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On symmetrizability of hyperbolic matrix spaces.

Abstract

We introduce a new general criterion of symmetrizability for linear matrix spaces over fields $\mathbb{R}$, $\mathbb{C}$, and give some applications to first order quasilinear systems.

Let $L \subset \mathrm{Mat}(n, k)$ be a linear subspace of the space of $n \times n$-matrices over a field k, where $k = \mathbb{R}$ or $\mathbb{C}$.

Definition 1. We shall call the space L *hyperbolic* if it satisfies the condition

$$A^2 \in L \quad \forall A \in L, \tag{1}$$

and any matrix in L has simple real spectrum (i.e. eigenvalues of any matrix $A \in L$ are real and there is a basis consisting of corresponding eigenvectors).

From (1) it follows that $AB + BA = (A + B)^2 - A^2 - B^2 \in L \; \forall A, B \in L$ (this means that L is a special Jordan algebra). Thus, we can define the linear operators S_A by the rule $S_A B = (AB + BA)$.

The hyperbolicity condition admits the following reformulation.

Proposition 1. *Suppose a space L satisfies (1) and contains the unit matrix E. Then it is hyperbolic if and only if the linear operators S_A in L have simple real spectra for all $A \in L$.*

Proof. Let L be a hyperbolic space. Then the spectrum $\sigma(A)$ of every matrix $A \in L$ is simple and real. We define the symmetric bilinear form $(A, B) = \mathrm{Tr}\, AB$. Since $(A, A) = \sum\limits_{\lambda \in \sigma(A)} \lambda^2 > 0$ for $A \neq 0$ then the form $(\cdot, \cdot)$ is positive definite and determines the scalar multiplication on L. As is directly verified the operators S_A are symmetric with respect to this scalar multiplication. Therefore they have simple real spectra.

Conversely, assume that the operators S_A have simple real spectra. From (1) and the condition $E \in L$ it follows that L contains all powers A^n, $n \geq 0$ for $A \in L$ and therefore functions $f(A) \in L$ are well-defined for every real function $f(z) \in C^{n-1}$. Let $A \in L$. Clearly, simplicity of spectrum of a matrix (or, an operator) A is equivalent to the existence of a polynomial $p(z) = \prod\limits_{k=1}^{m} (z - \lambda_k)$, which has distinct real roots λ_k, $k = 1, \ldots, m$, such that $p(A) = 0$ (we use that there are no nontrivial blocks in the Jordan form of A). By our assumption the operator S_A has simple real spectrum. Therefore, there exist a polynomial $p(z)$ having distinct real roots such that $p(S_A) = 0$. Since $p(2A) = p(S_A)E = 0$, and the polynomial $p(2z)$ has distinct real roots we conclude that the spectrum of A is real and simple. The proof is complete.

Remark that the assumption $E \in L$ is necessary for the inverse statement of Proposition 1. Indeed, let J be any nontrivial matrix such that $J^2 = 0$, and $L = \{ \lambda J \mid \lambda \in \mathbb{R} \}$. Obviously, L satisfies (1) but $E \notin L$. One can directly verify that operators $S_A = 0 \, \forall A \in L$ but clearly L is not a hyperbolic space.

As follows from Proposition 1, the hyperbolicity condition means that the system of conservation laws, generated by the Burgers-like equation $U_t + (U^2)_x = 0$, $U = U(t, x) \in L$, is hyperbolic. In papers [1, 2] the more general systems

$$U_t + f(U)_x = 0 \tag{2}$$

were studied, in which the unknown function $U = U(t, x)$ takes its values in the space S_n of symmetric or in the space H_n of Hermitian matrices of order n, and $U \to f(U)$ is the functional calculus operator. As was shown in these papers, the system (2) is hyperbolic. Clearly, systems like (2) can be considered also in the general case when U takes its values in an arbitrary matrix linear space L, which is invariant under functional calculus operators: $f(U) \in L \, \forall U \in L$, $f(u) \in C^n(\mathbb{R})$. In particular L must satisfy (1). As follows from our main Theorem 1 below, for nonlinear f system (2) is hyperbolic only in the case studied in [1, 2] when L consists of symmetric or Hermitian matrices (after appropriate choice of a basis).

