(u,v)\big)_x\,dx$$

$$= [v]\Big|_{x=\phi(t)}\dot{\phi}(t) - [G(u,v)]\Big|_{x=\phi(t)}$$

$$+G\big(u(-\infty,t),v(-\infty,t)\big) - G\big(u(+\infty,t),v(+\infty,t)\big).$$

Taking into account that

$$G\big(u(-\infty,t),v(-\infty,t)\big) = G\big(u(+\infty,t),v(+\infty,t)\big) = G(0,0)$$

and using the Rankine–Hugoniot conditions (2.5), we obtain

$$\dot{S}_v(t) = \bigg([v]\frac{[F(u,v)]}{[u]}\Big|_{x=\phi(t)} - [G(u,v)]\bigg)\Big|_{x=\phi(t)}.$$

The first relation (2.10) is the well-known relation for $L^1\cap L^\infty$-generalized solutions of conservation laws. The proof of this relation is carried out in the same way. Integrating expressions (2.10), we obtain (2.11). $\square$

From the second relation (2.11), we can see that the sense of amplitude $e(t)$ of δ function is the *"area" of the discontinuity line*. Moreover, the *"total area" $S_v(t)+e(t)$ is independent of time.*

3. δ'-shock wave type solutions.

3.1. Some misty reasoning. Let us consider a scalar conservation law

$$u_t + \big(f(u)\big)_x = 0, \tag{3.1}$$

where $f''(u) > 0$. Differentiating this equation with respect to x and denoting $v = u_x$, we obtain the another equation $v_t + \big(f'(u)v\big)_x = 0$. The pair of equations (1.5) constitutes so-called 2×2 "prolonged system".

Now consider the Cauchy problem for system (1.5) with the singular piecewise smooth initial data

$$u^0(x) = u^0_0(x) + u^0_1(x)H(-x), \quad v^0(x) = v^0_0(x) + v^0_1(x)H(-x) + e^0\delta(-x), \tag{3.2}$$

where $u^0_k(\mathrm{x})$, $v^0_k(x)$, $k=0,1$ are given smooth functions, e^0 is a given constant.

In [**6**]– [**9**] the following theorem was proved (see also [**11**], [**13**]).

THEOREM 3.1. *There is $T > 0$ such that for $t \in [0, T)$ the Cauchy problem* (1.5), (3.2)*, has a unique δ-shock wave type solution*

$$
\begin{aligned}
u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\
v(x,t) &= v_+(x,t) + [v(x,t)]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)),
\end{aligned} \tag{3.3}
$$

which satisfies the integral identities (2.1)*, where $\Gamma = \{(x,t) : x = \phi(t), \quad t \in [0, T)\}$, $V(x,t) = v_+ + [v]H(-x+\phi(t))$. Here functions $u_+(x,t)$, $v_+(x,t)$, $u_-(x,t) = u_+(x,t) + [u(x,t)]$, $v_-(x,t) = v_+(x,t) + [v(x,t)]$, and $\phi(t)$, $e(t)$ are defined by system*

$$
\begin{aligned}
(u_\pm)_t + \big(f(u_\pm)\big)_x &= 0, \quad \pm x > \pm\phi(t), \\
(v_\pm)_t + \big(f'(u_\pm)v_\pm\big)_x &= 0, \quad \pm x > \pm\phi(t), \\
\dot{\phi}(t) &= \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\
\dot{e}(t) &= \Big([vf'(u)] - [v]\frac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)},
\end{aligned} \tag{3.4}
$$

with the initial data determined by (3.2)*, $\phi(0) = 0$.*

Here, according to the results of Sec. 2, the third and fourth relations in (3.4) are the *Rankine–Hugoniot conditions for δ-shock*.

Now we consider the Cauchy problem for the first equation in system (1.5):

$$
u_t + \big(f(u)\big)_x = 0, \qquad u^0(x) = u_0^0(x) + u_1^0(x)H(-x). \tag{3.5}
$$

This Cauchy problem has a solution in the *shock wave* form

$$
u(x,t) = u_+(x,t) + [u(x,t)]H(-x+\phi(t)),
$$

where, according to the Rankine–Hugoniot condition,

$$
\dot{\phi}(t) = \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}. \tag{3.6}
$$

Equality (3.6) is the first Rankine–Hugoniot condition for δ-shock in (3.4).

Taking into account the fact that the second equation in system (1.5) was derived by differentiating of the first equation (in system (1.5)), one may expect that the second component v in (3.3) is obtained by differentiating $v = u_x$. However, in this way one can construct only a particular case of solution (3.3), (3.4) of the Cauchy problem (1.5), (3.2), i.e.,

$$
\begin{aligned}
u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\
v(x,t) &= u_x(x,t) = u_{+x}(x,t) + [u_x(x,t)]H(-x+\phi(t)) \\
&\qquad - [u(x,t)]\delta(-x+\phi(t)).
\end{aligned} \tag{3.7}
$$

Thus we construct the solution of the Cauchy problem (1.5), (3.2) such that

$$
v_+(x,t) = u_{+x}(x,t), \quad [v(x,t)] = [u_x(x,t)], \quad e(t) = -[u(x,t)]\big|_{x=\phi(t)}.
$$

Moreover, in this way we can derive the last relation in (3.4), i.e., the second Rankine–Hugoniot condition for δ-shock. Namely, from equalities

$$(u_\pm)_t + f'(u_\pm)v_\pm = 0, \qquad \pm x > \pm\phi(t)$$

it follows that $[u_t]\big|_{x=\phi(t)} = -[f'(u)v]\big|_{x=\phi(t)}$ and

$$\dot{e}(t) = -\frac{d[u(\phi(t),t)]}{dt} = -\big([u_t] + \dot{\phi}(t)[u_x]\big)\big|_{x=\phi(t)}$$

$$= \big([f'(u)v] - \dot{\phi}(t)[v]\big)\big|_{x=\phi(t)}. \tag{3.8}$$

Above *misty reasoning* is useful as a start for constructing δ'-shock type solutions. Differentiating the scalar conservation law (3.1) twice with respect to x and denoting $v = u_x$, $w = v_x = u_{xx}$, we obtain the 3×3 "prolonged system" (1.7).

Clearly, for the smooth solution $u(x,t) \in C^2$ of (3.1) in some domain Ω the triple (u, v, w) is a generalized solution of the prolonged system (1.7) in the same domain Ω.

Now we consider a piecewise smooth generalized solution $u(x,t)$ of (3.1), which contains a single shock $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$, and has the form

$$u(x,t) = u_+(x,t) + [u(x,t)]H(-x+\phi(t)).$$

as above. Denote $\Omega = \mathbb{R}\times(0,+\infty)$, $\Omega_\pm = \{(x,t) \in \Omega : \pm(x-\phi(t)) > 0\}$ and suppose that $u_\pm \in C^2(\Omega)$. Denote by $v_\pm = (u_\pm)_x$, $w_\pm = (v_\pm)_x = (u_\pm)_{xx}$ the derivatives of $u_\pm$, and define functions

$$\begin{aligned} V(x,t) &= v_+(x,t) + [v(x,t)]H(-x+\phi(t)), \\ W(x,t) &= w_+(x,t) + [w(x,t)]H(-x+\phi(t)). \end{aligned}$$

Observe firstly that $\dot{\phi}(t) = \frac{[f(u)]}{[u]}$ by the "standard" Rankine-Hugoniot condition for scalar equation (3.1). Further, as it is easy to see, in the sense of distributions on Ω

$$\begin{aligned} u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\ v(x,t) &= u_x = V(x,t) + e(t)\delta(-x+\phi(t)), \\ w(x,t) &= v_x = W(x,t) + g(t)\delta(-x+\phi(t)) + h(t)\delta'(-x+\phi(t)), \end{aligned} \tag{3.9}$$

where

$$e(t) = -[u(\phi(t),t)], \quad g(t) = -[v(\phi(t),t)], \quad h(t) = [u(\phi(t),t)]. \tag{3.10}$$

In this case, according to (3.6), (3.8),

$$\begin{aligned} \dot{\phi}(t) &= \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot{e}(t) &= \big([f'(u)v] - \dot{\phi}(t)[v]\big)\big|_{x=\phi(t)}. \end{aligned} \tag{3.11}$$

Similarly as above, from the equalities

$$(v_\pm)_t + f''(u_\pm)(v_\pm)^2 + f'(u_\pm)w_\pm = 0, \qquad \pm x > \pm\phi(t)$$

it follows that $[v_t]\big|_{x=\phi(t)} = -[f''(u)v^2 + f'(u)w]\big|_{x=\phi(t)}$ and

$$\dot g(t) = -\frac{d[v(\phi(t),t)]}{dt} = -\big([v_t] + \dot\phi(t)[v_x]\big)\big|_{x=\phi(t)}$$

$$= \big([f''(u)v^2 + f'(u)w] - \dot\phi(t)[w]\big)\big|_{x=\phi(t)}. \tag{3.12}$$

We also see that

$$h(t) = \frac{(-[u(\phi(t),t)])^2}{[u(\phi(t),t)]} = \frac{e^2(t)}{[u]}\bigg|_{x=\phi(t)}. \tag{3.13}$$

Since $v = u_x$, $w = v_x$, it is natural to expect that the triple (u, v, w) must be a *generalized δ'-shock type solution* of the prolonged system (1.7). Thus the "right" class of generalized solutions must involve solutions of the form (3.9)–(3.10). Here the equalities (3.11), (3.12), and a direct consequence of the equality (3.13), i.e.,

$$\begin{array}{rcl} \dot\phi(t) &=& \frac{[f(u)]}{[u]}\Big|_{x=\phi(t)}, \\ \dot e(t) &=& \big([f'(u)v] - \dot\phi(t)[v]\big)\big|_{x=\phi(t)}, \\ \dot g(t) &=& \big([f''(u)v^2 + f'(u)w] - \dot\phi(t)[w]\big)\big|_{x=\phi(t)}, \\ \dfrac{d}{dt}\Big(h(t)[u(\phi(t),t)]\Big) &=& \dfrac{de^2(t)}{dt} \end{array} \tag{3.14}$$

must constitute *δ'-shock Rankine–Hugoniot conditions* in the case of solution (3.9)–(3.10). But we ask: In which sense the δ'-shock type solution (3.9)–(3.10) satisfies the nonlinear system (1.7)?

