Effective Thermoviscoelasticity of a Saturated Porous Ground

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Abstract

The linearized model of joint motion of an elastic heat-conducting porous skeleton and a viscous compressible thermofluid filling the pore space is considered. It is assumed that the pore space has a periodic geometry and the model incorporates a small parameter, which is the ratio of micro- and macroscopic length scales. A homogenization procedure, i.e., a limiting transition as the small parameter tends to zero, is fulfilled. It is established as the result, that the limiting distributions of displacements and temperature solve a well-posed homogenized linear model of thermoviscoelasticity with shape and heat memory. Moreover, the coefficients of the homogenized model are uniquely defined by data given for microstructure. The homogenization procedure is fulfilled fully rigorously by means of the two-scale convergence method.

Introduction

We consider the linearized model of *thermoporoelasticity*, namely, the linear dynamical model of joint motion of an elastic porous ground and a viscous compressible fluid, entirely filling the pore space, with taking into account heat-conduction phenomena. Equations of thermoporoelasticity are in focus of specialists in various fields of mechanics because of many reasons: for example [4, 15], the enhanced recovery of gas, oil and geothermally heated water depends upon flow in porous strata; underwater acoustics involves propagation in the water-saturated bottom of the ocean; liquid waste disposed of underground seeps into pores; pore fluids in the ground are believed to play a role in the triggering earthquakes.

The major difficulty in studying thermoporoelasticity is due to the following fact. The thermomechanical properties of the solid skeleton and the porous fluid are very different, and at the same time pore diameters are very small comparing to the size of the entire porous body. This implies that, whenever domains occupied by the solid and liquid phases are distinguished in analysis, i.e., description of the thermomechanical system is fulfilled using *microscale*, the corresponding mathematical models exhibit rapidly oscillatory regimes. Therefore analysis of these models is likely to be ill-fated in applications, especially in numerical simulations, because amount of calculations may be impossible even for supercomputers.

However, it is well-known (see, for example, [6]) that on a length scale much bigger than diameters of single pores, in other words, on *macroscale*, porous media have stable physical properties (compressibility, heat-conductivity, viscosity, etc.), which are called average or effective characteristics of media and which are, in general, different from the corresponding characteristics of the distinct phases. Thus there arises the problem to evaluate effective characteristics and derive effective macroscopic equations, starting from data given for heterogeneous microstructure. This is called a homogenization problem. It consists of carrying-out and justification of a limiting transition in equations of microstructure, as the small parameter (say, ε) – ratio of characteristic micro- and macroscale lengthes – tends to zero.

The main aim of the present work is to solve the homogenization problem for the model of linear thermoporoelasticity under consideration in a mathematically rigorous way, under assumptions that the porous ground has a connected periodic geometric structure and that the physical characteristics of each phase (viscosity, elasticity, heat-conductivity, heat-capacity coefficients, etc) do not depend on ε. We fulfill and rigorously justify the homogenization procedure by means of the two-scale convergence method. As the result, we construct a well-posed model of linear thermoviscoelasticity with shape and heat memory effects. Moreover, all coefficients in the equations of the homogenized model, in other words, effective coefficients, are defined uniquely starting from data give for microstructure. This study is somewhat close to the work [9], devoted to application of the two-scale convergence method to homogenization of poroelasticity without thermal effects, and may be regarded as its continuation in the case when thermal conductivity is essential.

The rest of the paper is organized as follows. In Sec. 1 we give the description of the heterogeneous microstructure, which incorporates a small parameter $\varepsilon > 0$ and is our starting point. We recall the existence and uniqueness theorem for this model and state the uniform in ε bounds on its solutions. In Secs. 2.1–2.2 the main results of the paper are formulated as theorems 1 and 2 and the statement of averaged problem B. In Sec 2.3 the conclusions about physical sense and mathematical well-posedness of problem B are made. In Secs. 3–4 theorem 1 is proved and the homogenized model, i.e., statement of problem B, is constructed. Sec. 5 is devoted to justification of theorem 2 and ends the paper.

1. Heterogeneous model of linear thermoporoelasticity on microscale

According to the fundamentals of continuum mechanics [18, ch. I], the joint motion of a heat-conducting elastic porous body and a viscous thermofluid is described by the mass, linear momentum, and energy balance equations, the first and second laws of thermodynamics in each phase, individual state equations, determining thermomechanical behavior in each phase, and certain conditions on solid–liquid interface. Assuming a priori, that perturbations of the considered thermomechanical system are small about some rest state, applying in this view the classical linearization formalism [8, §8.1] to the equations of the model, passing to the proper dimensionless variables, and supplementing geometry of the porous body with connectivity and periodicity properties, we arrive eventually at the closed system of linear thermoporoelasticity equations. The initial-boundary value problem for this system is formulated below and is considered further throughout the paper.

Problem A. (Model of linear thermoporoelasticity on the microscale.) In the space-time prism $Q = \Omega \times (0,T)$, where T is a positive constant and Ω is an open unit cube in $\mathbb{R}^3$, i.e., $\Omega = (0,1)^3$, divided into two disjoint open sets Ω^ε_f and Ω^ε_s and the boundary $\Gamma_\varepsilon = \bar{\Omega}^\varepsilon_f \cap \bar{\Omega}^\varepsilon_s$ between them, find a displacement field $\boldsymbol{w}^\varepsilon = \boldsymbol{w}^\varepsilon(\boldsymbol{x},t)$ and a temperature distribution $\theta^\varepsilon = \theta^\varepsilon(\boldsymbol{x},t)$, satisfying the equations

$$\alpha_\tau \rho_f \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2} - \operatorname{div}_x\Big\{\alpha_\mu \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) + \Big(\alpha_p \operatorname{div}_x \boldsymbol{w}^\varepsilon + \alpha_\nu \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} - \alpha_{\theta f}\theta^\varepsilon\Big)\mathbb{I}\Big\} = \alpha_F \rho_f \boldsymbol{F}, \quad (\boldsymbol{x},t) \in \Omega^\varepsilon_f \times (0,T), \tag{1.1a}$$

$$c_{pf}\frac{\partial \theta^\varepsilon}{\partial t} = \operatorname{div}_x(\varkappa_f \nabla_x \theta^\varepsilon) - \alpha_{\theta f}\operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} + \Psi^\varepsilon_f, \quad (\boldsymbol{x},t) \in \Omega^\varepsilon_f \times (0,T), \tag{1.1b}$$

$$\alpha_\tau \rho_s \frac{\partial^2 \boldsymbol{w}^\varepsilon}{\partial t^2} - \operatorname{div}_x \big\{ \alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) + (\alpha_\eta \operatorname{div}_x \boldsymbol{w}^\varepsilon - \alpha_{\theta s} \theta^\varepsilon) \mathbb{I} \big\} = \alpha_F \rho_s \boldsymbol{F}, \quad (\boldsymbol{x}, t) \in \Omega_s^\varepsilon \times (0, T), \tag{1.1c}$$

$$c_{ps} \frac{\partial \theta^\varepsilon}{\partial t} = \operatorname{div}_x (\varkappa_s \nabla_x \theta^\varepsilon) - \alpha_{\theta s} \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} + \Psi_s^\varepsilon, \quad (\boldsymbol{x}, t) \in \Omega_s^\varepsilon \times (0, T), \tag{1.1d}$$

the relations on the interface Γ_ε

$$\theta^\varepsilon_{(s)}(\boldsymbol{x}_0, t) = \theta^\varepsilon_{(f)}(\boldsymbol{x}_0, t), \quad \boldsymbol{w}^\varepsilon_{(s)}(\boldsymbol{x}_0, t) = \boldsymbol{w}^\varepsilon_{(f)}(\boldsymbol{x}_0, t), \quad \boldsymbol{x}_0 \in \Gamma_\varepsilon, \quad t \geq 0, \tag{1.1e}$$

$$\begin{gathered} \big\{ \alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon_{(s)}) + (\alpha_\eta \operatorname{div}_x \boldsymbol{w}^\varepsilon_{(s)} - \alpha_{\theta s} \theta^\varepsilon_{(s)}) \mathbb{I} \big\} \cdot \boldsymbol{n}^\varepsilon = \\ \Big\{ \alpha_\mu \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon_{(f)}}{\partial t}\Big) + \Big(\alpha_p \operatorname{div}_x \boldsymbol{w}^\varepsilon_{(f)} + \alpha_\nu \operatorname{div}_x \frac{\partial \boldsymbol{w}^\varepsilon_{(f)}}{\partial t} - \alpha_{\theta f} \theta^\varepsilon_{(f)} \Big) \mathbb{I} \Big\} \cdot \boldsymbol{n}^\varepsilon, \quad \boldsymbol{x}_0 \in \Gamma_\varepsilon, \quad t \geq 0, \end{gathered} \tag{1.1f}$$

$$\varkappa_s \nabla_x \theta^\varepsilon_{(s)} \cdot \boldsymbol{n}^\varepsilon = \varkappa_f \nabla_x \theta^\varepsilon_{(f)} \cdot \boldsymbol{n}^\varepsilon, \quad \boldsymbol{x}_0 \in \Gamma_\varepsilon, \quad t \geq 0, \tag{1.1g}$$

initial data

$$\boldsymbol{w}^\varepsilon|_{t=0} = \boldsymbol{w}^\varepsilon_0, \quad \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big|_{t=0} = \boldsymbol{v}^\varepsilon_0, \quad \theta^\varepsilon|_{t=0} = \theta^\varepsilon_0, \quad \boldsymbol{x} \in \Omega, \tag{1.1h}$$

and the homogeneous boundary conditions on $\partial\Omega$:

$$\boldsymbol{w}^\varepsilon = 0, \quad \theta^\varepsilon = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad t \geq 0. \tag{1.1i}$$

Geometry of the domains Ω_f^ε and Ω_s^ε is given and depends on a small parameter $\varepsilon > 0$, which is the ratio of the characteristic lengthes l_0 and L_0 on the micro- and macroscales, respectively. Since Ω is the unit cube, we clearly have $l_0 = \varepsilon$ and $L_0 = 1$. The formal description of the geometry of the porous structure is given similarly to [9] as follows. Firstly, a structure inside the unit pattern periodicity cell $\mathcal{Y} = (0,1)^3$ is postulated: we assume that $\mathcal{Y}_s$, the solid part, is some open subset of $\mathcal{Y}$, and $\mathcal{Y}_f$, the liquid part, is the complement of the closure of $\mathcal{Y}_s$ in $\mathcal{Y}$, i.e., $\mathcal{Y}_f = \mathcal{Y} \setminus \bar{\mathcal{Y}}_s$. Secondly, the periodic repetition of $\mathcal{Y}_s$ over the whole space $\mathbb{R}^3$ is constructed and it is set that $\mathcal{Y}_s^k = \mathcal{Y}_s + \boldsymbol{k}$, $\boldsymbol{k} \in \mathbb{Z}^3$. Evidently, that thus obtained set $E_s = \bigcup_{k \in \mathbb{Z}^3} \mathcal{Y}_s^k$ and the complement of its closure $E_f = \mathbb{R}^3 \setminus \bar{E}_s$ both are open sets in $\mathbb{R}^3$. The following demand are imposed on geometry of $\mathcal{Y}_s$ and E_s:

- $\mathcal{Y}_s$ is a connected set of strictly positive measure with a Lipschitz boundary and $\mathcal{Y}_f$ has strictly positive measure in $\mathcal{Y}$ as well.

- E_s and E_f are open sets in $\mathbb{R}^3$ with a Lipschitz interface between them; E_s and E_f are locally situated on one side of their boundary; E_s is connected.