Before formulation of our main result we describe some useful constructions conserving the property of hyperbolicity.

Let L be a matrix space. Denote by L^* a matrix space consisting of conjugate matrices A^*, $A \in L$ (with respect to some scalar multiplication on k^n). Clearly, the space L^* is hyperbolic together with L. Now, suppose that $H \subset k^n$ is a linear subspace, which is invariant under the action of L that is $A(H) \subset H \, \forall A \in L$. Then we can define matrix spaces L_H and $L_{/H}$ consisting of matrices corresponding to the restricted operators $A|_H : H \to H$, $A \in L$ and to the factor-operators $A/H : k^n/H \to k^n/H$, $A \in L$ respectively. Clearly, the orthogonal complement $H^\perp$ is an invariant space for L^* and $(L_{/H})^* = L^*|_{H^\perp}$. We have the following simple

Lemma 1. *Let L be a hyperbolic matrix space and $H \subset k^n$ be an invariant subspace under the action of L. Then the matrix spaces $L|_H$ and $L_{/H}$ are hyperbolic as well.*

Proof. By the duality $L_{/H} = (L^*|_{H^\perp})^*$ and it is sufficient to prove the Lemma for the case of matrix space L_H. Clearly, L_H satisfies (1). As was shown in the proof of Proposition 1, for any matrix $A \in L$ there exist a polynomial $p(z)$ with distinct real roots such that $p(A) = 0$. Then also $p(A|_H) = p(A)|_H = 0$. Therefore, the spectrum of $A|_H$ is real and simple. The proof is complete.

Let us formulate our main result.

Theorem 1. *A space L, satisfying (1), is hyperbolic if and only if all matrices in L are symmetric (Hermitian) with respect to some scalar multiplication in k^n.*

We shall assume below that L is a space of matrices over the field $k = \mathbb{C}$. The case of real field $k = \mathbb{R}$ is reduced to the case $k = \mathbb{C}$ by the complexification procedure. Indeed, if a real matrix family L consists of Hermitian matrices with respect to a scalar multiplication $(\cdot, \cdot)$ on $\mathbb{C}^n$ then all matrices in L are symmetric with respect to the real scalar multiplication $\mathrm{Re}(\cdot, \cdot)$ on $\mathbb{R}^n$. We also observe that the condition of hyperbolicity for L remains valid after the complexification.

To prove Theorem 1 we need some preliminary results and constructions.

Lemma 2. *For all $A, B, C \in L$ the following identity*

$$[S_A, S_B]C = [[A, B], C]$$

holds. Here $[\cdot, \cdot]$ is a commutator of operators (matrices).

Proof. The claim of the Lemma directly follows from the equality

$$\begin{aligned}[S_A, S_B]C = S_A S_B C - S_B S_A C = \\ ABC + ACB + BCA + CBA - BAC - BCA - ACB - CAB = \\ (AB - BA)C - C(AB - BA) = [[A, B], C].\end{aligned}$$

Corollary 1. *1) $[[A, B], C] \in L \ \forall A, B, C \in L$; 2) Let $[L, L]$ be a linear hull of the set of commutators $[A, B]$, $A, B \in L$. Then $[L, L]$ is a Lie algebra (with multiplication $[\cdot, \cdot]$).*

Proof. The first statement directly follows from Lemma 2. To prove the second statement we have to verify that $[[A_1, B_1], [A_2, B_2]] \in [L, L]$

$\forall A_1, B_1, A_2, B_2 \in L$. But this property follows from the equality

$$[[A_1, B_1], [A_2, B_2]] = [[[A_1, B_1], A_2], B_2]] - [[[A_1, B_1], B_2], A_2] = [C_1, B_2] - [C_2, A_2],$$

and the fact that $C_1 = [[A_1, B_1], A_2], C_2 = [[A_1, B_1], B_2] \in L$ due to the statement 1). The proof is complete.