Below we give the following definition of a *δ'-shock type solution* for system (1.7).

3.2. Generalized solution and the Rankine–Hugoniot conditions. Let $\Gamma = \{\gamma_i : i \in I\}$ be a graph introduced in Sec. 2. Initial data $(u^0(x), v^0(x), w^0(x))$, where

$$v^0(x) = V^0(x) + e^0\delta(\Gamma_0), \quad w^0(x) = W^0(x) + g^0\delta(\Gamma_0) + h^0\delta'(\Gamma_0),$$

and $u^0, V^0, W^0 \in L^\infty\big(\mathbb{R};\mathbb{R}\big)$, we call *$\delta'$-shock wave type initial data*. Here, by definition, $e^0\delta(\Gamma_0) \stackrel{def}{=} \sum_{k\in I_0} e^0_k\delta(x-x^0_k)$, $g^0\delta(\Gamma_0) \stackrel{def}{=} \sum_{k\in I_0} g^0_k\delta(x-x^0_k)$, $h^0\delta(\Gamma_0) \stackrel{def}{=} \sum_{k\in I_0} h^0_k\delta'(x-x^0_k)$, where e^0_k, g^0_k, h^0_k are constants, $k \in I_0$.

DEFINITION 3.1. A triple of distributions $\big(u(x,t), v(x,t), w(x,t)\big)$ and graph Γ, where $v(x,t)$ and $w(x,t)$ are represented in the form of the sums

$$v(x,t) = V(x,t) + e(x,t)\delta(\Gamma), \quad w(x,t) = W(x,t) + g(x,t)\delta(\Gamma) + h(x,t)\delta'(\Gamma),$$

where $u, V, W \in L^\infty\big(\mathbb{R}\times(0,\infty);\mathbb{R}\big)$,

$$\begin{array}{rcl} e(x,t)\delta(\Gamma) &\stackrel{def}{=}& \sum_{i\in I} e_i(x,t)\delta(\gamma_i), \\ g(x,t)\delta(\Gamma) &\stackrel{def}{=}& \sum_{i\in I} g_i(x,t)\delta(\gamma_i), \\ h(x,t)\delta'(\Gamma) &\stackrel{def}{=}& \sum_{i\in I} h_i(x,t)\delta'(\gamma_i), \end{array}$$

and $e_i(x,t), g_i(x,t), h_i(x,t) \in C^1(\Gamma)$, $i \in I$, is called a *generalized δ'-shock wave type solution* of system (1.7) with the *δ'-shock wave type* initial data $(u^0(x), v^0(x), w^0(x))$ if the integral identities

$$
\begin{aligned}
\int_0^\infty \int \Big(u\varphi_t + f(u)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^\infty \int V\Big(\varphi_t + f'(u)\varphi_x\Big)\,dx\,dt + \sum_{i\in I}\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl &\\
+\int V^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} e_k^0\varphi(x_k^0,0) &= 0,\\
\int_0^\infty \int \Big(W\varphi_t + \big(f''(u)V^2 + f'(u)W\big)\varphi_x\Big)\,dx\,dt &\\
+\sum_{i\in I}\bigg(\int_{\gamma_i} g_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl &\\
+\int_{\gamma_i} h_i(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl + \int_{\gamma_i}\frac{\frac{\partial e_i^2(x,t)}{\partial\mathbf{l}} - h_i(x,t)\frac{\partial[u(x,t)]}{\partial\mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl\bigg) &\\
+\int W^0(x)\varphi(x,0)\,dx + \sum_{k\in I_0} g_k^0\varphi(x_k^0,0) + \sum_{k\in I_0} h_k^0\varphi_x(x_k^0,0) &= 0,
\end{aligned}
\tag{3.15}
$$

hold for all test functions $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\,\infty))$.

In the same way as in Theorem 2.1 we can derive explicit formulas for the Rankine-Hugoniot conditions (including the deficits) from integral identity (3.15). For simplicity assume that the graph Γ has the single wave form: $\Gamma = \{(x,t) : x = \phi(t)\}$, $\phi(t) \in C^1(0,+\infty)$ and $\Omega_\pm = \{(x,t) \in \Omega : \pm(x-\phi(t)) > 0\}$.

THEOREM 3.2. *Let us assume that $\Omega \subset \mathbb{R}\times(0,\,\infty)$ is some region cut by a smooth curve Γ into a left- and right-hand parts $\Omega_\mp$, $(u(x,t), v(x,t), w(x,t))$ and Γ is a generalized δ'-shock wave type solution of system* (1.7), *and functions $(u(x,t), V(x,t), W(x,t))$ are smooth in the domains $\Omega_\pm$ and have one-sided limits $u_\pm$, $V_\pm$, $W_\pm$ on the curve Γ, which are supposed to be continuous functions on Γ. Then the Rankine–Hugoniot conditions for δ'-shocks*

$$\dot\phi(t) = \frac{[f(u)]}{[u]}\bigg|_{x=\phi(t)}, \tag{3.16}$$

$$\dot e(t) = \Big([f'(u)v] - [v]\frac{[f(u)]}{[u]}\Big)\bigg|_{x=\phi(t)}, \tag{3.17}$$

$$\dot g(t) = \Big([f''(u)v^2 + f'(u)w] - [w]\frac{[f(u)]}{[u]}\Big)\bigg|_{x=\phi(t)}, \tag{3.18}$$

$$\frac{d}{dt}\Big(h(t)[u(\phi(t),t)]\Big) = \frac{de^2(t)}{dt} \tag{3.19}$$

hold along Γ. *Here the functions* e, g, h *can be treated as functions of the single variable* t, *so that* $e(t) \stackrel{def}{=} e(\phi(t), t)$, $g(t) \stackrel{def}{=} g(\phi(t), t)$, $h(t) \stackrel{def}{=} h(\phi(t), t)$.

PROOF. Setting $F(u, v) = f(u)$, $G(u, v) = f'(u)v$ and repeating the proof of Theorem 2.1 word for word, we prove conditions (3.16) and (3.17).

The proof of conditions (3.18) and (3.19) is carried out in the same way as the proofs of conditions (3.16) and (3.17). Since the triple (u, V, W) is a smooth solution of the system (1.7) in the domains $\Omega_{\pm}$, applying (3.15) to a test function $\varphi(x, t) \in \mathcal{D}(\mathbb{R} \times (0, +\infty))$, taking into account (2.2), (2.3), and integrating by parts, we obtain the identity

$$\begin{aligned}
0 &= \int_{\Omega} \Big(W\varphi_t + \big(f''(u)V^2 + f'(u)W\big)\varphi_x \Big)\, dx\, dt + \int_{\Gamma} g(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\, dl \\
&\qquad + \int_{\Gamma} h(x,t)\frac{\partial \varphi_x(x,t)}{\partial \mathbf{l}}\, dl + \int_{\Gamma} \frac{\partial}{\partial \mathbf{l}}\Big(\frac{e^2(x,t)}{[u]}\Big)\varphi_x(x,t)\, dl \\
&= \int_{\Gamma} \Big(\big[W\big]\nu_2 + \big[f''(u)V^2 + f'(u)W\big]\nu_1 \Big)\varphi(x,t)\, dl + \int_{\Gamma} g(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\, dl \\
&\qquad + \int_{\Gamma} h(x,t)\frac{\partial \varphi_x(x,t)}{\partial \mathbf{l}}\, dl + \int_{\Gamma} \frac{\frac{\partial e^2(x,t)}{\partial \mathbf{l}} - h(x,t)\frac{\partial [u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\, dl \\
&= \int_0^{\infty} \Big(-\big[W\big]\dot{\phi}(t) + \big[f''(u)V^2 + f'(u)W\big] \Big)\varphi(\phi(t),t)\, dt \\
&\qquad + \int_0^{\infty} g(t)\frac{d\varphi(\phi(t),t)}{dt}\, dt \\
&\qquad + \int_0^{\infty} h(t)\frac{d\varphi_x(\phi(t),t)}{dt}\, dt + \int_0^{\infty} \frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]}\varphi_x(\phi(t),t)\, dt \\
&= \int_0^{\infty} \Big(-\big[W\big]\dot{\phi}(t) + \big[f''(u)V^2 + f'(u)W\big] - \dot{g}(t) \Big)\varphi(\phi(t),t)\, dt \\
&\qquad + \int_0^{\infty} \Big(\frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]} - \dot{h}(t) \Big)\varphi_x(\phi(t),t)\, dt.
\end{aligned} \tag{3.20}$$

Here we take into account that integrating by parts, we can easily see that

$$\int_{\Gamma} \psi(x,t)\frac{\partial \varphi(x,t)}{\partial \mathbf{l}}\, dl = -\int_{\Gamma} \frac{\partial \psi(x,t)}{\partial \mathbf{l}}\varphi(x,t)\, dl = -\int_{\Gamma} \frac{d\psi(\phi(t),t)}{dt}\varphi(\phi(t),t)\, dt,$$

where $\psi(x, t)$ is a smooth function.

Since $\varphi(\phi(t), t)$, $\varphi_x(\phi(t), t)$ are arbitrary smooth functions, we conclude that conditions (3.18), (3.19) hold on the curve Γ.

The proof of Theorem 3.2 is complete. $\square$

REMARK 3.1. We can see that according to our expectations, the conditions (3.14) derived in Subsec. 3.1 for solution (3.9)–(3.10) coincide with the Rankine–Hugoniot conditions (3.16)–(3.19) for δ'-shock.

The equality (3.16) is the classical Rankine-Hugoniot condition, the right-hand sides of the equalities (3.17), (3.18) are the *first Rankine-Hugoniot deficits*, and the right-hand side of (3.19) is the *second Rankine-Hugoniot deficit*.