Basing on this construction, introduce a regular mesh of size ε, covering Ω, each cell being a cube $\mathcal{Y}_i^\varepsilon$ with an edge length equal to ε. For simplicity always assume that $1/\varepsilon$ is an integer positive. Each cube $\mathcal{Y}_i^\varepsilon$, $i = 1, 2, 3, \ldots 1/\varepsilon^3$ is homeomorphic to $\mathcal{Y}$ by linear homeomorphism Π_i^ε, being composed of a $1/\varepsilon$-fold compression and translation. Now define $\mathcal{Y}_{si}^\varepsilon = \Pi_i^\varepsilon(\mathcal{Y}_s)$, $\mathcal{Y}_{fi}^\varepsilon = \Pi_i^\varepsilon(\mathcal{Y}_f)$, and, finally,

$$\Omega_s^\varepsilon = \bigcup_{1 \leq k \leq 1/\varepsilon^3} \mathcal{Y}_{sk}^\varepsilon, \quad \Omega_f^\varepsilon = \bigcup_{1 \leq k \leq 1/\varepsilon^3} \mathcal{Y}_{fk}^\varepsilon, \quad \Gamma_\varepsilon = \bar{\Omega}_s^\varepsilon \cap \bar{\Omega}_f^\varepsilon.$$

Clearly, $\Omega_f^\varepsilon = \varepsilon E_f \cap \Omega$ and $\Omega_s^\varepsilon = \varepsilon E_s \cap \Omega$.

The domains Ω_f^ε and Ω_s^ε are occupied by a viscous compressible fluid and an elastic solid body, respectively. Equation (1.1a) is a system of three scalar Stokes equations of fluid dynamics, equation (1.1c) is a system of three scalar Lamé's equations of elasticity, and (1.1b) and (1.1d) are the heat equations in the liquid and solid domains, respectively. Relations (1.1e)–(1.1g) are the continuity equations on the interface Γ_ε for the temperature, displacement, and the normal stresses and heat fluxes, respectively. In (1.1a) and (1.1c) and further in the paper by $\mathbb{D}(x, \boldsymbol{\varphi})$ we denote the symmetric part of the gradient of some enough regular vector-function $\boldsymbol{\varphi}(\boldsymbol{x})$: $\mathbb{D}_{ij}(x, \boldsymbol{\varphi}) = (1/2)(\partial_{x_i}\varphi_j + \partial_{x_j}\varphi_i)$, $i, j = 1, 2, 3$. The rest of the notation for differential operators in the paper is standard. By $\mathbb{I}$ we denote the identical transformation in $\mathbb{R}^3$, i.e., $\mathbb{I} = (\delta_{ij})$, where δ_{ij} is Kronecker's delta. In (1.1e)–(1.1g) the following notation for values on the interface Γ_ε is used: for any $\boldsymbol{x}_0 \in \Gamma_\varepsilon$ and for any function $\varphi^\varepsilon(\boldsymbol{x})$, continuous inside Ω_s^ε and inside Ω_f^ε, set

$$\varphi^\varepsilon_{(s)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_s^\varepsilon}} \varphi^\varepsilon(\boldsymbol{x}), \quad \varphi^\varepsilon_{(f)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}_0 \\ \boldsymbol{x} \in \Omega_f^\varepsilon}} \varphi^\varepsilon(\boldsymbol{x}).$$

Vector $\boldsymbol{n}^\varepsilon(\boldsymbol{x}_0)$ is the unit normal to Γ_ε at a point $\boldsymbol{x}_0$, pointing into Ω_f^ε.

Constant positive dimensionless coefficients α_τ, α_μ, α_λ, α_p, α_η, α_ν, $\alpha_{\theta f}$, $\alpha_{\theta s}$, α_F, $\varkappa_f$, $\varkappa_s$, ρ_f, ρ_s, c_{pf}, c_{ps} do not depend on ε. Along with dimensionless functions $\Psi_s^\varepsilon(\boldsymbol{x}, t)$, $\Psi_f^\varepsilon(\boldsymbol{x}, t)$, and a vector-function $\boldsymbol{F}(\boldsymbol{x}, t)$, they are given and relate to the dimensional physical characteristics of the problem via the following identities:

$$\begin{aligned}
&\alpha_\nu = \frac{1}{p_0\tau_0}\Big(\nu - \frac{2}{3}\mu\Big), && \alpha_\mu = \frac{2\mu}{p_0\tau_0}, && \alpha_\lambda = \frac{2\lambda}{p_0}, && \alpha_p = \frac{K}{p_0}, && \alpha_F = \frac{\gamma_0 g L_0}{c_0^2}, \\
&\alpha_\eta = \frac{1}{p_0}\Big(\eta - \frac{2}{3}\lambda\Big), && \alpha_\tau = \frac{\gamma_0 L_0^2}{c_0^2\tau_0^2}, && \alpha_{\theta s} = \frac{\gamma_s \eta \vartheta_0}{p_0}, && \alpha_{\theta f} = \frac{\gamma_f K \vartheta_0}{p_0}, && \boldsymbol{F} = \boldsymbol{F}'/g, \\
&\varkappa_f = \frac{\tau_0\vartheta_0}{L_0^2 p_0}\varkappa'_f, && \varkappa_s = \frac{\tau_0\vartheta_0}{L_0^2 p_0}\varkappa'_s, && \rho_f = \frac{\rho'_f}{\rho_0}, && \Psi_s^\varepsilon = \frac{\tau_0}{p_0}\Psi'_{s\varepsilon}, \\
&\rho_s = \frac{\rho'_s}{\rho_0}, && c_{pf} = \frac{\vartheta_0}{p_0}c'_{pf}, && c_{ps} = \frac{\vartheta_0}{p_0}c'_{ps}, && \Psi_f^\varepsilon = \frac{\tau_0}{p_0}\Psi'_{f\varepsilon}.
\end{aligned}$$

Here $L_0 = 1$ is the characteristic size of Ω; τ_0 is a characteristic duration of physical processes; g is the gravity acceleration; p_0 is the atmosphere pressure; ρ_0, c_0, and $\gamma_0 = 7/5$ are respective mean density, speed of sound, and polytropic exponent in air at the temperature 273 and at the atmosphere pressure; ϑ_0 is the temperature difference between the boiling- and freezing-points of water at the atmosphere pressure. Coefficients μ, ν, K, ρ'_f, γ_f, $\varkappa'_f$, and c'_{pf} in the fluid phase are respective shear and bulk viscosities, hydrostatic compression modulus, mean density, heat extension, heat conductivity, and specific heat capacity at constant pressure. Coefficients λ, η, ρ'_s, γ_s, $\varkappa'_s$, and c'_{ps} in the solid phase are respective shear and bulk elasticity moduli, mean density, heat extension, heat conductivity, and specific heat capacity at constant pressure. These physical dimensional characteristics of the fluid and the solid are constant and correspond to the rest state, in the neighborhood of which the linearization procedure for the basic nonlinear model was fulfilled. Finally, $\boldsymbol{F}'$ is a density of distributed mass forces and $\Psi'_{f\varepsilon}$ and $\Psi'_{s\varepsilon}$ are volumetric densities of external heat application in the fluid and the solid phases, respectively.

Now let us introduce some notation and then formulate a definition of generalized solution to problem A. By χ and χ^ε denote the indicator functions of E_f in $\mathbb{R}^3$ and of Ω_f^ε

in Ω, respectively:

$$\chi(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{x} \in E_f, \\ 0 & \text{if} \quad \boldsymbol{x} \in \mathbb{R}^3 \setminus E_f, \end{cases} \qquad \chi^\varepsilon(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad \boldsymbol{x} \in \Omega^\varepsilon_f, \\ 0 & \text{if} \quad \boldsymbol{x} \in \Omega \setminus \Omega^\varepsilon_f. \end{cases}$$

In view of the structures of the sets E_f and Ω^ε_f it is clear that

$$\chi^\varepsilon(\boldsymbol{x}) = \chi\left(\frac{\boldsymbol{x}}{\varepsilon}\right), \quad \boldsymbol{x} \in \Omega,$$

and that χ is 1-periodic in $\mathbb{R}^3$. Set

$$\rho^\varepsilon = \chi^\varepsilon \rho_f + (1-\chi^\varepsilon)\rho_s, \quad \alpha^\varepsilon_\theta = \chi^\varepsilon \alpha_{\theta f} + (1-\chi^\varepsilon)\alpha_{\theta s}, \quad c^\varepsilon_p = \chi^\varepsilon c_{pf} + (1-\chi^\varepsilon)c_{ps},$$
$$\varkappa^\varepsilon = \chi^\varepsilon \varkappa_f + (1-\chi^\varepsilon)\varkappa_s, \quad \Psi^\varepsilon = \chi^\varepsilon \Psi^\varepsilon_f + (1-\chi^\varepsilon)\Psi^\varepsilon_s.$$

Definition 1. *A pair of functions $(\boldsymbol{w}^\varepsilon, \theta^\varepsilon)$ is called a generalized solution of problem A, if it satisfies the regularity demands*

$$\boldsymbol{w}^\varepsilon \in W^1_2(Q), \quad \chi^\varepsilon \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) \in L^2(Q), \quad \theta^\varepsilon \in L^2(0,T; W^1_2(\Omega)), \tag{1.2}$$

the boundary conditions (1.1i) *and initial condition $\boldsymbol{w}^\varepsilon|_{t=0} = \boldsymbol{w}^\varepsilon_0$ in the trace sense, and the integral equalities*

$$\int_Q \Big(\alpha_\tau \rho^\varepsilon \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} - \Big\{\chi^\varepsilon \Big[\alpha_\mu \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) + \Big(\alpha_p div_x \boldsymbol{w}^\varepsilon + \alpha_\nu div_x \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)\mathbb{I}\Big]$$
$$+ (1-\chi^\varepsilon)\Big[\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) + (\alpha_\eta div_x \boldsymbol{w}^\varepsilon)\mathbb{I}\Big] - \alpha^\varepsilon_\theta \theta^\varepsilon \mathbb{I}\Big\} : \nabla_x \boldsymbol{\varphi} + \alpha_F \rho^\varepsilon \boldsymbol{F} \cdot \boldsymbol{\varphi}\Big) d\boldsymbol{x}dt$$
$$+ \int_\Omega \alpha_\tau \rho^\varepsilon \boldsymbol{v}^\varepsilon_0 \cdot \boldsymbol{\varphi}(\boldsymbol{x}, 0) d\boldsymbol{x} = 0, \tag{1.3}$$

$$\int_Q \Big(c^\varepsilon_p \theta^\varepsilon \frac{\partial \psi}{\partial t} - \varkappa^\varepsilon \nabla_x \theta^\varepsilon \cdot \nabla_x \psi + \alpha^\varepsilon_\theta (div_x \boldsymbol{w}^\varepsilon)\frac{\partial \psi}{\partial t} + \Psi^\varepsilon \psi\Big) d\boldsymbol{x}dt$$
$$+ \int_\Omega \big(c^\varepsilon_p \theta^\varepsilon_0 + \alpha^\varepsilon_\theta div_x \boldsymbol{w}^\varepsilon_0\big)\psi(\boldsymbol{x}, 0) d\boldsymbol{x} = 0 \tag{1.4}$$

for all smooth test vector-functions $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ and scalar functions $\psi(\boldsymbol{x}, t)$ vanishing near $\partial\Omega$ and in a neighborhood of $t = T$.