We introduce a linear subspace $\mathcal{A} = [L, L] \oplus L$ and define a multiplication setting for $x = X \oplus A$, $y = Y \oplus B$

$$xy = ([X, Y] - [A, B]) \oplus ([X, B] - [Y, A]). \tag{3}$$

Observe firstly that by Corollary 1, $[X, Y], [A, B] \in [L, L]$, $[X, B], [Y, A] \in L$ $\forall X, Y \in [L, L], A, B \in L$ and the multiplication is well defined. We introduce the following subspaces

$$Z_1 = \{\, X \in [L, L] \mid [X, B] = 0 \; \forall B \in L \,\},\; Z_2 = \{\, A \in L \mid [A, B] = 0 \; \forall B \in L \,\}$$

of the spaces $[L, L]$ and L respectively.

Lemma 3. 1) *$\mathcal{A}$ is a Lie algebra, its centre $Z(\mathcal{A}) = Z_1 \oplus Z_2$;* 2) *maps*

$$f(X \oplus A)B = [X, B] + iS_A B,\; h(X \oplus A)v = Xv + iAv$$

are linear representations of $\mathcal{A}$ in the spaces $L \otimes \mathbb{C}$ and $\mathbb{C}^n$ respectively. Here $i^2 = -1$.

Proof. 1) The statement that $\mathcal{A}$ is a Lie algebra is directly verified using the known properties of commutators. We omit the corresponding tedious calculations. Let us describe the centre $Z(\mathcal{A})$. Suppose $x = X \oplus A \in Z(\mathcal{A})$. Then $xy = 0$ $\forall y \in \mathcal{A}$. Taking $y = 0 \oplus B$ we obtain that $[X, B] = [A, B] = 0$ $\forall B \in L$. Therefore $x \in Z_1 \oplus Z_2$. Conversely, if $x = X \oplus A \in Z_1 \oplus Z_2$ then X, A commute with all matrices in L and therefore they commute with matrices in $[L, L]$: $[X, Y] = [A, Y] = 0$ $\forall Y \in [L, L]$ (this easily follows from the Jacobi identity). By (3) we have that $xy = 0$ $\forall y \in \mathcal{A}$, i.e. $x \in Z(\mathcal{A})$.

2) For $X \in [L, L]$ we define the operator C_X acting in $L \otimes \mathbb{C}$ as follows: $C_X B = [X, B]$. As is directly verified, $[C_X, C_Y] = C_{[X,Y]}$, $[C_X, S_A] = S_{[X,A]}$ for $X, Y \in [L, L]$, $A \in L$. Let $x = X \oplus A, y = Y \oplus B \in \mathcal{A}$. Then, by the above relation and Lemma 2

$$\begin{aligned}&[f(x), f(y)] = [C_X + iS_A, C_Y + iS_B] = [C_X, C_Y] - [S_A, S_B] + \\ &\quad i([C_X, S_B] - [C_Y, S_A]) = C_{[X,Y]-[A,B]} + iS_{[X,B]-[Y,A]} = f(xy).\end{aligned}$$

Thus the map F is a homomorphism of the algebra $\mathcal{A}$ into the algebra $gl(L\otimes \mathbb{C})$ of linear operators in $L\otimes\mathbb{C}$, i.e. it determines a linear representation of $\mathcal{A}$ in $L\otimes\mathbb{C}$. Further,

$$[h(x),h(y)] = [X+iA, Y+iB] = [X,Y]-[A,B]+i([X,B]-[Y,A]) = h(xy)$$

that is h determines a linear representation of the algebra $\mathcal{A}$ in $\mathbb{C}^n$. The proof is complete.

Firstly, we prove the assertion of Theorem 1 in the case $Z_1=\{0\}$.