The integral identities (3.15) differ from classical integral identities (1.2) (in the case of $m = 3$) by additional terms in the second and third identities. Here the terms

$$\int_{\gamma_i} e_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl, \qquad \int_{\gamma_i} g_i(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl$$

appear due to the *first Rankine–Hugoniot deficit*, and the term

$$\int_{\gamma_i} h_i(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl + \int_{\gamma_i} \frac{\frac{\partial e_i^2(x,t)}{\partial\mathbf{l}} - h_i(x,t)\frac{\partial[u(x,t)]}{\partial\mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl$$

appears due to the *second Rankine–Hugoniot deficit*. Moreover, the first integral identity in (3.15) is a "standard" type integral identity (see (1.2)), while the first and second integral identities in (3.15) constitute δ-shock type integral identities (see Definition 2.1), and the third integral identity in (3.15) is a special type of δ'-shock type integral identity.

3.3. Geometrical sense of δ'-shock Rankine–Hugoniot conditions. Now we derive a "generalized" analog of conservation laws (2.8) for δ'-shock wave type solutions.

Denote by

$$\begin{aligned} S_w(t) &= \int_{-\infty}^{\phi(t)} w(x,t)\,dx + \int_{\phi(t)}^{+\infty} w(x,t)\,dx, \\ S_w(0) &= \int_{-\infty}^{0} w^0(x)\,dx + \int_{0}^{+\infty} w^0(x)\,dx \end{aligned} \tag{3.21}$$

the areas under the graphs $y = w(x,t)$ and $y = w^0(x)$, respectively, where $x = \phi(t)$ is a line in the upper half-plane $\{(x,t) : x \in \mathbb{R},\ t \in [0,\infty)\}$ issued from $\phi(0) = 0$.

Repeating the proof of Theorem 2.2 almost word for word, we obtain the following assertion.

THEOREM 3.3. *Let the triple of distributions* $\big(u(x,t), v(x,t), w(x,t)\big)$ *be a generalized δ'-shock wave type solution of the Cauchy problem* (1.7) *with δ-shock wave type initial data, where* $v(x,t) = V(x,t) + e(t)\delta(\Gamma)$, $\Gamma = \{(x,t) : x = \phi(t)\}$ *is the discontinuity line, and* $u(x,t)$, $V(x,t)$ *are compactly supported functions with respect to* x*. Then the following balance relations hold:*

$$\dot{S}_u(t) = 0, \quad \dot{S}_v(t) = -\dot{e}(t), \quad \dot{S}_w(t) = -\dot{g}(t), \tag{3.22}$$

where $S_u(t)$, $S_v(t)$, $S_w(t)$ *given by* (2.9), (3.21), *and*

$$\begin{aligned} \dot{e}(t) &= \Big([f'(u)v] - [v]\tfrac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \\ \dot{g}(t) &= \Big([f''(u)v^2 + f'(u)w] - [w]\tfrac{[f(u)]}{[u]}\Big)\Big|_{x=\phi(t)}, \end{aligned}$$

are the first Rankine–Hugoniot deficits given by (3.17)*,* (3.18)*. Thus*

$$
\begin{aligned}
&\int_{-\infty}^{\phi(t)} u(x,t)\,dx+\int_{\phi(t)}^{+\infty} u(x,t)\,dx \\
&\qquad\qquad =\int_{-\infty}^{0} u^0(x)\,dx+\int_{0}^{+\infty} u^0(x)\,dx, \\
&\int_{-\infty}^{\phi(t)} v(x,t)\,dx+\int_{\phi(t)}^{+\infty} v(x,t)\,dx+e(t) \\
&\qquad\qquad =\int_{-\infty}^{0} v^0(x)\,dx+\int_{0}^{+\infty} v^0(x)\,dx+e^0, \\
&\int_{-\infty}^{\phi(t)} w(x,t)\,dx+\int_{\phi(t)}^{+\infty} w(x,t)\,dx+g(t) \\
&\qquad\qquad =\int_{-\infty}^{0} w^0(x)\,dx+\int_{0}^{+\infty} w^0(x)\,dx+g^0,
\end{aligned}
\tag{3.23}
$$

where e^0 *and* g^0 *are initial amplitudes of the* δ*-functions in* v *and* w*, respectively.*

From relations (3.23), we see that the sense of amplitudes $e(t)$ and $g(t)$ of δ functions in v and w are the *"areas" of the discontinuity line*. Moreover, the *"total areas"* $S_v(t)+e(t)$ and $S_w(t)+g(t)$ *are independent of time*.

REMARK 3.2. The most unexpected result obtained by Theorem 3.3 is the fact that the *"area" balance relation* for w is independent of the *second Rankine-Hugoniot deficit*.

3.4. Weak asymptotic solution. Now we introduce the notion of a weak asymptotic solution, which is one of the most important in the weak asymptotics method.

DEFINITION 3.2. A triple of functions $\big(u(x,t,\varepsilon), v(x,t,\varepsilon), w(x,t,\varepsilon)\big)$ smooth as $\varepsilon>0$ is called a *weak asymptotic solution* of system (1.7) with the initial data $(u^0(x), v^0(x), w^0(x))$ if

$$
\begin{aligned}
\int L_1[u(x,t,\varepsilon)]\psi(x)\,dx &= o(1), \\
\int L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)]\psi(x)\,dx &= o(1), \\
\int L_3[u(x,t,\varepsilon), v(x,t,\varepsilon), w(x,t,\varepsilon)]\psi(x)\,dx &= o(1), \\
\int \Big(u(x,0,\varepsilon)-u^0(x)\Big)\psi(x)\,dx &= o(1), \\
\int \Big(v(x,0,\varepsilon)-v^0(x)\Big)\psi(x)\,dx &= o(1), \\
\int \Big(w(x,0,\varepsilon)-w^0(x)\Big)\psi(x)\,dx &= o(1), \quad \varepsilon\to+0,
\end{aligned}
$$

for all $\psi(x) \in \mathcal{D}(\mathbb{R})$. The last relations can be rewritten as

$$
\begin{array}{rcl}
L_1[u(x,t,\varepsilon)] & = & o_{\mathcal{D}'}(1), \\
L_2[u(x,t,\varepsilon), v(x,t,\varepsilon)] & = & o_{\mathcal{D}'}(1), \\
L_3[u(x,t,\varepsilon), v(x,t,\varepsilon), w(x,t,\varepsilon)] & = & o_{\mathcal{D}'}(1), \\
u(x,0,\varepsilon) & = & u^0(x) + o_{\mathcal{D}'}(1), \\
v(x,0,\varepsilon) & = & v^0(x) + o_{\mathcal{D}'}(1), \\
w(x,0,\varepsilon) & = & w^0(x) + o_{\mathcal{D}'}(1), \quad \varepsilon \to +0,
\end{array} \tag{3.24}
$$

where the first three estimates are uniform in t.

In (3.24), all distributions in u, v, w depend on t as a parameter.

Within the framework of the *weak asymptotics method*, we find a δ'-*shock wave type solution* of the Cauchy problem as a weak limit

$$
u(x,t) = \lim_{\varepsilon \to +0} u(x,t,\varepsilon), \quad v(x,t) = \lim_{\varepsilon \to +0} v(x,t,\varepsilon), \quad w(x,t) = \lim_{\varepsilon \to +0} w(x,t,\varepsilon), \tag{3.25}
$$

of the *weak asymptotic solution* $(u(x,t,\varepsilon), v(x,t,\varepsilon), w(x,t,\varepsilon))$ to this Cauchy problem.

Constructing the *weak asymptotic solution*, multiplying the first three relations (3.24) by a test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\infty))$, integrating these relations by parts and then passing to the limit as $\varepsilon \to +0$, we will see that the triple of distributions (1.12) satisfy the integral identities (3.15).

4. Propagation of δ'-shock in system (1.8)

4.1. Construction of a weak asymptotic solution. The first step of our approach is to find a *weak asymptotic solution* of the Cauchy problem (1.8), (1.9). In this case the graph Γ contains only one arc and has the form $\Gamma = \{(x,t) : x = \phi(t)\}$ and hence $e(x,t)\big|_{\Gamma} = e(t)$, $g(x,t)\big|_{\Gamma} = g(t)$, $h(x,t)\big|_{\Gamma} = h(t)$.

Here we choose the *corrections* in the special form

$$
\begin{array}{rcl}
R_u(x,t,\varepsilon) & = & 0, \\
R_v(x,t,\varepsilon) & = & 0, \\
R_w(x,t,\varepsilon) & = & P(t)\frac{1}{\varepsilon^2}\Omega'''_P\Big(\frac{-x+\phi(t)}{\varepsilon}\Big)
\end{array} \tag{4.1}
$$

where $P(t)$ is the desired function, $\frac{1}{\varepsilon^4}\Omega'''_P\big(x/\varepsilon\big)$ is a regularization of the distribution $\delta'''(x)$. Consequently, $R_w(x,t,\varepsilon) = \varepsilon^2 P(t)\delta'''_P(-x+\phi(t),\varepsilon) \in O_{\mathcal{D}'}(\varepsilon)$, i.e., estimates (1.13) hold.

Lemma 4.1. *Let* $\delta(x,\varepsilon) = \frac{1}{\varepsilon}\omega\Big(\frac{x}{\varepsilon}\Big)$, $\delta'(x,\varepsilon) = \frac{1}{\varepsilon^2}\omega'\Big(\frac{x}{\varepsilon}\Big)$, $\delta'''_P(x,\varepsilon) = \frac{1}{\varepsilon^4}\Omega'''_P\Big(\frac{x}{\varepsilon}\Big)$, *and* $H_j(x,\varepsilon) = \omega_{0j}\Big(\frac{x}{\varepsilon}\Big) = \int_{-\infty}^{\frac{x}{\varepsilon}} \omega_j(\eta)\,d\eta$ *be regularizations of the delta function* δ, δ', δ''', *and the Heaviside function* $H(x)$, $j = 1,2$, *respectively. Then we have the following* *weak asymptotic expansions:*

$$
\begin{array}{rcl}
\big(H_j(x,\varepsilon)\big)^r &=& H(x)+O_{\mathcal{D}'}(\varepsilon),\\
H_1(x,\varepsilon)H_2(x,\varepsilon) &=& H(x)+O_{\mathcal{D}'}(\varepsilon),\\
\Big(H_j(x,\varepsilon)\Big)^r\delta(x,\varepsilon) &=& \delta(x)\displaystyle\int \omega_{0j}^r(\eta)\omega(\eta)\,d\eta+O_{\mathcal{D}'}(\varepsilon),\\
\big(\delta(x,\varepsilon)\big)^2 &=& \dfrac{1}{\varepsilon}\delta(x)\displaystyle\int \omega^2(\eta)\,d\eta+O_{\mathcal{D}'}(\varepsilon),\\
H_j(x,\varepsilon)\delta'(x,\varepsilon) &=& -\dfrac{1}{\varepsilon}\delta(x)\displaystyle\int \omega_j(\eta)\omega(\eta)\,d\eta\\
&& +\delta'(x)\displaystyle\int \omega_{0j}(\eta)\omega(\eta)\,d\eta+O_{\mathcal{D}'}(\varepsilon),\\
H_j(x,\varepsilon)\varepsilon^2\delta_P'''(x,\varepsilon) &=& \dfrac{1}{\varepsilon}\delta(x)\displaystyle\int \omega_j'(\eta)\Omega_P'(\eta)\,d\eta+O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon\to+0,
\end{array}
\tag{4.2}
$$

$r=1,2,\dots;\ \ j,k=1,2.$

Proof. From (1.18), we obviously have the first two relations in (4.2).