Remark 1. *On the strength of regularity properties* (1.2) *and the well-known facts from the theory of Sobolev space $W^1_2(\Omega)$, a generalized solution of problem A (if any) necessarily satisfies conditions* (1.1e) *on Γ^ε in the trace sense. In view of condition* (1.1e)*, after integration by parts, integral equalities* (1.3) *and* (1.4) *yield equations* (1.1a) *and* (1.1b) *in $\Omega^\varepsilon_f \times (0,T)$ and equations* (1.1c) *and* (1.1d) *in $\Omega^\varepsilon_s \times (0,T)$ in the distributions sense, and initial relations $\boldsymbol{w}^\varepsilon_t|_{t=0} = \boldsymbol{v}^\varepsilon_0$ and $\theta^\varepsilon|_{t=0} = \theta^\varepsilon_0$, and conditions* (1.1f) (1.1g) *on Γ^ε in the trace sense. Thus the above introduced notion of generalized solutions is consistent with the formulation of problem A.*

The following statements of the well-posedness of problem A and the bounds on solutions are justified by the classical methods in the theory of generalized solutions of the problems in mathematical physics [11]. An extended proof can be found in [12].

Proposition 1. *For all fixed $\varepsilon > 0$, for any given data $\boldsymbol{w}_0^\varepsilon \in \overset{\circ}{W}{}_2^1(\Omega)$, $\boldsymbol{v}_0^\varepsilon, \theta_0^\varepsilon \in L^2(\Omega)$, $\boldsymbol{F}^\varepsilon, \Psi^\varepsilon \in L^2(Q)$ problem A has a unique generalized solution $(\boldsymbol{w}^\varepsilon, \theta^\varepsilon)$ in the sense of definition 1. This solution admits the energy estimate*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in[0,\tau]}\Big(\|\boldsymbol{w}^\varepsilon(t)\|_{2,\Omega}^2 + \|\nabla_x \boldsymbol{w}^\varepsilon(t)\|_{2,\Omega}^2 + \|\partial_t \boldsymbol{w}^\varepsilon(t)\|_{2,\Omega}^2\Big) + \|\chi^\varepsilon \mathbb{D}(x, \partial_t \boldsymbol{w}^\varepsilon)\|_{2,\Omega\times(0,\tau)}^2 \\
&\qquad\qquad + \operatorname*{ess\,sup}_{t\in[0,\tau]}\Big(\|\theta^\varepsilon(t)\|_{2,\Omega}^2 + \|\nabla_x \theta^\varepsilon(t)\|_{2,\Omega}^2\Big) \\
&\le C_0 \cdot \big(\|\boldsymbol{F}\|_{2,\Omega\times(0,\tau)}^2 + \|\Psi^\varepsilon\|_{2,\Omega\times(0,\tau)}^2 + \|\boldsymbol{w}_0^\varepsilon\|_{2,\Omega}^2 + \|\nabla_x \boldsymbol{w}_0^\varepsilon\|_{2,\Omega}^2 + \|\boldsymbol{v}_0^\varepsilon\|_{2,\Omega}^2 + \|\theta_0^\varepsilon\|_{2,\Omega}^2\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall\, \tau \in (0, T], \quad (1.5)
\end{aligned}$$

where $C_0 = C_0(T, \alpha_\tau, \alpha_\mu, \alpha_\lambda, \alpha_p, \alpha_\eta, \alpha_\nu, \alpha_{\theta f}, \alpha_{\theta s}, \alpha_F, \varkappa_f, \varkappa_s, \rho_f, \rho_s, c_{pf}, c_{ps})$ is a constant independent of ε.

2. Homogeneous macroscopic model of effective thermoviscoelasticity: formulation of the main results

Theorems 1 and 2 and the statement of averaged problem B below in Secs. 2.1–2.2 are the main results of the article. In Sec. 2.3 the conclusions about the physical sense and the mathematical well-posedness of problem B follow immediately from the assertions of theorems 1 and 2.

2.1. Convergence of the homogenization process. Averaged model.

Theorem 1. *Let functions $\boldsymbol{w}_0^\varepsilon$, $\boldsymbol{v}_0^\varepsilon$, θ_0^ε, $\boldsymbol{F}$, and Ψ^ε be given and satisfy the assumptions of proposition 1 and the limiting relations*

$$\boldsymbol{w}_0^\varepsilon \to \boldsymbol{w}_0^* \text{ weakly in } W_2^1(\Omega), \quad \Psi^\varepsilon \to \bar{\Psi} \text{ weakly in } L^2(Q), \tag{2.1}$$

$$\boldsymbol{v}_0^\varepsilon \to \boldsymbol{V}_0(\boldsymbol{x}, \boldsymbol{y}),\ \theta_0^\varepsilon \to \Theta_0(\boldsymbol{x}, \boldsymbol{y}) \text{ in the two-scale sense} \tag{2.2}$$

for $\varepsilon \searrow 0$ with some functions $\boldsymbol{w}_0 \in \overset{\circ}{W}{}_2^1(\Omega)$, $\bar{\Psi} \in L^2(Q)$, $\boldsymbol{V}_0, \Theta_0 \in L^2(\Omega \times \mathcal{Y})$. Let a pair of functions $(\boldsymbol{w}^\varepsilon, \theta^\varepsilon)$ be the generalized solution of problem A corresponding to the given functions $\boldsymbol{w}_0^\varepsilon$, $\boldsymbol{v}_0^\varepsilon$, θ_0^ε, $\boldsymbol{F}$ Ψ^ε for an arbitrary fixed $\varepsilon > 0$ such that $\varepsilon^{-1} \in \mathbb{N}$.

Then, as $\varepsilon \searrow 0$ ($\varepsilon^{-1} \in \mathbb{N}$), the sequence $(\boldsymbol{w}^\varepsilon, \theta^\varepsilon)$ weakly in $W_2^1(Q) \times L^2(0, T; W_2^1(\Omega))$ tends to a pair of functions $(\boldsymbol{w}^*, \theta^*)$, which is a generalized solution of problem B, stated below.

In the statement of problem B, the constant fourth-rank tensors $\mathbb{A}_0$ and $\mathbb{A}_1$, the constant 3×3-matrices $\mathbb{C}_0$, $\mathbb{E}_0$, and $\mathbb{E}_1$, the function $t \mapsto \mathbb{A}_2(t)$ with values in the space of fourth-rank tensors, and the function $t \mapsto \mathbb{C}_1(t)$ with values in the space of 3×3-matrices depend only on the geometry of domains $\mathcal{Y}_f$ and $\mathcal{Y}_s$ and on the quantities α_μ, α_p, α_ν, α_λ, α_η, $\alpha_{\theta f}$, $\alpha_{\theta s}$, $\varkappa_f$, and $\varkappa_s$, and are uniquely defined by equations (4.5)–(4.18) *(see in Sec. 4).*

The demand of two-scale convergence, imposed in limiting relation (2.2), is formulated explicitly in Sec. 3.

Problem B. In the space-time prism $Q = \Omega \times (0,T)$, where $T = \mathrm{const} > 0$ and $\Omega = (0,1)^3$, find a displacement field $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x},t)$ and a temperature distribution $\theta = \theta(\boldsymbol{x},t)$, satisfying the equations

$$\alpha_\tau \bar{\rho} \frac{\partial^2 \boldsymbol{w}}{\partial t^2} - \mathrm{div}_x \Big\{ \mathbb{A}_0 : \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) + \mathbb{A}_1 : \mathbb{D}(x, \boldsymbol{w}) - \mathbb{C}_0 \theta \\ + \int_0^t \mathbb{A}_2(t-\tau) : \mathbb{D}(x, \boldsymbol{w}(\tau)) d\tau - \int_0^t \mathbb{C}_1(t-\tau)\theta(\tau) d\tau \Big\} = \alpha_F \bar{\rho} \boldsymbol{F}, \quad (\boldsymbol{x},t) \in Q, \quad (2.3)$$

$$\alpha_\tau \bar{c}_p \frac{\partial \theta}{\partial t} - \mathrm{div}_x \Big\{ \mathbb{E}_0 \nabla_x \theta - \mathbb{E}_1 \frac{\partial \boldsymbol{w}}{\partial t} \Big\} = \bar{\Psi}, \quad (\boldsymbol{x},t) \in Q, \quad (2.4)$$

initial data

$$\boldsymbol{w}|_{t=0} = \boldsymbol{w}_0^*, \qquad \boldsymbol{x} \in \Omega, \quad (2.5)$$

$$(\partial \boldsymbol{w}/\partial t)|_{t=0} = \boldsymbol{v}_0^* \stackrel{def}{=} (1/\bar{\rho}) \langle (\chi \rho_f + (1-\chi)\rho_s) \boldsymbol{V}_0 \rangle_{\mathcal{Y}}, \qquad \boldsymbol{x} \in \Omega, \quad (2.6)$$

$$\theta|_{t=0} = \theta_0^* \stackrel{def}{=} (1/\bar{c}_p) \langle (\chi c_{pf} + (1-\chi) c_{ps}) \Theta_0 \rangle_{\mathcal{Y}}, \qquad \boldsymbol{x} \in \Omega \quad (2.7)$$

and homogeneous boundary conditions

$$\boldsymbol{w} = 0, \quad \theta = 0, \quad \boldsymbol{x} \in \partial\Omega, \quad t \geq 0. \quad (2.8)$$

Tensors $\mathbb{A}_0$, $\mathbb{A}_1$, and $\mathbb{A}_2(t)$, and matrices $\mathbb{C}_0$, $\mathbb{C}_1(t)$, $\mathbb{E}_0$, and $\mathbb{E}_1$ are referred to as given in the statement of problem B. From the assertion of theorem 1 it is clear that they are defined only by the data given for the microstructure.

In (2.5) and further in the paper the standard notation for mean value over the period $\mathcal{Y}$ for any 1-periodic in $\boldsymbol{y}$ integrable function $\phi(\boldsymbol{x},t,\boldsymbol{y})$ is used: $\langle \phi(\boldsymbol{x},t,\boldsymbol{y}) \rangle_{\mathcal{Y}} = \int_Y \phi(\boldsymbol{x},t,\boldsymbol{y}) d\boldsymbol{y}$. In particular, in (2.3) and (2.4) by $\bar{\rho}$ and $\bar{c}_p$ we denote the mean density and heat capacity, respectively: $\bar{\rho} = |\mathcal{Y}_f|\rho_f + |\mathcal{Y}_s|\rho_s$ and $\bar{c}_p = |\mathcal{Y}_f| c_{pf} + |\mathcal{Y}_s| c_{ps}$, where $|\mathcal{Y}_f| = \langle \chi \rangle_{\mathcal{Y}}$ and $|\mathcal{Y}_s| = \langle 1-\chi \rangle_{\mathcal{Y}}$.

As usually, for any fourth-rank tensor $\mathbb{A}_*$ and 3×3-matrices $\mathbb{B}_*$ and $\mathbb{C}_*$ by $\mathbb{A}_* : \mathbb{B}_*$ and $(\mathbb{A}_* : \mathbb{B}_*) : \mathbb{C}_*$ we denote inner tensor products in $\mathbb{R}^{3\times 3}$ and $\mathbb{R}$, respectively: $(\mathbb{A}_* : \mathbb{B}_*)_{kl} = \sum_{i,j=1}^3 A_*^{ijkl} B_{*ij}$ $(k,l = 1,2,3)$, $(\mathbb{A}_* : \mathbb{B}_*) : \mathbb{C}_* = \sum_{i,j,k,l=1}^3 A_*^{ijkl} B_{*ij} C_{*kl}$.