Proposition 2. *Let L be a hyperbolic matrix space such that $Z_1=\{0\}$. Then all matrices in L are Hermitian with respect to some scalar multiplication in $\mathbb{C}^n$.*

Proof. If L is a hyperbolic space then its extension $\{\ A+\lambda E \mid \lambda\in\mathbb{R}\ \}$ obtained by adjunction of the unit matrix E is also a hyperbolic space with the same algebra $[L,L]$. Therefore, without loss of generality we may suppose that $E\in L$. As is easy to see, operators $f(x)$ are skew-Hermitian in $L\otimes\mathbb{C}$ with respect to the scalar product $(A,B)=\operatorname{Tr} A\bar{B}$, where $B\to\bar{B}$ denotes the complex conjugation on $L\otimes\mathbb{C}$. Therefore, the symmetric bilinear form $(x,y)=-\operatorname{Tr} f(x)f(y)$ is nonnegative definite. Moreover, if $x=X\oplus A\in\mathcal{A}$ and $(x,x)=0$ then $f(x)=0$. In particular $A=-\frac{i}{2}f(x)E=0$. Then $f(x)B=[X,B]=0\ \forall B\in L$, i.e. $X\in Z_1$. Since $Z_1=\{0\}$ then $X=0$. We conclude that $x=0$ and the form $(\cdot,\cdot)$ is non-degenerate. Thus, this form determines a scalar product on $\mathcal{A}$. Besides, operators $ad_x y = xy$ are skew-symmetric with respect to this scalar product. Indeed,

$$(ad_x y, z) = -\operatorname{Tr} f(xy)f(z) = -\operatorname{Tr}[f(x),f(y)]f(z) = \\ \operatorname{Tr} f(y)[f(x),f(z)] = \operatorname{Tr} f(y)f(xz) = -(y, ad_x z).$$

The above property means that $\mathcal{A}$ is a compact Lie algebra. By known properties of compact Lie algebras (see for instance [4]) $\mathcal{A}=\mathcal{A}_1\oplus Z(\mathcal{A})$, where $\mathcal{A}_1$ is a semi-simple compact Lie algebra, which is the Lie algebra of a unique simple connected compact Lie group G. Moreover, the homomorphism $h:\mathcal{A}_1\to gl(\mathbb{C}^n)$ induces a homomorphism of the Lie groups $\tilde{h}: G\to \mathrm{GL}(\mathbb{C}^n)$. Here $\mathrm{GL}(\mathbb{C}^n)$ is the Lie group of non-degenerate linear operators on $\mathbb{C}^n$ with corresponding Lie algebra $gl(\mathbb{C}^n)$ of all linear operators on $\mathbb{C}^n$. Thus, G acts linearly on $\mathbb{C}^n$: $gv=\tilde{h}(g)v$. The space $\mathbb{C}^n$ is decomposed into a direct sum of indecomposable invariant subspaces under the acting of L: $\mathbb{C}^n=\bigoplus\limits_{k=1}^{m} V_k$. If $x\in Z(\mathcal{A})$ then by Lemma 3 and the condition $Z_1=0$ we claim that $x = 0 \oplus A$, where $[A, B] = 0 \ \forall B \in L$. The latter implies that A acts trivially on the spaces V_k: $A = \lambda_k E$ on V_k. Indeed, in the opposite case V_k can be decomposed into a direct sum of proper subspaces consisting of eigenvectors of A, and these subspaces are invariant for all matrices in L, in view of the condition $[A, B] = 0 \ \forall B \in L$. But this contradicts the fact that V_k are indecomposable. Clearly, all V_k, $k = 1, \ldots, m$ are invariant subspace for matrices in the algebra $\mathcal{A}$ and consequently they are invariant for action of the group G as well. We may suppose that the scalar product in $\mathbb{C}^n$ is chosen in such way that the spaces V_k, $k = 1, \ldots, m$ are one-by-one orthogonal. Then, we define the new invariant scalar product $(u, v)_i$ in $\mathbb{C}^n$, setting $(u, v)_i = \int_G (gu, gv) d\mu(g)$, where μ is the Haar measure in G. Under this new scalar product $\tilde{h}$ takes its values in the group $U(n)$ of unitary operators (matrices) and consequently for $x \in \mathcal{A}_1$ the image $h(x)$ lays in the corresponding algebra of skew-Hermitian matrices $u(n)$. As is easy to see, the spaces V_k, $k = 1, \ldots, m$ remain one-by-one orthogonal under the new scalar product. Therefore, matrices $h(x)$ are skew-Hermitian also for $x = 0 \oplus A \in Z(\mathcal{A})$ because they are skew-Hermitian on the one-by-one orthogonal subspaces V_k (we recall that $h(x) = iA = i\lambda_k E$ on V_k, $k = 1, \ldots, m$). Thus, the image $h(\mathcal{A})$ consists of Hermitian matrices and since $h(0 \oplus A) = iA$ for $A \in L$ we conclude that all matrices in L are Hermitian.