Consider the asymptotics of the product $\Big(H_j(x,\varepsilon)\Big)^r\delta(x,\varepsilon)$. Using (1.16), (1.18), and making the change of variables $x=\varepsilon\eta$, we obtain

$$
\Big\langle\frac{1}{\varepsilon}\omega\big(\frac{x}{\varepsilon}\big)\Big(\omega_{0j}\big(\frac{x}{\varepsilon}\big)\Big)^r,\psi(x)\Big\rangle
$$
$$
=\int\omega_{0j}^r(\eta)\omega(\eta)\psi(\varepsilon\eta)\,d\eta=\psi(0)\int\omega_{0j}^r(\eta)\omega(\eta)\,d\eta+O(\varepsilon),\quad \varepsilon\to+0,
$$

for all $\psi(x)\in\mathcal{D}(\mathbb{R})$, i.e., the third relation is proved.

Analogously, making the change of variables $\xi=\varepsilon\eta$, we obtain

$$
\Big\langle\big(\delta(x,\varepsilon)\big)^2,\psi(x)\Big\rangle=\frac{1}{\varepsilon}\int\omega^2(\eta)\psi(\varepsilon\eta)\,d\eta
$$
$$
=\frac{\psi(0)}{\varepsilon}\int\omega^2(\eta)\,d\eta+\psi'(0)\int\omega^2(\eta)\eta\,d\eta+O(\varepsilon),\quad \varepsilon\to+0,
$$

where, according to property (e), $\int\omega^2(\eta)\eta\,d\eta=0$.

Next, we calculate the asymptotics of the product $H_j(x,\varepsilon)\delta'(x,\varepsilon)$. Using (1.17), (1.18), and making the change of variables $\xi=\varepsilon\eta$, we have

$$
\Big\langle H_j(x,\varepsilon)\delta'(x,\varepsilon),\psi(x)\Big\rangle=\frac{1}{\varepsilon}\int\omega_{0j}(\eta)\omega'(\eta)\psi(\varepsilon\eta)\,d\eta
$$
$$
=\frac{\psi(0)}{\varepsilon}\int\omega_{0j}(\eta)\omega'(\eta)\,d\eta+\psi'(0)\int\omega_{0j}(\eta)\omega'(\eta)\eta\,d\eta+O(\varepsilon),\quad \varepsilon\to+0.
$$

Since $\int\omega_{0j}(\eta)\omega'(\eta)\,d\eta=-\int\omega_j(\eta)\omega(\eta)\,d\eta$ and, according to property (e),

$$
\int\omega_{0j}(\eta)\omega'(\eta)\eta\,d\eta=-\int\omega(\eta)\big(\omega_{0j}(\eta)+\eta\omega_j(\eta)\big)\,d\eta=-\int\omega(\eta)\omega_{0j}(\eta)\,d\eta,
$$

we prove the fifth relation in (4.2).

In a similar way we can prove the last relation in (4.2). □

THEOREM 4.1. *Suppose that inequality* (1.11) *holds for* $t = 0$, *then there exists* $T > 0$ *such that the Cauchy problem* (1.8), (1.9) *for* $t \in [0, T)$ *has a weak asymptotic solution* (1.15), (4.1) *if and only if*

$$
\begin{aligned}
L_{11}[u_\pm] &= 0, \quad \pm x > \pm\phi(t), \\
L_{12}[u_\pm, v_\pm] &= 0, \quad \pm x > \pm\phi(t), \\
L_{13}[u_\pm, v_\pm, w_\pm] &= 0, \quad \pm x > \pm\phi(t), \\
\dot\phi(t) &= \frac{[u^2]}{[u]}\Big|_{x=\phi(t)} = (u_- + u_+)\big|_{x=\phi(t)}, \\
\dot e(t) &= \Big(2[vu] - [v]\tfrac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)} \\
&= [u](v_- + v_+)\big|_{x=\phi(t)}, \\
\dot g(t) &= \Big(2[v^2 + uw] - [w]\tfrac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)} \\
&= \big(2[v](v_- + v_+) + [u](w_- + w_+)\big)\big|_{x=\phi(t)}, \\
\frac{d\big(h(t)[u(\phi(t),t)]\big)}{dt} &= \frac{de^2(t)}{dt},
\end{aligned}
\tag{4.3}
$$

$P(t)$, *and mollifiers* ω_u, ω_v, ω_e, ω_g, ω_h *are such that*

$$
\begin{aligned}
P(t) &= \frac{\big(u_1^0(0)h^0 - (e^0)^2\big)\int\omega_e^2(\eta)\,d\eta}{\int\omega_u'(\eta)\Omega_P'(\eta)\,d\eta}\frac{1}{[u(\phi(t),t)]}, \\
\int\omega_{0u}(\eta)\omega_j(\eta)\,d\eta &= \int\omega_{0v}(\eta)\omega_e(\eta)\,d\eta = \frac{1}{2}, \quad j = e, g, h, \\
\int\omega_u(\eta)\omega_h(\eta)\,d\eta &= \int\omega_e^2(\eta)\,d\eta,
\end{aligned}
\tag{4.4}
$$

where $u_+ = u_0$, $v_+ = v_0$, $w_+ = w_0$, $u_- = u_0 + u_1$, $v_- = v_0 + v_1$, $w_- = w_0 + w_1$. *The initial data for system* (4.3) *are determined by* (1.9), *and* $\phi(0) = 0$.

PROOF. Using the first three relations in (4.2) (of Lemma 4.1), one can show that

$$
\big(u(x,t,\varepsilon)\big)^2 = u_0^2 + [u^2]H(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \tag{4.5}
$$

$$
\begin{aligned}
&u(x,t,\varepsilon)v(x,t,\varepsilon) = u_0v_0 + [uv]H(-x+\phi(t)) \\
&\quad + e(t)\Big(u_0 + u_1\int\omega_{0u}(\eta)\omega_e(\eta)\,d\eta\Big)\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0.
\end{aligned}
\tag{4.6}
$$

Analogously, using relations (4.2) from Lemma 4.1, one can calculate

$$
\begin{aligned}
\big(v(x,t,\varepsilon)\big)^2 &= v_0^2 + [v^2]H(-x+\phi(t)) \\
&\quad + 2e(t)\Big(v_0 + v_1\int\omega_{0v}(\eta)\omega_e(\eta)\,d\eta\Big)\delta(-x+\phi(t)) \\
&\quad + \Big(e^2(t)\int\omega_e^2(\eta)\,d\eta\Big)\frac{1}{\varepsilon}\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0.
\end{aligned}
\tag{4.7}
$$

Using relations (4.2) from Lemma 4.1 and taking into account that $R_w(x,t,\varepsilon)\in O_{\mathcal{D}'}(\varepsilon)$, we obtain

$$
\begin{aligned}
u(x,t,\varepsilon)w(x,t,\varepsilon) &= u_0w_0+[uw]H(-x+\phi(t))\\
&\quad+\Big\{g(t)\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_g(\eta)\,d\eta\Big)\\
&\quad\quad+h(t)\Big\{u_0+u_1\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big\}\delta'(-x+\phi(t))\\
&\quad+\Big\{-u_1h(t)\int\omega_u(\eta)\omega_h(\eta)\,d\eta+u_1P(t)\int\omega'_u(\eta)\Omega'_P(\eta)\,d\eta\Big\}\\
&\quad\times\frac{1}{\varepsilon}\delta(-x+\phi(t)))+O_{\mathcal{D}'}(\varepsilon),\quad \varepsilon\to+0.
\end{aligned}
\tag{4.8}
$$

Relations (4.7), (4.8), and the obvious identity

$$
p(x,t)\delta'(-x+\phi(t))=p(\phi(t),t)\delta'(-x+\phi(t))+p_x(\phi(t),t)\delta(-x+\phi(t))
$$

$\forall p\in C^1$ imply

$$
\begin{aligned}
&2\Big((v(x,t,\varepsilon))^2+u(x,t,\varepsilon)w(x,t,\varepsilon)\Big)\\
&\quad=2(v_0^2+u_0w_0)+2[v^2+uw]H(-x+\phi(t))\\
&\quad\quad+2\Big\{2e(t)\Big(v_0+v_1\int\omega_{0v}(\eta)\omega_e(\eta)\,d\eta\Big)\\
&\quad\quad\quad+g(t)\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_g(\eta)\,d\eta\Big)\\
&\quad\quad+h(t)\Big(u_{0x}+u_{1x}\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big\}\delta(-x+\phi(t))\\
&\quad\quad+2h(t)\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big|_{x=\phi(t)}\delta'(-x+\phi(t))\\
&\quad\quad+2\Big\{e^2(t)\int\omega_e^2(\eta)\,d\eta-u_1h(t)\int\omega_u(\eta)\omega_h(\eta)\,d\eta\\
&\quad\quad+u_1P(t)\int\omega'_u(\eta)\Omega'_P(\eta)\,d\eta\Big\}\frac{1}{\varepsilon}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),\quad\varepsilon\to+0.
\end{aligned}
\tag{4.9}
$$