Definition 2. *A pair of functions $(\boldsymbol{w},\theta)$ is called a generalized solution of problem B, if it satisfies the regularity conditions $\boldsymbol{w} \in W_2^1(Q)$ and $\theta \in L^2(0,T; W_2^1(\Omega))$, boundary conditions (2.5) and (2.8) in the trace sense, and integral equalities*

$$\int_Q \Big\{ \alpha_\tau \bar{\rho} \frac{\partial \boldsymbol{w}}{\partial t} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial t} - \Big[\mathbb{A}_0 : \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}}{\partial t}\Big) + \mathbb{A}_1 : \mathbb{D}(x,\boldsymbol{w}) - \mathbb{C}_0 \theta \\ + \int_0^t \mathbb{A}_2(t-\tau) : \mathbb{D}(x, \boldsymbol{w}(\tau)) d\tau - \int_0^t \mathbb{C}_1(t-\tau)\theta(\tau) d\tau \Big] : \nabla_x \boldsymbol{\varphi} + \alpha_F \bar{\rho} \boldsymbol{F} \cdot \boldsymbol{\varphi} \Big\} d\boldsymbol{x} dt \\ + \int_\Omega \alpha_\tau \bar{\rho} \boldsymbol{v}_0^*(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x},0) d\boldsymbol{x} = 0, \quad (2.9)$$

$$\int_Q \Big\{ \alpha_\tau \bar{c}_p \theta \frac{\partial \psi}{\partial t} - \Big[\mathbb{E}_0 \nabla_x \theta - \mathbb{E}_1 \frac{\partial \boldsymbol{w}}{\partial t} \Big] \cdot \nabla_x \psi + \bar{\Psi} \psi \Big\} d\boldsymbol{x} dt + \int_\Omega \alpha_\tau \bar{c}_p \theta_0^*(\boldsymbol{x}) \psi(\boldsymbol{x},0) d\boldsymbol{x} = 0 \quad (2.10)$$

for all smooth test vector-functions $\boldsymbol{\varphi}(\boldsymbol{x},t)$ and scalar functions $\psi(\boldsymbol{x},t)$ vanishing near $\partial\Omega$ and in a neighborhood of $t = T$.

2.2. Properties of the effective coefficients.

Theorem 2. 1. *The tensors $\mathbb{A}_0$, $\mathbb{A}_1$, $\mathbb{A}_2(t)$, and the matrices $\mathbb{C}_0$, $\mathbb{C}_1(t)$, $\mathbb{E}_0$, and $\mathbb{E}_1$ are symmetric, i.e., their components satisfy the equalities*

$$A_r^{ijkl} = A_r^{jikl} = A_r^{jilk} = A_r^{klij} \quad (r = 0, 1, 2), \quad C_r^{ij} = C_r^{ji}, \quad E_r^{ij} = E_r^{ji} \quad (r = 0, 1). \quad (2.11)$$

2. *The fourth-rank tensor $\mathcal{A}^\gamma \stackrel{def}{=} \gamma\mathbb{A}_0 + \mathbb{A}_1 + \hat{\mathbb{A}}_2(\gamma)$ and the 3×3-matrix $\mathcal{C}^\gamma \stackrel{def}{=} \mathbb{C}_0 + \hat{\mathbb{C}}_1(\gamma)$ are strictly positively defined for $\gamma > 0$.*

3. *If the both sets $\mathcal{Y}_f$ and E_f are connected then $\mathbb{A}_0$ is strictly positively defined.*

4. *If the set $\partial\mathcal{Y} \cap \partial\mathcal{Y}_f$ is empty, in other words, the porous space Ω_f^ε consists only of trapped pores, then $\mathbb{A}_0$ is zero tensor and $\mathbb{A}_1$ is strictly positively defined.*

5. *The matrices $\mathbb{E}_0$, $\mathbb{E}_1$, and $\mathbb{C}_0$ are strictly positively defined.*

In item 2 of the theorem by $\hat{\mathbb{A}}_2(\gamma)$ and $\hat{\mathbb{C}}_1(\gamma)$ the respective Laplace transforms of $\mathbb{A}_2(t)$ and $\mathbb{C}_1(t)$ are denoted, and at the same time it is assumed that $\mathbb{A}_2(t) = 0$ and $\mathbb{C}_1(t) = 0$ for $t > 0$. Recall that the Laplace transform of an arbitrary locally integrable and not fast-increasing on the semi-axis $(0, \infty)$ function $\varphi(t)$ is defined by the formula

$$\hat{\varphi}(\gamma) = \mathcal{L}[\varphi](\gamma) = \int_0^\infty \varphi(t)e^{-\gamma t}dt, \quad \gamma > 0.$$

2.3. On physical significance and mathematical well-posedness of problem B. In view of the obtained in theorem 2 properties of symmetry and positive definiteness for the tensors and matrices of the effective coefficients, problem B is identified as an initial-boundary value problem for a model of linear thermoviscoelasticity with memory of shape and heat, except for the case of the trapped pores (see item 4 of the theorem), in which the homogenized model takes the form of a model of linear thermoelasticity.

Comparing with the well-known formulations in the linear theory of thermoviscoelasticity (see., for example, [6, ch. 4, 9] and [19, ch. 6]), we conclude that $\mathbb{A}_0$ is the effective viscosity tensor of the averaged medium, $\mathbb{A}_1$ is the effective instantaneous elasticity tensor, $\mathbb{C}_0$ is the matrix of effective heat extension, $\mathbb{E}_0$ is the matrix of effective heat conductivity, $\mathbb{E}_1$ is the matrix of effective coefficients characterizing irreversible heat generation due to viscosity friction, and $\mathbb{A}_2(t)$ and $\mathbb{C}_1(t)$ are the relaxation functions determining influence of thermomechanical history of the medium during the period $(0, t)$ on the current state at the moment t.

Following [3, 5], we may notice that in the case, when the pore space is connected (see item 3 in theorem 2), the fluid viscosity terms dominate the solid stress terms. Such a thermomechanical system can be compared to an unconsolidated, saturated heat-conducting marine sediment. Such a sediment possesses low skeletal rigidity. Nevertheless, such a medium possesses a dissipative rigidity that is capable of supporting shear. Quite the contrary, in view of item 4 of the theorem, in the case, when all pores are trapped, the viscosity phenomena become subtle in effective macroscopic behavior of the medium and the solid stress terms dominate.

On the strength of theorem 1, problem B is solvable in the sense of definition 2, provided with the condition that the coefficients of equations (2.3) and (2.4) admit certain relations with data of the microstructure, since some solution of problem B can be constructed as a limit of solutions of problem A, as $\varepsilon \searrow 0$. At the same time, it should be noticed that, if we assume that the coefficients of equations (2.3) and (2.4) a priori satisfy the properties in assertions of theorem 2, then the conclusion about well-posedness of problem B is correct independently of whether problem B is connected with the microstructure, or not. More precisely, the following proposition holds true.

Corollary of theorem 2. *Assume that the tensors and matrices of coefficients of equations* (2.3) *and* (2.4) *have the properties, stated in items 1, 2, and 5 of theorem 2, and satisfy the inclusions* $A_2^{ijkl}, C_1^{ij} \in L^2(0,T)$ $(i,j,k,l=1,2,3)$. *Let all of them be, in principle, irrelevant to the data given for problem A.*

Then for any given initial distributions $\boldsymbol{w}_0^* \in \overset{\circ}{W}{}_2^1(\Omega)$, $\boldsymbol{v}_0^*, \theta_0^* \in L^2(\Omega)$ *and right-hand sides* $\boldsymbol{F}, \bar{\Psi} \in L^2(Q)$ *of equations* (2.3) *and* (2.4), *there exists a unique generalized solution of problem B, in the sense of definition 2.*

Proof of this corollary is by the quite standard considerations, therefore we confine ourselves to a brief scheme of the proof.

On the strength of the well-known properties of Laplace's transform (see, for example, [2, ch. 4], [14, ch. 3]), applying formally Laplace's transform to equations (2.3) and (2.4) and taking into account the given initial data we arrive at the Dirichlet problem for the system of two second-order partial differential equations as follows:

$$\begin{aligned}\operatorname{div}_x\big\{\mathcal{A}^\gamma : \mathbb{D}(x,\hat{\boldsymbol{w}}^\gamma) - \mathcal{C}^\gamma\hat{\theta}^\gamma\big\} - \alpha_\tau\bar{\rho}\gamma^2\hat{\boldsymbol{w}}^\gamma &= \\ \operatorname{div}_x\big\{\mathbb{A}_0 : \mathbb{D}(x,\boldsymbol{w}_0^*)\big\} - \bar{\rho}(\alpha_\tau\gamma\boldsymbol{v}_0^* + \alpha_\tau\boldsymbol{w}_0^* + \alpha_F\hat{\boldsymbol{F}}^\gamma), &\quad \boldsymbol{x}\in\Omega, \end{aligned} \tag{2.12a}$$

$$\operatorname{div}_x\big\{\mathbb{E}_0\nabla_x\hat{\theta}^\gamma - \gamma\mathbb{E}_1\hat{\boldsymbol{w}}^\gamma\big\} - \alpha_\tau\bar{c}_p\gamma\hat{\theta}^\gamma = -\operatorname{div}_x\big\{\mathbb{E}_1\boldsymbol{w}_0^*\big\} - \alpha_\tau\bar{c}_p\theta_0^* - \hat{\Psi}^\gamma, \quad \boldsymbol{x}\in\Omega, \tag{2.12b}$$

$$\hat{\boldsymbol{w}}^\gamma = 0, \quad \hat{\theta}^\gamma = 0, \quad \boldsymbol{x}\in\partial\Omega. \tag{2.12c}$$

Variable $\gamma > 0$ enters this problem as a parameter.

On the strength of the strict positive definiteness of tensor $\mathcal{A}^\gamma$ and matrix $\mathbb{E}_0$, equation (2.12a) is uniformly elliptic with respect to the unknown function $\hat{\boldsymbol{w}}^\gamma$, and equation (2.12b) is uniformly elliptic with respect to the unknown function $\hat{\theta}^\gamma$. Due to this and the strict positive definiteness and symmetry of matrices $\mathbb{E}_1$ and $\mathcal{C}^\gamma$, it is true that problem (2.12) has exactly one generalized solution $(\hat{\boldsymbol{w}}^\gamma, \hat{\theta}^\gamma) \in \overset{\circ}{W}{}_2^1(\Omega)$ for any given $\boldsymbol{w}_0^* \in \overset{\circ}{W}{}_2^1(\Omega)$, $\hat{\boldsymbol{F}}^\gamma, \hat{\Psi}^\gamma, \theta_0^*, \boldsymbol{v}_0^* \in L^2(\Omega)$, for any fixed $\gamma > 0$.