Now we prove that our assumption $Z_1 = 0$ is in fact always satisfied.

Proposition 3. *Let L be a hyperbolic matrix space. Then $Z_1 = \{0\}$.*

Proof. We shall draw the proof by induction in dimension n. If $n = 0, 1$ then $[L, L] = \{0\}$ and there is nothing to prove. Now suppose that $n > 1$ and the statement of our Proposition is true for dimensions less than n. Let $X \in [L, L]$ and $[X, B] = 0 \ \forall B \in L$. We should derive that $X = 0$. Let $\mu \in \mathbb{C}$ be an eigenvalue of X, and $H \subset \mathbb{C}^n$ be the corresponding subspace of eigenvectors. If $H = \mathbb{C}^n$ then $X = \mu E = 0$ (by the condition $\operatorname{Tr} X = 0$), as was to be proved. Thus, suppose that H is a proper linear subspace of $\mathbb{C}^n$. Then H is invariant under the action of L, as it follows from the equality $XAv = AXv = \mu Av$ for all $v \in H$, $A \in L$. Clearly, H is also invariant under the action of $[L, L]$. Therefore, we can define homomorphisms of restriction $A \to A|_H$ of the spaces L, $[L, L]$ into the spaces L_H, $[L_H, L_H]$ respectively. By Lemma 1 L_H is a hyperbolic space of less order $m = \dim H < n$ and $X|_H$ commutes with L_H. Since $m < n$ then by the induction hypothesis $X|_H = 0$ that is $\mu = 0$ and $H = \operatorname{Ker} X$. If $V \subset \mathbb{C}^n$ is a proper linear subspace invariant under the action of L then V is also invariant under the action of $[L, L]$ and