Substituting the smooth ansatzs (1.15) and weak asymptotics (4.5), (4.6), (4.9) into system (1.7), we obtain up to $O_{\mathcal{D}'}(\varepsilon)$ the following relations

$$
\begin{aligned}
L_{11}[u(x,t,\varepsilon)] &= L_{11}[u_+]+\big[L_{11}[u]\big]H(-x+\phi(t))\\
&\quad+\Big\{[u]\dot\phi(t)-[u^2]\Big\}\delta(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon),
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
L_{12}[u(x,t,\varepsilon),v(x,t,\varepsilon)] &= L_{12}[u_+,v_+]+\big[L_{12}[u,v]\big]H(-x+\phi(t))\\
&\quad+\Big\{[v]\dot\phi(t)+\dot e(t)-2[uv]\Big\}\delta(-x+\phi(t))
\end{aligned}
$$

$$+e(t)\Big\{\dot\phi(t)-2\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_e(\eta)\,d\eta\Big)\Big\}\Big|_{x=\phi(t)}\delta'(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon), \tag{4.11}$$

$$\begin{aligned}
&L_{13}[u(x,t,\varepsilon),v(x,t,\varepsilon),w(x,t,\varepsilon)]\\
&\quad=L_{13}[u_+,v_+,w_+]+\big[L_{13}[u,v,w]\big]H(-x+\phi(t))\\
&\quad+\Big\{[w]\dot\phi(t)+\dot g(t)-2[v^2+uw]\Big\}\delta(-x+\phi(t))\\
&\quad+\Big\{g(t)\dot\phi(t)+\dot h(t)-2\Big(2e(t)\Big(v_0+v_1\int\omega_{0v}(\eta)\omega_e(\eta)\,d\eta\Big)\\
&\qquad+g(t)\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_g(\eta)\,d\eta\Big)\\
&\qquad+h(t)\Big(u_{0x}+u_{1x}\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big\}\Big|_{x=\phi(t)}\delta'(-x+\phi(t))\\
&\quad+h(t)\Big\{\dot\phi(t)-2\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big\}\Big|_{x=\phi(t)}\delta''(-x+\phi(t))\\
&\quad-2\Big\{e^2(t)\int\omega_e^2(\eta)\,d\eta-u_1h(t)\int\omega_u(\eta)\omega_h(\eta)\,d\eta\\
&\qquad+u_1P(t)\int\omega'_u(\eta)\Omega'_P(\eta)\,d\eta\Big\}\Big|_{x=\phi(t)}\frac{1}{\varepsilon}\delta'(-x+\phi(t))+O_{\mathcal{D}'}(\varepsilon).
\end{aligned}\tag{4.12}$$

Equating the coefficients of $\delta(-x+\phi(t))$, $\delta'(-x+\phi(t))$, $\delta''(-x+\phi(t))$, $\frac{1}{\varepsilon}\delta'(-x+\phi(t))$ with zero, we obtain the necessary and sufficient conditions for the relations

$$\begin{aligned}
L_{11}[u(x,t,\varepsilon)]&=O_{\mathcal{D}'}(\varepsilon),\\
L_{12}[u(x,t,\varepsilon),v(x,t,\varepsilon)]&=O_{\mathcal{D}'}(\varepsilon),\\
L_{13}[u(x,t,\varepsilon),v(x,t,\varepsilon),w(x,t,\varepsilon)]&=O_{\mathcal{D}'}(\varepsilon),
\end{aligned}\tag{4.13}$$

i.e., the first six equations in system (4.3) and the following system of equations

$$\begin{aligned}
\dot\phi(t)&=2\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_e(\eta)\,d\eta\Big)\Big|_{x=\phi(t)},\\
\dot\phi(t)&=2\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big|_{x=\phi(t)},\\
\dot h(t)&=2\Big\{2e(t)\Big(v_0+v_1\int\omega_{0v}(\eta)\omega_e(\eta)\,d\eta\Big)\\
&\qquad+g(t)\Big(u_0+u_1\int\omega_{0u}(\eta)\omega_g(\eta)\,d\eta\Big)\\
&\qquad+h(t)\Big(u_{0x}+u_{1x}\int\omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big\}\Big|_{x=\phi(t)}\\
&\qquad-g(t)\dot\phi(t),\\
h(t)\int\omega_u(\eta)\omega_h(\eta)\,d\eta&=\frac{e^2(t)}{u_1(\phi(t),t)}\int\omega_e^2(\eta)\,d\eta+P(t)\int\omega'_u(\eta)\Omega'_P(\eta)\,d\eta.
\end{aligned}\tag{4.14}$$

Comparing the first two relations in system (4.14) and fourth relation in system (4.3), we readily see that $\int \omega_{0u}(\eta)\omega_e(\eta)\,d\eta = \int \omega_{0u}(\eta)\omega_h(\eta)\,d\eta = \frac{1}{2}$.

Taking a derivative of the last equality in (4.14) and using the relation $\dot{e}(t) = \big(2[uv] - [v]\dot{\phi}(t)\big)\big|_{x=\phi(t)} = u_1(2v_0+v_1)\big|_{x=\phi(t)}$, we derive that

$$\dot{h}(t) = 2e(t)\big(2v_0(\phi(t),t) + v_1(\phi(t),t)\big)\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta} - \frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}\frac{e^2(t)\dot{u}_1(\phi(t),t)}{u_1^2(\phi(t),t)} + \dot{P}(t)\frac{\int \omega_u'(\eta)\Omega_P'(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}.$$

Let us subtract the third relation in (4.14) from the latter equality. Taking into account the equalities $\dot{\phi}(t) = \big(2u_0+u_1\big)\big|_{x=\phi(t)}$, $\int \omega_{0u}(\eta)\omega_h(\eta)\,d\eta = \frac{1}{2}$, and

$$h(t) = \frac{e^2(t)}{u_1(\phi(t),t)}\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta} + P(t)\frac{\int \omega_u'(\eta)\Omega_P'(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta},$$

we obtain that

$$\begin{aligned}
&2e(t)\Bigg\{2v_0(\phi(t),t)\bigg(\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta} - 1\bigg) \\
&\quad + v_1(\phi(t),t)\bigg(\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta} - 2\int \omega_{0v}(\eta)\omega_e(\eta)\,d\eta\bigg)\Bigg\} \\
&\quad - u_1(\phi(t),t)g(t)\bigg(2\int \omega_{0u}(\eta)\omega_g(\eta)\,d\eta - 1\bigg) \\
&\quad = \frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}\frac{e^2(t)}{u_1^2(\phi(t),t)} \\
&\qquad \times\bigg(\frac{du_1(\phi(t),t)}{dt} + \big(2u_{0x}(\phi(t),t) + u_{1x}(\phi(t),t)\big)u_1(\phi(t),t)\bigg) \\
&\qquad - \frac{\int \omega_u'(\eta)\Omega_P'(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}\Big(\dot{P}(t) - P(t)\big(2u_{0x}(\phi(t),t) + u_{1x}(\phi(t),t)\big)\Big).
\end{aligned} \tag{4.15}$$

Taking into account the first equality in (4.3) and the limit properties of the function $u_\pm$, we readily obtain that $u_{0t}(\phi(t),t) + 2u_0(\phi(t),t)u_{0x}(\phi(t),t) = 0$, $\big(u_{0t}(\phi(t),t) + u_{1t}(\phi(t),t)\big) + 2\big(u_0(\phi(t),t) + u_1(\phi(t),t)\big)\big(u_{0x}(\phi(t),t) + u_{1x}(\phi(t),t)\big) = 0$. Hence,

$$\begin{aligned}
u_{1t}(\phi(t),t) &+ u_{1x}(\phi(t),t)\big(2u_0(\phi(t),t) + u_1(\phi(t),t)\big) \\
&+ u_1(\phi(t),t)\big(2u_{0x}(\phi(t),t) + u_{1x}(\phi(t),t)\big) = 0.
\end{aligned}$$

Since $\dot{\phi}(t) = \big(2u_0+u_1\big)\big|_{x=\phi(t)}$, the last relation can be rewritten as

$$\frac{du_1(\phi(t),t)}{dt} + \big(2u_{0x}(\phi(t),t) + u_{1x}(\phi(t),t)\big)u_1(\phi(t),t) = 0. \tag{4.16}$$

Thus (4.15) can be transformed as

$$
\begin{aligned}
&2e(t)\Big\{2v_0(\phi(t),t)\Big(\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}-1\Big)\\
&\qquad +v_1(\phi(t),t)\Big(\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}-2\int \omega_{0v}(\eta)\omega_e(\eta)\,d\eta\Big)\Big\}\\
&\qquad -u_1(\phi(t),t)g(t)\Big(2\int \omega_{0u}(\eta)\omega_g(\eta)\,d\eta-1\Big)\\
&\qquad\qquad = -\frac{\int \omega_u'(\eta)\Omega_P'(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}\Big(\dot P(t)+P(t)\frac{\dot u_1(\phi(t),t)}{u_1(\phi(t),t)}\Big). \qquad (4.17)
\end{aligned}
$$

Note that we construct the asymptotic solution of the Cauchy problem which is *suitable* for any entropy initial data.

Taking $t=0$ in relation (4.17), we see that this relation holds for any initial data if and only if

$$
\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u(\eta)\omega_h(\eta)\,d\eta}=1, \qquad \int \omega_{0v}(\eta)\omega_e(\eta)\,d\eta=\frac{1}{2}, \qquad \int \omega_{0u}(\eta)\omega_g(\eta)\,d\eta=\frac{1}{2}, \qquad (4.18)
$$

and $\dot P(t)+P(t)\frac{\dot u_1(\phi(t),t)}{u_1(\phi(t),t)}=0$, i.e.,

$$
P(t)=\frac{A}{u_1(\phi(t),t)}, \qquad (4.19)
$$

where A is a constant.

Using relations (4.18), (4.16), and $\dot e(t)=u_1(2v_0+v_1)\big|_{x=\phi(t)}$, we rewrite the third relation in (4.14) as

$$
\frac{du_1(\phi(t),t)h(t)}{dt}=\frac{de^2(t)}{dt}. \qquad (4.20)
$$

The fourth relation in (4.14) and relations (4.19), (4.20) imply that

$$
A=\big(u_1^0(0)h^0-(e^0)^2\big)\frac{\int \omega_e^2(\eta)\,d\eta}{\int \omega_u'(\eta)\Omega_P'(\eta)\,d\eta}.
$$

Thus the first relation in (4.4) holds.