Verification of this assertion is fulfilled within the framework of the well-known theory of generalized solutions to elliptic equations [11, ch. 2]. Indeed, multiply equation (2.12a) by $\gamma\mathbb{E}_1(\mathcal{C}^\gamma)^{-1}\hat{\boldsymbol{w}}^\gamma$ and equation (2.12b) by $\hat{\theta}^\gamma$, integrate the obtained equations with respect to $\boldsymbol{x}$ on Ω, integrate by parts in $\boldsymbol{x}$ in all summands involving operator div_x, except for just one arising from (2.12a) and having the integrand $-\gamma\mathbb{E}_1(\mathcal{C}^\gamma)^{-1}\hat{\boldsymbol{w}}^\gamma\cdot\operatorname{div}_x(\mathcal{C}^\gamma\hat{\theta}^\gamma)$, sum up the resulting equations, and fulfill some certain simple algebraic transformations of the integrands, using the facts that two positive definite and symmetric matrices $\mathbb{E}_1$ and $\mathcal{C}^\gamma$ can be brought to a diagonal form by the same orthogonal transformation (say, $\mathfrak{Q}^\gamma$) and that $(\mathcal{C}^\gamma)^{-1}$, as well as $\mathcal{C}^\gamma$, is a symmetric matrix and can be brought to a diagonal form by the transformation $\mathfrak{Q}^\gamma$. These technical procedures, in particular, bring the integrand $-\gamma\mathbb{E}_1(\mathcal{C}^\gamma)^{-1}\hat{\boldsymbol{w}}^\gamma\cdot\operatorname{div}_x(\mathcal{C}^\gamma\hat{\theta}^\gamma)$ to the form $-\gamma\mathbb{E}_1\hat{\boldsymbol{w}}^\gamma\cdot\nabla_x\hat{\theta}^\gamma$. The latter integrand cancels with the similar term arising from equation (2.12b), and, thanks to this, we eventually arrive at the energy identity

$$\|\mathbb{E}_0^{1/2}\nabla_x\hat{\theta}^\gamma\|^2_{2,\Omega}+\gamma\alpha_\tau\bar{c}_p\|\hat{\theta}^\gamma\|^2_{2,\Omega}+\gamma^3\alpha_\tau\bar{\rho}\|\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}\hat{\boldsymbol{w}}^\gamma\|^2_{2,\Omega}$$
$$+\int_\Omega\big(\mathcal{A}^\gamma:\mathbb{D}(x,\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}\hat{\boldsymbol{w}}^\gamma)\big):\mathbb{D}(x,\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}\hat{\boldsymbol{w}}^\gamma)d\boldsymbol{x}$$
$$=\int_\Omega\gamma\big(\mathbb{A}_0:\mathbb{D}(x,\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}\boldsymbol{w}_0^*)\big):\mathbb{D}(x,\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}\hat{\boldsymbol{w}}^\gamma)d\boldsymbol{x}$$
$$+\int_\Omega\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}\hat{\boldsymbol{w}}^\gamma\cdot\big\{\bar{\rho}\mathbb{E}_1^{1/2}(\mathcal{C}^\gamma)^{-1/2}(\gamma\alpha_\tau\boldsymbol{v}_0^*+\alpha_\tau\boldsymbol{w}_0^*+\alpha_F\hat{\boldsymbol{F}}^\gamma)\big\}d\boldsymbol{x}$$
$$+\int_\Omega\big\{-(\mathbb{E}_1\boldsymbol{w}_0^*)\cdot\nabla_x\hat{\theta}^\gamma+\alpha_\tau\bar{c}_p\theta_0^*\hat{\theta}^\gamma+\hat{\Psi}^\gamma\hat{\theta}^\gamma\big\}d\boldsymbol{x},\quad\gamma>0.$$

Applying Cauchy's, Holder's, and Korn's [16, ch. 3, §3.2] inequalities and taking into account strict positive definiteness of tensor $\mathcal{A}^\gamma$ and matrices $\mathbb{E}_0$, $\mathbb{E}_1$, and $(\mathcal{C}^\gamma)^{-1}$, from this identity we derive the energy inequality

$$c_1^\gamma\|\hat{\boldsymbol{w}}^\gamma\|^2_{2,\Omega}+c_2^\gamma\|\nabla_x\hat{\boldsymbol{w}}^\gamma\|^2_{2,\Omega}+c_3^\gamma\|\hat{\theta}^\gamma\|^2_{2,\Omega}+c_4^\gamma\|\nabla_x\hat{\theta}^\gamma\|^2_{2,\Omega}$$
$$\le c_5^\gamma\|\boldsymbol{w}_0^*\|^2_{2,\Omega}+c_6^\gamma\|\nabla_x\boldsymbol{w}_0^*\|^2_{2,\Omega}+c_7^\gamma\|\theta_0^*\|^2_{2,\Omega}+c_8^\gamma\|\hat{\boldsymbol{F}}^\gamma\|^2_{2,\Omega}+c_9^\gamma\|\hat{\Psi}^\gamma\|^2_{2,\Omega},\quad\gamma>0.$$

Here all constants c_i^γ are nonnegative and depend merely on γ. Moreover, the constants with indices $i=1,2,3,4$ are strictly positive. Relying on this estimate we finish the proof of the unique solvability of problem (2.12) precisely following the lines of [11, ch. 2, §2, theorem 2.1, §3].

Finally, applying the inverse Laplace transform in γ to the solution of problem (2.12) and following the considerations from [11, ch. 3, §4; ch. 4, §7] or from [7] we deduce the solution of problem B in the form

$$\boldsymbol{w}(\boldsymbol{x},t)=\mathcal{L}^{-1}[\hat{\boldsymbol{w}}^\gamma]=\frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\hat{\boldsymbol{w}}^\gamma(\boldsymbol{x})e^{\gamma t}d\gamma,$$

$$\theta(\boldsymbol{x},t)=\mathcal{L}^{-1}[\hat{\theta}^\gamma]=\frac{1}{2\pi i}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\hat{\theta}^\gamma(\boldsymbol{x})e^{\gamma t}d\gamma.$$

This solution is unique due to one-to-oneness of $\mathcal{L}$ and $\mathcal{L}^{-1}$. □

3. Proof of theorem 1 (part I): the two-scale convergence method and weak and two-scale limits of solutions of problem A

In this section we outline the notion of two-scale convergence and then derive the system of two-scale averaged equations from equations of problem A with the help of this notion by a limiting transition as $\varepsilon\searrow0$. This derivation is the first step in the proof of theorem 1.

Definition 3. *(G. Nguetseng [13].) The sequence $\{\varphi^\varepsilon\}\subset L^2(Q)$ is said to two-scale converge to a limit $\varphi\in L^2(Q\times\mathcal{Y})$, if and only if for any 1-periodic in $\boldsymbol{y}$ function $\sigma=\sigma(\boldsymbol{x},t,\boldsymbol{y})$ such that $\sigma\in L^2(Q\times\mathcal{Y})$ one has*

$$\lim_{\varepsilon\searrow0}\int_Q\varphi^\varepsilon(\boldsymbol{x},t)\sigma\Big(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\Big)d\boldsymbol{x}dt=\int_{Q\times\mathcal{Y}}\varphi(\boldsymbol{x},t,\boldsymbol{y})\sigma(\boldsymbol{x},t,\boldsymbol{y})d\boldsymbol{x}d\boldsymbol{y}dt.$$

Existence and the basic properties of two-scale convergent sequences is established in the following fundamental theorem [1, 13].

Theorem TS. 1. *From each bounded sequence in $L^2(Q)$ one can extract a subsequence which two-scale converges to a limit $\varphi \in L^2(Q \times \mathcal{Y})$.*
2. *If a sequence in $L^2(Q)$ two-scale converges to two functions $\varphi_1, \varphi_2 \in L^2(Q \times \mathcal{Y})$ simultaneously, then $\varphi_1 = \varphi_2$ a.e. in $Q \times \mathcal{Y}$.*
3. *Let $\{\varphi^\varepsilon\}$ and $\{\nabla_x \varphi_\varepsilon\}$ be bounded sequences in $L^2(Q)$. Then there exist functions $\varphi \in L^2(Q)$ and $\psi \in L^2(Q \times \mathcal{Y})$ and a subsequence $\{\varphi^\varepsilon\}$ such that ψ is 1-periodic in $\boldsymbol{y}$, $\nabla_y \psi \in L^2(Q \times \mathcal{Y})$, and both $\{\varphi^\varepsilon\}$ and $\{\nabla_x \varphi^\varepsilon\}$ two-scale converge to φ and $\nabla_x \varphi(\boldsymbol{x}, t) + \nabla_y \psi(\boldsymbol{x}, t, \boldsymbol{y})$, respectively.*

Remark 2. *Let $\sigma \in L^\infty(\mathcal{Y})$, continue σ from $\mathcal{Y}$ onto the whole space $\mathbb{R}^3$ by periodic repetition, define $\sigma^\varepsilon(\boldsymbol{x}) = \sigma(\boldsymbol{x}/\varepsilon)$ $(\boldsymbol{x} \in \Omega)$, and let the sequence $\{\varphi^\varepsilon\} \subset L^2(Q)$ two-scale converge to a limit $\varphi \in L^2(Q \times \mathcal{Y})$. Then from definition 3 and theorem TS it is easy to see that $\{\sigma^\varepsilon \varphi^\varepsilon\}$ two-scale converges to the limit $\sigma(\boldsymbol{y})\varphi(\boldsymbol{x}, t, \boldsymbol{y})$.*

Now turn to consideration of the limiting transition in the equations of problem A, as $\varepsilon \searrow 0$ $(\varepsilon^{-1} \in \mathbb{N})$. On the strength of proposition 1, the sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{\partial_t \boldsymbol{w}^\varepsilon\}$, $\{\theta^\varepsilon\}$, $\{\nabla_x \theta^\varepsilon\}$, $\{\chi^\varepsilon \mathbb{D}(x, \partial_t \boldsymbol{w}^\varepsilon)\}$ are uniformly bounded in $L^2(Q)$. From this, theorem TS, and remark 2 it follows that there exist a subsequence from $\{\varepsilon > 0 \,|\, \varepsilon^{-1} \in \mathbb{N}\}$ and four functions $\{\boldsymbol{w}^* \in W_2^1(Q),\ \theta^* \in L^2(0, T; \overset{\circ}{W}{}_2^1(\Omega)),\ \boldsymbol{W}, \Theta \in L^2(Q \times \mathcal{Y})\}$ such that

$$
\begin{gathered}
\chi(\boldsymbol{y})(\mathbb{D}(x, \partial_t \boldsymbol{w}^*) + \mathbb{D}(y, \partial_t \boldsymbol{W})), \nabla_y \boldsymbol{W}, \nabla_y \Theta \in L^2(Q \times \mathcal{Y}); \\
\boldsymbol{W}, \Theta \quad \text{are 1-periodic in } \boldsymbol{y};
\end{gathered}
\tag{3.1}
$$

$$
\begin{aligned}
\boldsymbol{w}^\varepsilon &\to \boldsymbol{w}^* \text{ weakly in } W_2^1(Q), \\
\theta^\varepsilon &\to \theta^* \text{ weakly in } L^2(0, T; W_2^1(\Omega)),
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
\nabla_x \boldsymbol{w}^\varepsilon &\to \nabla_x \boldsymbol{w}^*(\boldsymbol{x}, t) + \nabla_y \boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y}), \\
\nabla_x \theta^\varepsilon &\to \nabla_x \theta^*(\boldsymbol{x}, t) + \nabla_y \Theta(\boldsymbol{x}, t, \boldsymbol{y}), \\
\chi^\varepsilon \mathbb{D}(x, \partial_t \boldsymbol{w}^\varepsilon) &\to \chi(\boldsymbol{y})\big(\mathbb{D}(x, \partial_t \boldsymbol{w}^*) + \mathbb{D}(y, \partial_t \boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y}))\big) \\
& \text{ in the two-scale sense, as } \varepsilon \searrow 0.
\end{aligned}
\tag{3.3}
$$