$X|_V \in [L_V, L_V]$, $[X|_V, A|_V] = 0$ $\forall A \in L$. Again by the induction hypothesis we see that $X|_V = 0$ that is $V \subset H$. Thus, H contains all proper invariant subspaces. Now, observe that $H_1 = \operatorname{Im} X$ is also an invariant subspace. Indeed, this directly follows from the commutation identity $AXv = XAv$ $\forall A \in L$, $v \in \mathbb{C}^n$. Therefore $H_1 \subset H$, i.e. $X^2 = 0$. As was shown above, L_H is a hyperbolic space, for which the space $Z_1 = 0$. By Proposition 2 we can choose a scalar multiplication on H such that matrices $A|_H$ are Hermitian for all $A \in L$. Let $H_2 = H \ominus H_1$ be an orthogonal complement to H_1 in H. Since matrices $A \in L$ are Hermitian on H then H_2 is invariant under the action of spaces L and $[L, L]$. Thus we can consider the space $L_{/H_2}$, which is hyperbolic by Lemma 1. We see also that $X/H_2 \in [L_{/H_2}, L_{/H_2}]$ and commutes with $L_{/H_2}$. Assuming that $H_2 \neq \{0\}$ we claim that $\dim \mathbb{C}^n/H_2 < n$ and by the induction hypothesis $X/H_2 = 0$. But this is not true since $\operatorname{Im} X/H_2 = H/H_2 \simeq H_1 \neq \{0\}$. We conclude that $H_2 = 0$, i.e. $H_1 = H$. This implies that $\mathbb{C}^n = (\mathbb{C}^n/H) \oplus H$ and the operator X determines the isomorphism $X : \mathbb{C}^n/H \to H$. We can identify $\mathbb{C}^n/H$ and H due to this isomorphism. After such identification we have that $\mathbb{C}^n = H \oplus H$ and $X(u, v) = (0, u)$. Any operator $A \in L$ can be represented as follows: $A(u, v) = (A_1u, A_2u + A_3v)$ because the space $0 \oplus H$ is invariant. Since $0 = [X, A](u, v) = (0, A_1u - A_3u)$ then $A_3 = A_1$. Further, suppose $A, B \in L$ and $A(u, v) = (A_1u, A_2u + A_1v)$, $B(u, v) = (B_1u, B_2u + B_1v)$. Then, as is directly computed, $[A, B](u, v) = (C_1u, C_2u + C_1v)$, where $C_1 = [A_1, B_1]$, $C_2 = [A_2, B_1] + [A_1, B_2]$. In particular, $\operatorname{Tr} C_1 = \operatorname{Tr} C_2 = 0$. Clearly, this property holds for all matrices from $[L, L]$ because they are linear combination of commutators $[A, B]$, $A, B \in L$. Since $X \in [L, L]$ and $X(u, v) = (0, Eu)$, where E is a unit matrix, we obtain the equality $\operatorname{Tr} E = 0$, which is not true. Thus, our assumption $X \neq 0$ fails, and $X = 0$, as was to be proved.

Now we are ready to finish the **proof of Theorem 1.** The direct statement of Theorem 1 immediately follows from Propositions 2,3. Conversely, if all matrices in a linear matrix space L are Hermitian then they have simple real spectra and the space L is hyperbolic. The proof of Theorem 1 is complete.

We give below one application of the obtained result to the problem of symmetrizability for a first order system

$$u_t + \sum_{k=1}^{m} A_k u_{x_k} = 0, \quad A_k = A_k(t, x, u) \in \operatorname{Mat}(n, \mathbb{R}), \quad k = 1, \ldots, m. \tag{4}$$

Recall that system (4) is symmetrizable if for fixed t, x, u all matrices A_k, $k = 1, \dots, m$ can be simultaneously symmetrized by the appropriate choice of a basis or, what is the same, by the choice of a scalar product (Bu, v), where the matrix B is positive definite. Multiplying system (4) by B from the left, we obtain the following symmetric form of this system

$$Bu_t + \sum_{k=1}^{m} C_k u_{x_k} = 0,$$

where matrices B, C_k, $k = 1, \dots, m$ are symmetric and B is positive definite.

Denote by M a linear hull of the matrices A_k, $k = 1, \dots, m$. Symmetrizability of system (4) can be formulated as possibility to symmetrize all matrices from M. Clearly, the hyperbolicity condition

$$A \text{ has simple real spectrum } \forall A \in M \tag{5}$$

is necessary for symmetrizability of real linear matrix subspace $M \subset \mathrm{Mat}(n, \mathbb{C})$. In the case of complex matrices, we say that M is symmetrizable if it is possible to reduce all matrices in M to Hermitian form. When $m = 1$ or $n = 2$ condition (5) seems to be also sufficient for symmetrizability (see for instance [3]).

It turns out that this statement remains true only in the indicated cases. If $n > 2$ then condition (5) and even the more restrictive condition of strict hyperbolicity

$$A \text{ has distinct and real eigenvalues } \forall A \in M, A \neq 0 \tag{6}$$

is not sufficient for symmetrizability of a matrix space M with $dim M > 1$. The corresponding example was constructed in [3]. For the sake of completeness we give this example below .