At the second stage of the proof, we show that system (4.3) is solvable.

Let us consider the shock-front problem

$$
\begin{aligned}
L_{11}[u_\pm] &= (u_\pm)_t+(u_\pm^2)_x=0, \quad \pm x>\pm\phi(t),\\
\dot\phi(t) &= \frac{[u^2]}{[u]}\Big|_{x=\phi(t)}=(u_-+u_+)\big|_{x=\phi(t)}, \qquad (4.21)\\
u(x,0) &= u_0^0(x)+u_1^0(x)H(-x), \quad \phi(0)=0,
\end{aligned}
$$

assuming that condition (1.11) holds for $t=0$, i.e., $u^0(x)$ is entropy initial data.

In order to solve this problem, we consider the Cauchy problem $L_{11}[u]=u_t+\big(u^2\big)_x=0$, $u(x,0)=u^0(x)$. According to [**16**, Ch.2.1.], we extend $u_+^0(x)=u_0^0(x)$

$(u^0_-(x) = u^0_0(x) + u^0_1(x))$ to $x \le 0$ $(x \ge 0)$ in a bounded C^1 fashion and continue to denote the extended functions by $u^0_\pm(x)$. By $u_\pm(x,t)$ we denote the C^1 solutions of the problems

$$L_{11}[u] = u_t + \big(u^2\big)_x = 0, \quad u_\pm(x,0) = u^0_\pm(x)$$

which exist for small enough time interval $[0,\ T_1]$ and can be determined by integration along characteristics. The functions $u_\pm(x,t)$ determine a two-sheeted covering of the plane (x,t). Next, we define the function $x = \phi(t)$ as a solution of the problem

$$\dot{\phi}(t) = \frac{u^2_-(x,t) - u^2_+(x,t)}{u_-(x,t) - u_+(x,t)}\bigg|_{x=\phi(t)}, \quad \phi(0) = 0.$$

It is clear that there exists a unique function $\phi(t)$ for sufficiently short times $[0,\ T_2]$. To this end, for $T = \min(T_1, T_2)$ we define the shock solution by

$$u(x,t) = \begin{cases} u_+(x,t), & x > \phi(t), \\ u_-(x,t), & x < \phi(t). \end{cases}$$

Thus we define a solution of the front-problem (4.21) for $t \in [0,\ T)$. This solution is determined by the first and fourth equations in system (4.3).

Solving problem (4.21), we obtain $u(x,t)$, $\phi(t)$. Then, substituting these functions into the other equations in system (4.3), we obtain $v_\pm(x,t)$, $w_\pm(x,t)$, and $e(t)$, $g(t)$, $h(t)$. Next, we determine a function $P(t)$ given by the first equation in (4.4).

The proof of Theorem 4.1 is complete. □

4.2. Construction of generalized solution. We obtain a generalized solution of the Cauchy problem (1.8), (1.9) as a weak limit of a weak asymptotic solution constructed by Theorem 4.1.

THEOREM 4.2. *Suppose that inequality* (1.11) *holds for* $t = 0$. *Then for* $t \in [0, T)$, *where* $T > 0$ *is given by Theorem* 4.1, *the Cauchy problem* (1.8), (1.9) *has a unique generalized solution* (1.12)*:*

$$\begin{aligned}
u(x,t) &= u_+(x,t) + [u(x,t)]H(-x+\phi(t)), \\
v(x,t) &= v_+(x,t) + [v(x,t)]H(-x+\phi(t)) + e(t)\delta(-x+\phi(t)), \\
w(x,t) &= w_+(x,t) + [w(x,t)]H(-x+\phi(t)) + g(t)\delta(-x+\phi(t)) \\
&\qquad + h(t)\delta'(-x+\phi(t)),
\end{aligned}$$

which satisfies the integral identities (3.15)*:*

$$
\begin{aligned}
\int_0^T \int \Big(u(x,t)\varphi_t + u^2(x,t)\varphi_x\Big)\,dx\,dt + \int u^0(x)\varphi(x,0)\,dx &= 0,\\
\int_0^T \int \Big(V(x,t)\varphi_t + 2u(x,t)V(x,t)\varphi_x\Big)\,dx\,dt + \int_\Gamma e(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl & \\
+ \int V^0(x)\varphi(x,0)\,dx + e^0\varphi(0,0) &= 0,\\
\int_0^T \int \Big(W(x,t)\varphi_t + 2\big(V^2(x,t) + u(x,t)W(x,t)\big)\varphi_x\Big)\,dx\,dt & \\
+ \int_\Gamma g(t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl & \\
+ \int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl + \int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial\mathbf{l}} - h(x,t)\frac{\partial[u(x,t)]}{\partial\mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl & \\
+ \int W^0(x)\varphi(x,0)\,dx + g^0\varphi(0,0) + h^0\varphi_x(0,0) &= 0,
\end{aligned} \tag{4.22}
$$

for all $\varphi(x,t) \in \mathcal{D}(\mathbb{R}\times[0,\ \infty))$*, where functions* $u_\pm(x,t)$*,* $v_\pm(x,t)$*,* $w_\pm(x,t)$*,* $\phi(t)$*,* $e(t)$*,* $g(t)$*,* $h(t)$ *are defined by the system*

$$
\begin{aligned}
L_{11}[u_\pm] &= 0, \quad \pm x > \pm\phi(t),\\
L_{12}[u_\pm, v_\pm] &= 0, \quad \pm x > \pm\phi(t),\\
L_{13}[u_\pm, v_\pm, w_\pm] &= 0, \quad \pm x > \pm\phi(t),\\
\dot\phi(t) &= \frac{[u^2]}{[u]}\Big|_{x=\phi(t)} = (u_- + u_+)\big|_{x=\phi(t)},\\
\dot e(t) &= \Big(2[vu] - [v]\frac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)}\\
&= [u](v_- + v_+)\big|_{x=\phi(t)},\\
\dot g(t) &= \Big(2[v^2 + uw] - [w]\frac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)}\\
&= \big(2[v](v_- + v_+) + [u](w_- + w_+)\big)\big|_{x=\phi(t)},\\
\frac{d\big(h(t)[u(\phi(t),t)]\big)}{dt} &= \frac{de^2(t)}{dt},
\end{aligned} \tag{4.23}
$$

where initial data for system (4.23) *are determined by* (1.9)*, and* $\phi(0) = 0$*. Here* $\Gamma = \{(x,t) : x = \phi(t),\ 0 \le t \le T\}$*,* $V(x,t) = v_+ + [v]H(-x+\phi(t))$*,* $W(x,t) = w_+ + [w]H(-x+\phi(t))$*, and (see* (2.3)*)*

$$
\int_\Gamma e(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl = \int_0^T e(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,
$$

$$
\int_\Gamma g(x,t)\frac{\partial\varphi(x,t)}{\partial\mathbf{l}}\,dl = \int_0^T g(t)\frac{d\varphi(\phi(t),t)}{dt}\,dt,
$$

$$
\int_\Gamma h(x,t)\frac{\partial\varphi_x(x,t)}{\partial\mathbf{l}}\,dl = \int_0^T h(t)\frac{d\varphi_x(\phi(t),t)}{dt}\,dt,
$$

$$\int_\Gamma \frac{\frac{\partial e^2(x,t)}{\partial \mathbf{l}} - h(x,t)\frac{\partial [u(x,t)]}{\partial \mathbf{l}}}{[u(x,t)]}\varphi_x(x,t)\,dl = \int_0^T \frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]}\varphi_x(\phi(t),t)\,dt.$$

PROOF. By substituting relations (4.4), which determine mollifiers ω_u, ω_v, ω_e, ω_g, ω_h, and $P(t)$ into relations (4.6), (4.9), and taking into account (4.16) and relations $\dot\phi(t) = (2u_0 + u_1)\big|_{x=\phi(t)}$, $\dot e(t) = u_1(2v_0 + v_1)\big|_{x=\phi(t)}$ (see (4.3)), we obtain

$$\begin{aligned} 2u(x,t,\varepsilon)v(x,t,\varepsilon) &= 2u_+v_+ + 2[uv]H(-x+\phi(t)) \\ &\quad + e(t)\dot\phi(t)\delta(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} 2\big((v(x,t,\varepsilon))^2 &+ u(x,t,\varepsilon)w(x,t,\varepsilon)\big) \\ &= 2(v_+^2 + u_+w_+) + 2[v^2 + uw]H(-x+\phi(t)) \\ &\quad + \left(\frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]} + g(t)\dot\phi(t)\right)\delta(-x+\phi(t)) \\ &\quad + h(t)\dot\phi(t)\delta'(-x+\phi(t)) + O_{\mathcal{D}'}(\varepsilon), \quad \varepsilon \to +0, \end{aligned} \tag{4.25}$$

By Theorem 4.1 we have estimates (4.13). Let us apply the left-hand and right-hand sides of relations (4.13) to an arbitrary test function $\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0,\infty))$. Then, integrating by parts, we have

$$\begin{aligned} \int_0^T \int \Big(u(x,t,\varepsilon)\varphi_t(x,t) + (u(x,t,\varepsilon))^2\varphi_x(x,t)\Big)\,dxdt & \\ + \int u(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), &\quad \varepsilon \to +0, \\ \int_0^T \int \Big(v(x,t,\varepsilon)\varphi_t(x,t) + 2u(x,t,\varepsilon)v(x,t,\varepsilon)\varphi_x(x,t)\Big)\,dxdt & \\ + \int v(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), &\quad \varepsilon \to +0, \\ \int_0^T \int \Big(w(x,t,\varepsilon)\varphi_t(x,t) + 2\big((v(x,t,\varepsilon))^2 + u(x,t,\varepsilon)w(x,t,\varepsilon)\big)\varphi_x(x,t)\Big)\,dxdt & \\ + \int w(x,0,\varepsilon)\varphi(x,0)\,dx = O(\varepsilon), &\quad \varepsilon \to +0. \end{aligned}$$

Passing to the limit as $\varepsilon \to +0$ in each of these integrals, and taking into account relations (4.5), (4.24), (4.25), and the fact that

$$\lim_{\varepsilon\to+0}\int_0^T \int f(t)\delta\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt = \int_0^T f(t)\varphi(\phi(t),t)\,dt, \tag{4.26}$$

$$\lim_{\varepsilon\to+0}\int_0^T \int f(t)\delta'\big(-x+\phi(t),\varepsilon\big)\varphi(x,t)\,dxdt = \int_0^T f(t)\varphi_x(\phi(t),t)\,dt, \tag{4.27}$$

$\varphi(x,t) \in \mathcal{D}(\mathbb{R} \times [0, T))$, we derive the integral identities (4.22).