Substitute the test functions of the forms

$$
\boldsymbol{\varphi} = \boldsymbol{\varphi}_1(\boldsymbol{x}, t) + \varepsilon \boldsymbol{\varphi}_2\Big(\boldsymbol{x}, t, \frac{\boldsymbol{x}}{\varepsilon}\Big), \quad \psi = \psi_1(\boldsymbol{x}, t) + \varepsilon \psi_2\Big(\boldsymbol{x}, t, \frac{\boldsymbol{x}}{\varepsilon}\Big),
$$

where $\boldsymbol{\varphi}_1(\boldsymbol{x}, t)$, $\boldsymbol{\varphi}_2(\boldsymbol{x}, t, \boldsymbol{y})$, $\psi_1(\boldsymbol{x}, t)$, and $\psi_2(\boldsymbol{x}, t, \boldsymbol{y})$ are arbitrary smooth functions, vanishing near $\partial\Omega$ and in a neighborhood of $t = T$ and such that $\boldsymbol{\varphi}_2$ and ψ_2 are 1-periodic in $\boldsymbol{y}$, into integral equalities (1.3) and (1.4). Now, due to such choice of test functions and on the strength of relations (2.1), (2.2), (3.2), and (3.3), extracting a proper subsequence from $\{\varepsilon > 0,\ \varepsilon^{-1} \in \mathbb{N}\}$ (if necessary) and passing in integral equalities (1.3) and (1.4) to the limit as $\varepsilon \searrow 0$, we deduce the system of the *averaged two-scale equations*, which consists of the following four integral equalities:

$$\int_Q \Big\{ \alpha_\tau \bar{\rho} \frac{\partial \boldsymbol{w}^*}{\partial t} \cdot \frac{\partial \boldsymbol{\varphi}_1}{\partial t} - \Big[|\mathcal{Y}_f| \Big(\alpha_\mu \mathbb{D}(x, \frac{\partial \boldsymbol{w}^*}{\partial t}) + \big(\alpha_p \mathrm{div}_x \boldsymbol{w}^* + \alpha_\nu \mathrm{div}_x \frac{\partial \boldsymbol{w}^*}{\partial t} - \alpha_{\theta f} \theta^* \big) \mathbb{I} \Big)$$
$$+ |\mathcal{Y}_s| \big(\alpha_\lambda \mathbb{D}(x, \boldsymbol{w}^*) + (\alpha_\eta \mathrm{div}_x \boldsymbol{w}^* - \alpha_{\theta s} \theta^*) \mathbb{I} \big)$$
$$+ \Big\langle \chi(\boldsymbol{y}) \Big(\alpha_\mu \mathbb{D}(y, \frac{\partial \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})}{\partial t}) + \big(\alpha_p \mathrm{div}_y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) + \alpha_\nu \mathrm{div}_y \frac{\partial \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})}{\partial t} \big) \mathbb{I} \Big)$$
$$+ (1-\chi(\boldsymbol{y})) \big(\alpha_\lambda \mathbb{D}(y, \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})) + (\alpha_\eta \mathrm{div}_y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})) \mathbb{I} \big) \Big\rangle_{\mathcal{Y}} \Big] : \nabla_x \boldsymbol{\varphi}_1 + \alpha_F \bar{\rho} \boldsymbol{F} \cdot \boldsymbol{\varphi}_1 \Big\} d\boldsymbol{x} dt$$
$$+ \int_\Omega \alpha_\tau \big\langle (\chi(\boldsymbol{y}) \rho_f + (1-\chi(\boldsymbol{y})) \rho_s) \boldsymbol{V}_0(\boldsymbol{x},\boldsymbol{y}) \big\rangle_{\mathcal{Y}} \cdot \boldsymbol{\varphi}_1(\boldsymbol{x},0) d\boldsymbol{x} = 0, \quad (3.4)$$

$$\int_{Q \times \mathcal{Y}} \Big\{ \chi(\boldsymbol{y}) \Big[\alpha_\mu \mathbb{D}(x, \frac{\partial \boldsymbol{w}^*}{\partial t}) + \alpha_\mu \mathbb{D}(y, \frac{\partial \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})}{\partial t})$$
$$+ \big(\alpha_p \mathrm{div}_x \boldsymbol{w}^* + \alpha_p \mathrm{div}_y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) + \alpha_\nu \mathrm{div}_x \frac{\partial \boldsymbol{w}^*}{\partial t} + \alpha_\nu \mathrm{div}_y \frac{\partial \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})}{\partial t} - \alpha_{\theta f} \theta^* \big) \mathbb{I} \Big]$$
$$+ (1-\chi(\boldsymbol{y})) \big[\alpha_\lambda (\mathbb{D}(x, \boldsymbol{w}^*) + \mathbb{D}(y, \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})))$$
$$+ (\alpha_\eta \mathrm{div}_x \boldsymbol{w}^* + \alpha_\eta \mathrm{div}_y \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) - \alpha_{\theta s} \theta^*) \mathbb{I} \big] \Big\} : \nabla_y \boldsymbol{\varphi}_2(\boldsymbol{x},t,\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} dt = 0, \quad (3.5)$$

$$\int_Q \Big\{ \alpha_\tau \bar{c}_p \theta^* \frac{\partial \psi_1}{\partial t} - \Big[\big(|\mathcal{Y}_f| \varkappa_f + |\mathcal{Y}_s| \varkappa_s \big) \nabla_x \theta^* - \big(|\mathcal{Y}_f| \alpha_{\theta f} + |\mathcal{Y}_s| \alpha_{\theta s} \big) \frac{\partial \boldsymbol{w}^*}{\partial t}$$
$$+ \big\langle (\chi(\boldsymbol{y}) \varkappa_f + (1-\chi(\boldsymbol{y})) \varkappa_s) \nabla_y \Theta(\boldsymbol{x},t,\boldsymbol{y}) \big\rangle_{\mathcal{Y}} \Big] \cdot \nabla_x \psi_1 + \bar{\Psi} \psi_1 \Big\} d\boldsymbol{x} dt$$
$$+ \int_\Omega \alpha_\tau \big\langle (\chi(\boldsymbol{y}) c_{pf} + (1-\chi(\boldsymbol{y})) c_{ps}) \Theta_0(\boldsymbol{x},\boldsymbol{y}) \big\rangle_{\mathcal{Y}} \psi_1(\boldsymbol{x},0) d\boldsymbol{x} = 0, \quad (3.6)$$

$$\int_{Q \times \mathcal{Y}} \Big\{ \chi(\boldsymbol{y}) \Big[\varkappa_f \nabla_x \theta^* + \varkappa_f \nabla_y \Theta(\boldsymbol{x},t,\boldsymbol{y}) - \alpha_{\theta f} \frac{\partial \boldsymbol{w}^*}{\partial t} \Big]$$
$$+ (1-\chi(\boldsymbol{y})) \Big[\varkappa_s \nabla_x \theta^* + \varkappa_s \nabla_y \Theta(\boldsymbol{x},t,\boldsymbol{y}) - \alpha_{\theta s} \frac{\partial \boldsymbol{w}^*}{\partial t} \Big] \Big\} \cdot \nabla_y \psi_2(\boldsymbol{x},t,\boldsymbol{y}) d\boldsymbol{x} d\boldsymbol{y} dt = 0. \quad (3.7)$$

Due to the sufficient arbitrariness of the functions $\boldsymbol{\varphi}_1$, $\boldsymbol{\varphi}_2$, ψ_1, and ψ_2, system (3.4)–(3.7) is closed, because it is equivalent in the distributions sense to the initial-boundary value problem for the system of eight scalar equations involving eight unknown functions $w_i^*(\boldsymbol{x},t)$, $W_i(\boldsymbol{x},t,\boldsymbol{y})$ $(i=1,2,3)$, $\theta^*(\boldsymbol{x},t)$, and $\Theta(\boldsymbol{x},t,\boldsymbol{y})$.

The following assertion holds true.

Proposition 2. *For any given* $\boldsymbol{w}^*|_{t=0} \in \overset{\circ}{W}{}^1_2(\Omega)$, $\boldsymbol{V}_0, \Theta_0 \in L^2(\Omega \times \mathcal{Y})$, *and* $\boldsymbol{F}, \bar{\Psi} \in L^2(Q)$, *system* (3.4)–(3.7) *has a unique solution* $\boldsymbol{w}^* \in W^1_2(Q)$, $\theta^* \in L^2(0,T; \overset{\circ}{W}{}^1_2(\Omega))$, $\boldsymbol{W}, \Theta \in L^2(Q \times \mathcal{Y})$. *This solution possesses the regularity properties* (3.1).

Existence of solutions has been already proved by the limiting transition as $\varepsilon \searrow 0$. The proof of uniqueness follows the lines of [9, lemma 5]. $\square$

4. Proof of theorem 1 (part II): derivation of homogeneous equations (2.3) and (2.4), structure of effective coefficients

Let us resolve equations (3.5) and (3.7) with respect to the functions $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ and $\Theta(\boldsymbol{x},t,\boldsymbol{y})$, assuming that $\boldsymbol{w}^*$ and θ^* are given. To this end we employ the method of separation of variables. That is, we seek for solutions $\boldsymbol{W}$ and Θ having the forms

$$\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i,j=1}^{3}\Big(\mathbb{D}_{ij}(x,\boldsymbol{w}^{(}\boldsymbol{x},t))\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \int_0^t \mathbb{D}_{ij}(x,\boldsymbol{w}^*(\boldsymbol{x},\tau))\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t-\tau)d\tau\Big)$$
$$+\int_0^t \theta^*(\boldsymbol{x},\tau)\boldsymbol{Z}_3(\boldsymbol{y},t-\tau)d\tau, \quad (4.1)$$

$$\Theta(\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i=1}^{3}\Big(\frac{\partial\theta^*}{\partial x_i}g_1^i(\boldsymbol{y}) + \frac{\partial w_i^*}{\partial t}g_2^i(\boldsymbol{y})\Big), \quad (4.2)$$

where the vector-functions $\boldsymbol{Z}_1^{ij}$, $\boldsymbol{Z}_2^{ij}$, and $\boldsymbol{Z}_3$ and the scalar functions g_1^i and g_2^i are to be determined. Substituting (4.1) into (3.5) and (4.2) into (3.7) after some rather simple technical transformations we arrive at the integral equalities