Example. For $n = 3$ we consider the following two-dimensional linear matrix space M consisting of matrices

$$A = \begin{pmatrix} 0 & 0 & a-b \\ 0 & 0 & b \\ a-b & a & 0 \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

The eigenvalues of A are easily computed: $\lambda_1 = 0$, $\lambda_{2,3} = \pm\sqrt{(a-b)^2 + ab}$. They are real and distinct for $A \neq 0$ since the quadratic form $(a-b)^2 + ab$ is positive definite. Thus, condition (6) is satisfied. Let us prove that this "strictly hyperbolic" family can not be symmetrized. Assuming the contrary we can find a scalar product (Px, y) corresponding to some positive definite matrix $P = \begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{pmatrix}$ such that all matrices $A \in M$ are symmetric. This means that $(PAx, y) = (x, PAy)$, i.e. the matrices PA are symmetric under the original scalar product. Expanding equalities $(PA)_{12} = (PA)_{21}$, $(PA)_{13} = (PA)_{31}$, $(PA)_{23} = (PA)_{32}$ we get relations $ap_3 = (a-b)p_5$, $(a-b)p_1 + bp_2 = (a-b)p_6$, $(a-b)p_2 + bp_4 = ap_6$ for all $a, b \in \mathbb{R}$. The latter relations easily imply that $p_i = 0$, $i = 1, \ldots, 6$, i.e. $P = 0$. But this contradicts the condition $P > 0$. Therefore the family M is not symmetrizable. Taking basis matrices A_1, A_2, corresponding to values $a = 1, b = 0$ and $a = b = 1$ we obtain the strictly hypebolic but not simmetrizable system $q_t = A_1 q_x + A_2 q_y = 0$, $q = (u, v, w)^\top$, which can be written in the explicit form as follows

$$\begin{cases} u_t = w_x \\ v_t = w_y \\ w_t = (u+v)_x + v_y \end{cases} .$$

The above Example stimulates ones to search general conditions of symmetrizability. One criterion was found in [3]. Namely, as was shown in [3], a space M can be symmetrized if and only if all matrices in the minimal Lie algebra containing iM (with $i^2 = -1$) have simple imaginary spectra.

Now we are able to give another criterion following from Theorem 1. Let $L = L(M)$ be the minimal linear matrix subspace, which contains M and satisfies condition (1).

Theorem 2. *The family M is symmetrizable if and only if the space L is hyperbolic.*

Proof. If all matrices $A \in M$ are symmetric (Hermitian) under some scalar product then the same property holds for matrices from L and consequently these matrices have simple real spectra, i.e. the space L is hyperbolic.

The converse directly follows from Theorem 1.

Remark that the result of the Example above follows from Theorem 2. Indeed, let $A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ are basis matrices defined in the Example. Then the matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = A_2 A_1 A_2 = \frac{1}{2}[(S_{A_2})^2 A_1 - S_{A_2^2} A_1] \in L$$

but its spectrum is not simple.

Acknowledgments. The author is grateful to the University of Franche-Comté (Besançon, France), where this paper was finished, for invitation and very stimulating working atmosphere. He had also acknowledged the partial support of the Russian Foundation for Basic Research (grant No. 03-01-00444) and the Program "Universities of Russia" (grant No. UR.04.01.184).

References

[1] *Panov E.Yu.* A class of systems of quasilinear conservation laws. Sbornik: Mathematics. 188 (1997), no. 5, 725–751.

[2] *Panov E.Yu.* On a nonlocal theory of generalized entropy solutions of the Cauchy problem for a class of hyperbolic systems of conservation laws. Izvestiya: Mathematics. 63 (1999), no. 1, 129–179.

[3] *Panov E.Yu.* Symmetrizability of first-order hyperbolic systems. Doklady. Mathematics. 69 (2004), no. 3, 341–343.

[4] *Serre J.-P.* Lie algebras and Lie groups. New York - Amsterdam: W. A. Benjamin, Inc. 1965.