In view of above arguments in the second stage of the proof of Theorem 4.1, the Cauchy problem (1.8), (1.9) has a unique generalized solution.

Observe that the function $u(x,t)$ is the unique entropy solution to the Cauchy problem of the scalar equation $L_{11}[u] = 0$ (recall that the admissibility condition $u_+ < u_-$ is assumed to be satisfied). In particular, the shock line $x = \phi(t)$ is uniquely determined. Then, using the method of characteristics, we find the functions $v_\pm$, $w_\pm$ in the domains $\pm(x - \phi(t)) > 0$. Remark that all characteristics are incoming at the shock line, therefore these functions $v_\pm$, $w_\pm$ are unique. Finally, the functions e, g, h are unique solutions of the Rankine-Hugoniot relations in (4.23) with the corresponding initial data e^0, g^0, h^0. We see that the generalized solution is unique. Naturally, the admissibility condition $u_+ < u_-$ is essential for uniqueness (see Subsec. 4.4 below).

The proof is complete. $\square$

The system of equations

$$
\begin{array}{rcl}
\dot{\phi}(t) & = & \frac{[u^2]}{[u]}\Big|_{x=\phi(t)} = (u_- + u_+)\big|_{x=\phi(t)}, \\
\dot{e}(t) & = & \Big(2[vu] - [v]\frac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)} = [u](v_- + v_+)\big|_{x=\phi(t)}, \\
\dot{g}(t) & = & \Big(2[v^2 + uw] - [w]\frac{[u^2]}{[u]}\Big)\Big|_{x=\phi(t)} \\
 & = & \big(2[v](v_- + v_+) + [u](w_- + w_+)\big)\big|_{x=\phi(t)}, \\
\frac{d}{dt}\Big(h(t)[u(\phi(t),t)]\Big) & = & \frac{de^2(t)}{dt},
\end{array}
\tag{4.28}
$$

that determines the trajectory $x = \phi(t)$ of a δ'-shock wave and the coefficients $e(t)$, $g(t)$, $h(t)$ of the singularities, constitute the *Rankine–Hugoniot conditions for δ'-shock*. Moreover, the first equation in system (4.28) is the “standard” Rankine–Hugoniot condition for the *shock*, while the first and second equations are the “standard” *Rankine–Hugoniot conditions for δ-shock* (2.5). The right-hand sides of the second and third equations in system (4.28) are called the *first Rankine–Hugoniot deficits*. The right-hand side of the fourth equation in system (4.28) is called the *second Rankine–Hugoniot deficit*.

If the initial data (1.9) are piecewise constant then Theorem 4.2 implies the following corollary.

COROLLARY 4.1. *Suppose that inequality* (1.11) *holds for $t = 0$. Then for $t \in [0, \infty)$, the Cauchy problem* (1.8), (1.9) *with piecewise constant initial data has a unique generalized solution* (1.12)*:*

$$
\begin{array}{rcl}
u(x,t) & = & u_+ + [u]H(-x + \phi(t)), \\
v(x,t) & = & v_+ + [v]H(-x + \phi(t)) + e(t)\delta(-x + \phi(t)), \\
w(x,t) & = & w_+ + [w]H(-x + \phi(t)) + g(t)\delta(-x + \phi(t)) \\
 & & \qquad + h(t)\delta'(-x + \phi(t)),
\end{array}
$$

which satisfies the integral identities (4.22), *where*

$$
\begin{array}{rcl}
\phi(t) & = & \frac{[u^2]}{[u]}t = \big(u_- + u_+\big)t, \\
e(t) & = & e^0 + \Big(2[vu] - [v]\frac{[u^2]}{[u]}\Big)t = e^0 + [u](v_- + v_+)t, \\
g(t) & = & g^0 + \Big(2[v^2 + uw] - [w]\frac{[u^2]}{[u]}\Big)t \\
& = & g^0 + \big(2[v](v_- + v_+) + [u](w_- + w_+)\big)t, \\
h(t) & = & h^0 - \frac{(e^0)^2}{[u]} + \frac{e^2(t)}{[u]} \\
& = & h^0 + 2e^0(v_- + v_+)t + [u](v_- + v_+)^2 t^2.
\end{array}
\tag{4.29}
$$

REMARK 4.1. To find a generalized solution of the Cauchy problem (1.8), (1.9), we construct a weak asymptotic solution (1.15), (4.1) of the problem, where the functions $u_\pm(x,t)$, $v_\pm(x,t)$, $w_\pm(x,t)$, $\phi(t)$, $e(t)$, $g(t)$, $h(t)$, the mollifiers ω_u, ω_v, ω_e, ω_g, ω_h, $\Omega_P(\eta)$, $\Omega_Q(\eta)$ and function $P(t)$ are determined by systems (4.3), (4.4).

Without introducing a correction (4.1), i.e., setting $P(t) = 0$, we derive from (4.4) that we can solve the Cauchy problem (1.8), (1.9) if the initial data satisfy the following condition

$$
[u(0,0)]h^0 = (e^0)^2. \tag{4.30}
$$

In particular, in the case of solution (3.9)–(3.10), condition (4.30) holds, according to (3.13). Condition (4.30) makes the Cauchy problem *overdetermined*, so we cannot solve this Cauchy problem with an *arbitrary* jump.

4.3. "Right" singular superpositions. As mentioned in the Introduction, the problem of defining a δ'-shock wave type solution of the Cauchy problem is connected with the construction of *singular superpositions* (*products*) of distributions.

For example, it seems natural to introduce a product of the Heaviside function and delta function as the weak limit of the product of their regularizations. Then, according to the third relation (4.2), we have

$$
\overbrace{H(x)\delta(x)} \stackrel{def}{=} \lim_{\varepsilon\to+0} H(x,\varepsilon)\delta(x,\varepsilon) = B\delta(x), \tag{4.31}
$$

where $B = \int \omega_0(\eta)\omega_\delta(\eta)\,d\eta$. The product (4.31) defined in this way *depends* on the mollifiers ω, ω_δ, i.e., on the regularizations of distributions $H(x)$, $\delta(x)$.

In a similar way, using regularizations $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$, $w(x,t,\varepsilon)$ of distributions $u(x,t)$, $v(x,t)$, $w(x,t)$ given by (1.15), and asymptotics (4.5), (4.6), (4.9), one can introduce *singular superpositions* $\big(u(x,t)\big)^2$, $2u(x,t)v(x,t)$, and $2\big((v(x,t))^2 + u(x,t)w(x,t)\big)$ by the following definition:

$$
\overbrace{\big(u(x,t)\big)^2} \stackrel{def}{=} \lim_{\varepsilon\to+0} \big(u(x,t,\varepsilon)\big)^2 = u_+^2 + [u^2]H(-x+\phi(t)), \tag{4.32}
$$

$$
\overbrace{2u(x,t)v(x,t)} \stackrel{def}{=} \lim_{\varepsilon\to+0} 2u(x,t,\varepsilon)v(x,t,\varepsilon) = 2u_+v_+ + 2[uv]H(-x+\phi(t))
$$

$$+2e(t)\Big(u_0 + u_1 \int \omega_{0u}(\eta)\omega_e(\eta)\,d\eta\Big)\delta(-x+\phi(t)), \tag{4.33}$$

$$\begin{aligned}
&\overbrace{2\big((v(x,t))^2 + u(x,t)w(x,t)\big)} \stackrel{def}{=} \lim_{\varepsilon\to+0} 2\big((v(x,t,\varepsilon))^2 + u(x,t,\varepsilon)w(x,t,\varepsilon)\big) \\
&\quad = 2(v_+^2 + u_+w_+) + 2[v^2+uw]H(-x+\phi(t)) \\
&\qquad +2\Big\{2e(t)\Big(v_+ + [v]\int \omega_{0v}(\eta)\omega_e(\eta)\,d\eta\Big) \\
&\qquad\qquad +g(t)\Big(u_+ + [u]\int \omega_{0u}(\eta)\omega_g(\eta)\,d\eta\Big) \\
&\qquad\qquad +h(t)\Big(u_{+x} + [u_x]\int \omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big\}\delta(-x+\phi(t)) \\
&\qquad +2h(t)\Big(u_+ + [u]\int \omega_{0u}(\eta)\omega_h(\eta)\,d\eta\Big)\Big|_{x=\phi(t)}\delta'(-x+\phi(t)) \\
&\quad + \lim_{\varepsilon\to+0}\frac{2}{\varepsilon}\Big\{e^2(t)\int \omega_e^2(\eta)\,d\eta - [u]h(t)\int \omega_u(\eta)\omega_h(\eta)\,d\eta \\
&\qquad\qquad +[u]P(t)\int \omega_u'(\eta)\Omega_P'(\eta)\,d\eta\Big\}\delta(-x+\phi(t)).
\end{aligned} \tag{4.34}$$

It is easy to see that the *singular superpositions* (4.33)–(4.34) *depend* on the regularizations of the Heaviside function, delta function, its derivative, and function $P(t)$. Moreover, the right-hand side of (4.34) is *unbounded*. This means that the above introduced *singular superpositions* are *not unique*.

However, in the context of constructing *weak asymptotic solutions* of the Cauchy problems, we can define explicit formulas for the *"right" singular superpositions*.