$$\int_{Q\times\mathcal{Y}_f}\Big\{\sum_{i,j=1}^{3}\mathbb{D}_{ij}\Big(x,\frac{\partial\boldsymbol{w}^*}{\partial t}\Big)\big[\alpha_\mu\mathbb{D}(y,\boldsymbol{Z}_1^{ij}) + (\alpha_\nu \mathrm{div}_y\boldsymbol{Z}_1^{ij})\mathbb{I} + \alpha_\mu\mathbb{J}^{ij} + \alpha_\nu\delta_{ij}\mathbb{I}\big] : \nabla_y\boldsymbol{\varphi}_2$$
$$+\sum_{i,j=1}^{3}\mathbb{D}_{ij}(x,\boldsymbol{w}^*(\boldsymbol{x},t))\big[\alpha_\mu\mathbb{D}(y,\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0)) + (\alpha_\nu\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0))\mathbb{I}$$
$$+\alpha_p\big(\mathrm{div}_y\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \delta_{ij}\big)\mathbb{I}\big] : \nabla_y\boldsymbol{\varphi}_2 - \Bigg[\int_0^t\sum_{i,j=1}^{3}\mathbb{D}_{ij}(x,\boldsymbol{w}^*(\boldsymbol{x},\tau))\Big[\alpha_\mu\mathbb{D}\Big(y,\frac{\partial\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t-\tau)}{\partial\tau}\Big)$$
$$-\big(\alpha_p\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t-\tau)\big)\mathbb{I} + \Big(\alpha_\nu\mathrm{div}_y\frac{\partial\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t-\tau)}{\partial\tau}\Big)\mathbb{I}\Big]d\tau\Bigg] : \nabla_y\boldsymbol{\varphi}_2$$
$$+\theta^*(\boldsymbol{x},t)\big[\alpha_\mu\mathbb{D}(y,\boldsymbol{Z}_3(\boldsymbol{y},0)) + \big(\alpha_\nu\mathrm{div}_y\boldsymbol{Z}_3(\boldsymbol{y},0)\big)\mathbb{I} - \alpha_{\theta f}\mathbb{I}\big] : \nabla_y\boldsymbol{\varphi}_2$$
$$-\Bigg[\int_0^t\theta^*(\boldsymbol{x},\tau)\Big[\alpha_\mu\mathbb{D}\Big(y,\frac{\partial\boldsymbol{Z}_3(\boldsymbol{y},t-\tau)}{\partial\tau}\Big) - \big(\alpha_p\mathrm{div}_y\boldsymbol{Z}_3(\boldsymbol{y},t-\tau)\big)\mathbb{I}$$
$$+\Big(\alpha_\nu\mathrm{div}_y\frac{\partial\boldsymbol{Z}_3(\boldsymbol{y},t-\tau)}{\partial\tau}\Big)\mathbb{I}\Big]d\tau\Bigg] : \nabla_y\boldsymbol{\varphi}_2\Big\}d\boldsymbol{x}d\boldsymbol{y}dt$$
$$+\int_{Q\times\mathcal{Y}_s}\Big\{\sum_{i,j=1}^{3}\mathbb{D}_{ij}(x,\boldsymbol{w}^*)\big[\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_1^{ij}) + \big(\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_1^{ij}\big)\mathbb{I} + \alpha_\lambda\mathbb{J}^{ij} + \alpha_\eta\delta_{ij}\mathbb{I}\big] : \nabla_y\boldsymbol{\varphi}_2$$
$$+\Bigg[\int_0^t\sum_{i,j=1}^{3}\mathbb{D}_{ij}(x,\boldsymbol{w}^*(\boldsymbol{x},\tau))\big[\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t-\tau)) + \big(\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t-\tau)\big)\mathbb{I}\big]d\tau\Bigg] : \nabla_y\boldsymbol{\varphi}_2$$
$$+\Bigg[\int_0^t\theta^*(\boldsymbol{x},\tau)\big[\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_3(\boldsymbol{y},t-\tau)) + (\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_3(\boldsymbol{y},t-\tau))\mathbb{I}\big]d\tau\Bigg] : \nabla_y\boldsymbol{\varphi}_2$$
$$-\theta^*(\alpha_{\theta s}\mathrm{div}_y\boldsymbol{\varphi}_2)\Big\}d\boldsymbol{x}d\boldsymbol{y}dt = 0, \quad (4.3)$$

$$\int_{Q\times\mathcal{Y}_f}\Big\{\sum_{j=1}^{3}\frac{\partial\theta^*}{\partial x_j}(\varkappa_f\nabla_y g_1^j+\varkappa_f\boldsymbol{e}^j)\cdot\nabla_y\psi_2+\sum_{j=1}^{3}\frac{\partial w_j^*}{\partial t}(\varkappa_f\nabla_y g_2^j-\alpha_{\theta f}\boldsymbol{e}^j)\cdot\nabla_y\psi_2\Big\}d\boldsymbol{x}d\boldsymbol{y}dt$$

$$+\int_{Q\times\mathcal{Y}_s}\Big\{\sum_{j=1}^{3}\frac{\partial\theta^*}{\partial x_j}(\varkappa_s\nabla_y g_1^j+\varkappa_s\boldsymbol{e}^j)\cdot\nabla_y\psi_2+\sum_{j=1}^{3}\frac{\partial w_j^*}{\partial t}(\varkappa_s\nabla_y g_2^j-\alpha_{\theta s}\boldsymbol{e}^j)\cdot\nabla_y\psi_2\Big\}d\boldsymbol{x}d\boldsymbol{y}dt=0. \tag{4.4}$$

In (4.3) and (4.4) by $\boldsymbol{e}^j$ the standard vectors of Cartesian basis in $\mathbb{R}^3$ are denoted; $\mathbb{J}^{ij}\overset{def}{=}(1/2)(\boldsymbol{e}^i\otimes\boldsymbol{e}^j+\boldsymbol{e}^j\otimes\boldsymbol{e}^i)$ is the 3×3-matrix, in whose definition the expression $\boldsymbol{e}^k\otimes\boldsymbol{e}^l$ stands for the diad of two basis vectors, i.e., $(\boldsymbol{e}^k\otimes\boldsymbol{e}^l)\boldsymbol{a}\overset{def}{=}a_l\boldsymbol{e}^k$ for any $\boldsymbol{a}\in\mathbb{R}^3$.

From the structure of integral equalities (4.3) and (4.4) it follows that they hold true independently of all possible solutions $\boldsymbol{w}^*$ and θ^* and test functions $\boldsymbol{\varphi}_2$ and ψ_2, whenever we demand that the functions $\boldsymbol{Z}_1^{ij}$, $\boldsymbol{Z}_2^{ij}$, $\boldsymbol{Z}_3$, g_1^i, and g_2^i solve the following boundary value problems in the pattern cell $\mathcal{Y}$. (We state these problems using the variational formulations, like in [9, 17].)

Vector-function $\boldsymbol{Z}_1^{ij}$ $(i,j=1,2,3)$ is determined by the linear system

$$\int_{\mathcal{Y}_f}\big(\alpha_\mu\mathbb{D}(y,\boldsymbol{Z}_1^{ij})+(\alpha_\nu\mathrm{div}_y\boldsymbol{Z}_1^{ij})\mathbb{I}+\alpha_\mu\mathbb{J}^{ij}+\alpha_\nu\delta_{ij}\mathbb{I}\big):\nabla_y\boldsymbol{\varphi}(\boldsymbol{y})d\boldsymbol{y}=0, \tag{4.5a}$$

$\forall\,\boldsymbol{\varphi}\in W_2^1(\mathcal{Y})$ ($\boldsymbol{\varphi}$ is 1-periodic),

$$\int_{\mathcal{Y}_s}\big(\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_1^{ij})+(\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_1^{ij})\mathbb{I}+\alpha_\lambda\mathbb{J}^{ij}+\alpha_\eta\delta_{ij}\mathbb{I}\big):\nabla_y\boldsymbol{\beta}(\boldsymbol{y})d\boldsymbol{y}=$$
$$\int_{\partial\mathcal{Y}_s\setminus\partial\mathcal{Y}}\big(\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_1^{ij})+(\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_1^{ij})\mathbb{I}+\alpha_\lambda\mathbb{J}^{ij}+\alpha_\eta\delta_{ij}\mathbb{I}\big)\boldsymbol{n}(\sigma_y)\cdot\boldsymbol{\beta}(\sigma_y)d\sigma_y, \tag{4.5b}$$

$\forall\,\boldsymbol{\beta}\in W_2^1(\mathcal{Y})$ ($\boldsymbol{\beta}$ is 1-periodic),

$$\boldsymbol{Z}_1^{ij}\in W_2^1(\mathcal{Y})/\mathbb{R},\quad(\partial\boldsymbol{Z}_1^{ij}/\partial t)\in W_2^1(\mathcal{Y}_f)/\mathbb{R},\quad\boldsymbol{Z}_1^{ij}:\ \mathbb{R}^3\mapsto\mathbb{R}^3\text{ is 1-periodic.} \tag{4.5c}$$

Here by $\boldsymbol{n}$ the unit normal to $\partial\mathcal{Y}_s$, inward with respect to $\mathcal{Y}_f$, is denoted.

Next the problem for the initial value of the kernel $\boldsymbol{Z}_2^{ij}$ $(i,j=1,2,3)$ is formulated:

$$\int_{Y_f}\big(\alpha_\mu\mathbb{D}(y,\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0))+(\alpha_\nu\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0))\mathbb{I}+(\alpha_p\mathrm{div}_y\boldsymbol{Z}_1^{ij}(\boldsymbol{y}))\mathbb{I}+\alpha_p\delta_{ij}\mathbb{I}\big):\nabla_y\boldsymbol{\varphi}(\boldsymbol{y})d\boldsymbol{y}=$$
$$-\int_{\partial\mathcal{Y}_s\setminus\partial\mathcal{Y}}\big(\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_1^{ij})+(\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_1^{ij})\mathbb{I}+\alpha_\lambda\mathbb{J}^{ij}+\alpha_\eta\delta_{ij}\mathbb{I}\big)\boldsymbol{n}(\sigma_y)\cdot\boldsymbol{\varphi}(\sigma_y)d\sigma_y, \tag{4.6a}$$

$\forall\,\boldsymbol{\varphi}\in W_2^1(\mathcal{Y}_f)$ ($\boldsymbol{\varphi}$ is 1-periodic),

$$\boldsymbol{Z}_2^{ij}(\cdot,0)\in W_2^1(\mathcal{Y}_f)/\mathbb{R},\quad\boldsymbol{Z}_2^{ij}(\cdot,0):\ \mathbb{R}^3\mapsto\mathbb{R}^3\text{ is 1-periodic in }\boldsymbol{y}. \tag{4.6b}$$

In (4.6) the vector-function $\boldsymbol{Z}_1^{ij}$ is assumed given.

The value of the kernel $\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t)$ $(i,j=1,2,3)$ is determined in $\mathcal{Y}\times(0,T)$ by the system

$$\int_{\mathcal{Y}_f}\Big\{\alpha_\mu\mathbb{D}\Big(y,\frac{\partial\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t)}{\partial t}\Big)+(\alpha_p\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t))\mathbb{I}+\Big(\alpha_\nu\mathrm{div}_y\frac{\partial\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t)}{\partial t}\Big)\mathbb{I}\Big\}:\nabla_y\boldsymbol{\varphi}(\boldsymbol{y})d\boldsymbol{y}$$
$$+\int_{\mathcal{Y}_s}\big\{\alpha_\lambda\mathbb{D}(y,\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t))+(\alpha_\eta\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t))\mathbb{I}\big\}:\nabla_y\boldsymbol{\varphi}(\boldsymbol{y})d\boldsymbol{y}=0, \tag{4.7a}$$

$\forall\, \boldsymbol{\varphi} \in W_2^1(\mathcal{Y})$ ($\boldsymbol{\varphi}$ is 1-periodic),

$$\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0) \text{ is given in } \mathcal{Y}_f \text{ by (4.6)}, \tag{4.7b}$$

$$\boldsymbol{Z}_2^{ij} \in L^\infty(0,T;W_2^1(\mathcal{Y})/\mathbb{R}), \quad (\partial \boldsymbol{Z}_2^{ij}/\partial t) \in L^2(0,T;W_2^1(\mathcal{Y}_f)),$$
$$\boldsymbol{Z}_2^{ij}:\ \mathbb{R}^3 \times (0,T) \mapsto \mathbb{R}^3 \text{ is 1-periodic in } \boldsymbol{y}. \tag{4.7c}$$