Recall that relations (4.24), (4.25) were obtained by substituting relations (4.4), which determine ω_e, ω_g, ω_h, ω_u, ω_v, $P(t)$ into relations (4.6), (4.9). Now, using relations (4.5) (4.24), (4.25), we can define explicit formulas for the *"right" singular superpositions*:

$$\big(u(x,t)\big)^2 \stackrel{def}{=} \lim_{\varepsilon\to+0}\big(u(x,t,\varepsilon)\big)^2 = u_+^2 + [u^2]H(-x+\phi(t)), \tag{4.35}$$

$$\begin{aligned}
&2u(x,t)v(x,t) \stackrel{def}{=} \lim_{\varepsilon\to+0} 2u(x,t,\varepsilon)v(x,t,\varepsilon) \\
&\qquad = 2u_+v_+ + 2[uv]H(-x+\phi(t)) + e(t)\dot\phi(t)\delta(-x+\phi(t)),
\end{aligned} \tag{4.36}$$

$$\begin{aligned}
&2\big((v(x,t))^2 + u(x,t)w(x,t)\big) \stackrel{def}{=} \lim_{\varepsilon\to+0} 2\big((v(x,t,\varepsilon))^2 + u(x,t,\varepsilon)w(x,t,\varepsilon)\big) \\
&\qquad = 2(v_+^2 + u_+w_+) + 2[v^2+uw]H(-x+\phi(t)) \\
&\qquad\quad + \left(\frac{\frac{de^2(t)}{dt} - h(t)\frac{d[u(\phi(t),t)]}{dt}}{[u(\phi(t),t)]} + g(t)\dot\phi(t)\right)\delta(-x+\phi(t)) \\
&\qquad\qquad +h(t)\dot\phi(t)\delta'(-x+\phi(t)),
\end{aligned} \tag{4.37}$$

where distributions $u(x,t)$, $v(x,t)$, $w(x,t)$ given by (1.15).

Note that, although according to (4.7), (4.8), the terms $\lim_{\varepsilon\to+0} 2(v(x,t,\varepsilon))^2$, $\lim_{\varepsilon\to+0} 2u(x,t,\varepsilon)w(x,t,\varepsilon)$ are *unbounded*, the right-hand side of (4.37), (like the right-hand sides of (4.35), (4.36)) is a *well defined distribution*.

In contrast to (4.32)–(4.34), where $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$, $w(x,t,\varepsilon)$ are regularizations of distributions (1.12), in (4.35)–(4.37) $u(x,t,\varepsilon)$, $v(x,t,\varepsilon)$ give a *weak asymptotic solution* of the Cauchy problem (1.8), (1.9).

4.4. On the uniqueness of generalized solutions. Consider the initial data of the form

$$u^0(x) = -1+2H(-x), \quad v^0(x) = u_{0x} = -2\delta(-x), \quad w^0(x) = v_{0x} = 2\delta'(-x). \tag{4.38}$$

It is clear that the Cauchy problem (1.8), (4.38) has the stationary δ'-shock wave type solution $u(x,t) \equiv u^0(x)$, $v(x,t) \equiv v^0(x)$, $w(x,t) \equiv w^0(x)$.

Nevertheless, there are infinitely many solutions of this problem which satisfy the integral identities. For example,

$$\begin{aligned}
u(x,t) &= -1 + 3H(-x+t) - 4H(-x) + 3H(-x-t), \\
v(x,t) &= -3\delta(-x+t) + 4\delta(-x) - 3\delta(-x-t), \\
w(x,t) &= 3\delta'(-x+t) - 4\delta'(-x) + 3\delta'(-x-t).
\end{aligned}$$

Thus we need entropy conditions for δ'-shocks.

5. On $\delta^{(n)}$-shock wave type solutions.

Starting from the scalar conservation law $u_t + \big(f(u)\big)_x = 0$, by successive differentiation with respect to x variable and setting $u_1 = u$, $u_2 = u_{1x}$, $u_3 = u_{2x}$, $\dots$, $u_{n+2} = u_{n+1x}$ just as above (see Subsec. 3.1), we obtain a n-"prolonged system"

$$\begin{aligned}
(u_1)_t + \big(f(u_1)\big)_x &= 0, \\
(u_2)_t + \big(f'(u_1)u_2\big)_x &= 0, \\
(u_3)_t + \big(f''(u_1)u_2^2 + f'(u_1)u_3\big)_x &= 0, \\
(u_4)_t + \big(f'''(u_1)u_2^3 + 3f''(u_1)u_2u_3 + f'(u_1)u_4\big)_x &= 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots &\ \ . \ \ . \\
(u_{n+2})_t + \big(f^{(n+1)}(u_1)u_2^{n+1} + \dots\dots\dots + f'(u_1)u_{n+2}\big)_x &= 0.
\end{aligned} \tag{5.1}$$

It is clear that by repeating the constructions from Sec. 5, a $\delta^{(n)}$-shock wave type solution of system (5.1) can be constructed in the same way as the δ'-shock wave type solution of system (1.7). To solve this problem, we need to derive integral identities for defining this type of solution.

References

[1] F. Bouchut, *On zero pressure gas dynamics*, in "Advances in Kinetic Theory and Computing", Series on Advances in Mathematics for Applied Sciences, Vol. 22, 171–190, World Scientific, Sigapore, 1994.

[2] V. G. Danilov, V. P. Maslov, V. M. Shelkovich, *Algebra of singularities of singular solutions to first-order quasilinear strictly hyperbolic systems*, Theor. Math. Phys., **114**, no 1, (1998), 1–42.

[3] V. G. Danilov, G. A. Omel′yanov, V. M. Shelkovich, *Weak Asymptotics Method and Interaction of Nonlinear Waves*, in Mikhail Karasev (ed.), "Asymptotic Methods for Wave and Quantum Problems", Amer. Math. Soc. Transl., Ser. 2, **208**, 2003, 33–165.

[4] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of nonlinear waves to quasilinear equations*, Hyperbolic problems: Theory, Numerics, Applications (Eighth International Conference in Magdeburg, February/March 2000, v.I). International Series of Numerical Mathematics, v. 140, Birkhäuser Verlag Basel/Switzerland, 2001, 267–276.

[5] V. G. Danilov and V. M. Shelkovich, *Propagation and interaction of shock waves of quasilinear equation*, Nonlinear Studies, **8**, no. 1, (2001), 135–169.

[6] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of delta-shock waves of a hyperbolic system of conservation laws*, In Hou, Thomas Y.; Tadmor, Eitan (Eds.), Hyperbolic Problems: Theory, Numerics, Applications. Proceedings of the Ninth International Conference on Hyperbolic Problems held in CalTech, Pasadena, March 25-29, 2002, Springer Verlag, 2003, 483–492.

[7] V. G. Danilov, V. M. Shelkovich, *Propagation and interaction of δ-shock waves to hyperbolic systems of conservation laws*, Dokl. Ross. Akad. Nauk, **394**, no. 1, (2004), 10–14. English transl. in Russian Doklady Mathematics., **69**, no. 1, (2004).

[8] V. G. Danilov, V. M. Shelkovich, *Delta-shock wave type solution of hyperbolic systems of conservation laws*, (to appear in the Quart. Appl. Math.). (See also Preprint 2003-052 at the url:
http://www.math.ntnu.no/conservation/2003/052.html)

[9] V. G. Danilov, V. M. Shelkovich, *Dynamics of propagation and interaction of delta-shock waves in conservation law systems*, (to appear in the Journal of Differential Equations). (See also Preprint 2003-068 at the url: http://www.math.ntnu.no/conservation/2003/068.html)

[10] Weinan E., Yu. Rykov, Ya. G. Sinai, *Generalized variational principles, global weak solutions and behavior with random initial data for dystems of conservation laws arising in adhesion particlae dynamics*, Comm. Math. Phys., **177**, (1996), 349–380.

[11] G. Ercole, *Delta-shock waves as self-similar viscosity limits*, Quart. Appl. Math., **LVIII**, no. 1, (2000), 177–199.

[12] B. Lee Keyfitz and H. C. Kranzer, *Spaces of weighted measures for conservation laws with singular shock solutions*, Journal of Differential Equations, **118**, (1995), 420–451.

[13] P. Le Floch, *An existence and uniqueness result for two nonstrictly hyperbolic systems*, Nonlinear Evolution Equations That Change Type, IMA Vol. Math. Appl. 27 (1990), Springer-Verlag, 126–138.

[14] Philippe Le Floch, *Entropy weak solutions to nonlinear hyperbolic systems under nonconservative form*, Commun. in partial differential equations, **13**, no. 6, (1988), 669–727.

[15] Philippe Le Floch, Tai-Ping Liu, *Existense theory for nonlinear hyperbolic systems in nonconservative form*, Forum Math., **5**, (1993), no. 3, 261–280.

[16] A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Springer-Verlag New York, Berlin, Heidelberg, Tokyo, 1984.

[17] G. Dal Maso, P. G. Le Floch, and F. Murat, *Definition and weak stability of nonconservative products*, J. Math. Pures Appl., **74**, (1995), 483–548.

[18] V. M. Shelkovich, *An associative-commutative algebra of distributions that includes multiplicators, generalized solutions of nonlinear equations*, Mathematical Notices, **57**, no. 5, (1995), 765–783.

[19] V. M. Shelkovich, *Delta-shock waves of a class of hyperbolic systems of conservation laws*, in A. Abramian, S. Vakulenko, V. Volpert (Eds.), "Patterns and Waves", Akadem Print, St. Petersburg, 2003, 155–168.

[20] V. M. Shelkovich, *A specific hyperbolic system of conservation laws admitting delta-shock wave type solutions*, Preprint 2003-059 at the url:
http://www.math.ntnu.no/conservation/2003/059.html

[21] V. M. Shelkovich, *Delta-shocks, the Rankine–Hugoniot conditions, and singular superposition of distributions*, Proceedings of International Seminar Days on Difraction'2004, June 29– July 2, 2004, St.Petersburg, 2004, 175–196. (See also Preprint 2004-059 at the url: http://www.math.ntnu.no/conservation/2004/059.html)

[22] V. M. Shelkovich, *Delta-shocks, the Rankine–Hugoniot conditions and the balance laws for δ-shocks*, (to appear in Fundamental and Applied Mathematics).

[23] Dechun Tan, Tong Zhang and Yuxi Zheng, *Delta-shock waves as limits of vanishing viscosity for hyperbolic systems of conservation laws*, Journal of Differential Equations, **112**, (1994), 1–32.

[24] A. I. Volpert, *The space BV and quasilinear equations*, Math. USSR Sb. **2**, (1967), 225–267.

[25] Hanchun Yang, *Riemann problems for class of coupled hyperbolic systems of conservation laws*, Journal of Differential Equations, **159**, (1999), 447–484.

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