Analogously we formulate the problems for $\boldsymbol{Z}_3(\boldsymbol{y},0)$ and $\boldsymbol{Z}_3(\boldsymbol{y},t)$:

$$\int_{\mathcal{Y}_f} \big\{\alpha_\mu \mathbb{D}(y,\boldsymbol{Z}_3(\boldsymbol{y},0)) + (\alpha_\nu \operatorname{div}_y \boldsymbol{Z}_3(\boldsymbol{y},0))\mathbb{I} - \alpha_{\theta f}\mathbb{I}\big\} : \nabla_y \boldsymbol{\varphi}(\boldsymbol{y}) d\boldsymbol{y} - \int_{\mathcal{Y}_s} \alpha_{\theta s} \operatorname{div}_y \boldsymbol{\varphi}(\boldsymbol{y}) d\boldsymbol{y} = 0, \tag{4.8a}$$

$\forall\, \boldsymbol{\varphi} \in W_2^1(\mathcal{Y})$ ($\boldsymbol{\varphi}$ is 1-periodic),

$$\boldsymbol{Z}_3(\cdot,0) \in W_2^1(\mathcal{Y}_f)/\mathbb{R}, \quad \boldsymbol{Z}_3(\cdot,0):\ \mathbb{R}^3 \mapsto \mathbb{R}^3 \text{ is 1-periodic in } \boldsymbol{y}, \tag{4.8b}$$

and, correspondingly,

$$\boldsymbol{Z}_3(\boldsymbol{y},t) \text{ satisfies the system of (4.7a) and (4.7c)}, \tag{4.9a}$$

$$\boldsymbol{Z}_3(\boldsymbol{y},0) \text{ is given in } \mathcal{Y}_f \text{ by (4.8)}. \tag{4.9b}$$

Finally, the functions $g_1^j(\boldsymbol{y})$ and $g_2^j(\boldsymbol{y})$ are determined by the problems

$$\int_{\mathcal{Y}_f} \varkappa_f \big(\nabla_y g_1^j(\boldsymbol{y}) + \boldsymbol{e}^j\big) \cdot \nabla_y \psi(\boldsymbol{y}) d\boldsymbol{y} + \int_{\mathcal{Y}_s} \varkappa_s \big(\nabla_y g_1^j(\boldsymbol{y}) + \boldsymbol{e}^j\big) \cdot \nabla_y \psi(\boldsymbol{y}) d\boldsymbol{y} = 0, \tag{4.10a}$$

$\forall\, \psi \in W_2^1(\mathcal{Y})$ (ψ is 1-periodic),

$$g_1^j \in W_2^1(\mathcal{Y})/\mathbb{R}, \quad g_1^j:\ \mathbb{R}^3 \mapsto \mathbb{R} \text{ is 1-periodic}, \tag{4.10b}$$

and

$$\int_{\mathcal{Y}_f} \big(\varkappa_f \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^j\big) \cdot \nabla_y \psi(\boldsymbol{y}) d\boldsymbol{y} + \int_{\mathcal{Y}_s} \big(\varkappa_s \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^j\big) \cdot \nabla_y \psi(\boldsymbol{y}) d\boldsymbol{y} = 0, \tag{4.11a}$$

$\forall\, \psi \in W_2^1(\mathcal{Y})$, ($\psi$ is 1-periodic),

$$g_2^j \in W_2^1(\mathcal{Y})/\mathbb{R}, \quad g_2^j:\ \mathbb{R}^3 \mapsto \mathbb{R} \text{ is 1-periodic}. \tag{4.11b}$$

The following proposition implies the demand from the functions $\boldsymbol{Z}_1^{ij}$, $\boldsymbol{Z}_2^{ij}$, $\boldsymbol{Z}_3$, g_1^i, and g_2^i to solve the above stated problems makes sense. Hence the two-scale limiting functions $\boldsymbol{W}$ and Θ admit representations (4.1) and (4.2). Moreover, these representations are unique.

Proposition 3. *Let geometry of the sets $\mathcal{Y}_f$ and $\mathcal{Y}_s$ be prescribed and the coefficients α_μ, α_p, α_ν, α_λ, α_η, $\alpha_{\theta f}$, $\alpha_{\theta s}$, $\varkappa_f$, and $\varkappa_s$ be given. Then each of problems* (4.5)–(4.11) *has a unique solution.*

PROOF. Formulations of problems (4.5)–(4.9) are just slight modifications of the statements of the problems in the pattern cell $\mathcal{Y}$ from [9, Sec. 2.4]. Accordingly, verification of the well-posedness of problems (4.5)–(4.9) follows the lines of [9, Sec. 2.4, see lemmas 6, 7, 9, 10] without essential modifications, relying on the well-known facts and methods in the linear theory of partial differential equations, like the Lax–Milgram lemma, Korn's and Poincaré's inequalities, the Galerkin method, etc. Problems (4.10) and (4.11) are particular cases of the simplest periodic elliptic problem, whose unique solvability is well-known and can be found, for example, in [10, ch. 1]. $\square$

Substituting (4.1) into (3.4) and (4.2) into (3.6), we immediately arrive at the integral equalities (2.9) and (2.10) for $\boldsymbol{w}(\boldsymbol{x},t) = \boldsymbol{w}^*(\boldsymbol{x},t)$ and $\theta(\boldsymbol{x},t) = \theta^*(\boldsymbol{x},t)$ such that the components of tensors $\mathbb{A}_0$, $\mathbb{A}_1$, and $\mathbb{A}_2(t)$ and matrices $\mathbb{C}_0$, $\mathbb{C}_1(t)$, $\mathbb{E}_0$, and $\mathbb{E}_1$ in these integral equalities are given by

$$A_0^{ijkl} = |\mathcal{Y}_f|(\alpha_\mu\delta_{il}\delta_{jk} + \alpha_\nu\delta_{ij}\delta_{kl}) + \alpha_\mu\big\langle\chi(\boldsymbol{y})\mathbb{D}_{kl}(y, \boldsymbol{Z}_1^{ij}(\boldsymbol{y}))\big\rangle_{\mathcal{Y}} + \alpha_\nu\delta_{kl}\big\langle\chi(\boldsymbol{y})\mathrm{div}_y\boldsymbol{Z}_1^{ij}(\boldsymbol{y})\big\rangle_{\mathcal{Y}}, \tag{4.12}$$

$$\begin{aligned} A_1^{ijkl} = |\mathcal{Y}_f|\alpha_p\delta_{ij}\delta_{kl} + |\mathcal{Y}_s|(\alpha_\lambda\delta_{il}\delta_{kj} + \alpha_\eta\delta_{ij}\delta_{kl}) \\ + \delta_{kl}\big\langle\chi(\boldsymbol{y})\big(\alpha_p\mathrm{div}_y\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \alpha_\nu\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0)\big)\big\rangle_{\mathcal{Y}} \\ + \alpha_\mu\big\langle\chi(\boldsymbol{y})\mathbb{D}_{kl}(y, \boldsymbol{Z}_2^{ij}(\boldsymbol{y},0))\big\rangle_{\mathcal{Y}} + \alpha_\eta\delta_{kl}\big\langle(1-\chi(\boldsymbol{y}))\mathrm{div}_y\boldsymbol{Z}_1^{ij}(\boldsymbol{y})\big\rangle \\ + \alpha_\lambda\big\langle(1-\chi(\boldsymbol{y}))\mathbb{D}_{kl}(y, \boldsymbol{Z}_1^{ij}(\boldsymbol{y}))\big\rangle_{\mathcal{Y}}, \end{aligned} \tag{4.13}$$

$$\begin{aligned} A_2^{ijkl}(t) = \delta_{kl}\left\langle\chi(\boldsymbol{y})\Big(\alpha_p\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t) + \alpha_\nu\mathrm{div}_y\frac{\partial\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t)}{\partial t}\Big)\right\rangle_{\mathcal{Y}} \\ + \alpha_\mu\left\langle\chi(\boldsymbol{y})\mathbb{D}_{kl}\Big(y, \frac{\partial\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t)}{\partial t}\Big)\right\rangle_{\mathcal{Y}} + \alpha_\eta\delta_{kl}\big\langle\chi(\boldsymbol{y})\mathrm{div}_y\boldsymbol{Z}_2^{ij}(\boldsymbol{y},t)\big\rangle_{\mathcal{Y}} \\ + \alpha_\lambda\big\langle\chi(\boldsymbol{y})\mathbb{D}_{kl}(y, \boldsymbol{Z}_2^{ij}(\boldsymbol{y},t))\big\rangle_{\mathcal{Y}}, \end{aligned} \tag{4.14}$$

$$C_0^{ij} = |\mathcal{Y}_f|\alpha_{\theta f}\delta_{ij} + |\mathcal{Y}_s|\alpha_{\theta s}\delta_{ij} - \big\langle\chi(\boldsymbol{y})\big(\alpha_\nu\delta_{ij}\mathrm{div}_y\boldsymbol{Z}_3(\boldsymbol{y},0) + \alpha_\mu\mathbb{D}_{ij}(y, \boldsymbol{Z}_3(\boldsymbol{y},0))\big)\big\rangle_{\mathcal{Y}}, \tag{4.15}$$

$$\begin{aligned} C_1^{ij}(t) = -\delta_{ij}\left\langle\chi(\boldsymbol{y})\Big(\alpha_p\mathrm{div}_y\boldsymbol{Z}_3(\boldsymbol{y},t) + \alpha_\nu\mathrm{div}_y\frac{\partial\boldsymbol{Z}_3(\boldsymbol{y},t)}{\partial t}\Big)\right\rangle_{\mathcal{Y}} \\ - \alpha_\mu\left\langle\chi(\boldsymbol{y})\mathbb{D}_{ij}\Big(y, \frac{\partial\boldsymbol{Z}_3(\boldsymbol{y},t)}{\partial t}\Big)\right\rangle_{\mathcal{Y}} \\ - \alpha_\eta\delta_{ij}\big\langle(1-\chi(\boldsymbol{y}))\mathrm{div}_y\boldsymbol{Z}_3(\boldsymbol{y},t)\big\rangle_{\mathcal{Y}} - \alpha_\lambda\big\langle(1-\chi(\boldsymbol{y}))\mathbb{D}_{ij}(y, \boldsymbol{Z}_3(\boldsymbol{y},t))\big\rangle_{\mathcal{Y}}, \end{aligned} \tag{4.16}$$

$$E_0^{ij} = |\mathcal{Y}_f|\varkappa_f\delta_{ij} + |\mathcal{Y}_s|\varkappa_s\delta_{ij} + \left\langle\big(\chi(\boldsymbol{y})\varkappa_f + (1-\chi(\boldsymbol{y}))\varkappa_s\big)\frac{\partial g_1^j(\boldsymbol{y})}{\partial y_i}\right\rangle_{\mathcal{Y}}, \tag{4.17}$$

$$E_1^{ij} = |\mathcal{Y}_f|\alpha_{\theta f}\delta_{ij} + |\mathcal{Y}_s|\alpha_{\theta s}\delta_{ij} - \left\langle\big(\chi(\boldsymbol{y})\varkappa_f + (1-\chi(\boldsymbol{y}))\varkappa_s\big)\frac{\partial g_2^j(\boldsymbol{y})}{\partial y_i}\right\rangle_{\mathcal{Y}}, \tag{4.18}$$

$i, j, k, l = 1, 2, 3$.

Since the vector-functions $\boldsymbol{Z}_2^{ij}$ and $\boldsymbol{Z}_3$ satisfy the regularity conditions (4.7c), one has $A_2^{ijkl}, C_1^{ij} \in L^2(0,T)$. Due to this and on the strength of the regularity properties of functions $\boldsymbol{w}^*$ and θ^* (see in Sec. 3), all the integrals in (2.9) and (2.10) are well-defined. Also notice that, due to the uniqueness assertion in proposition 2, all convergent subsequences of $\{\boldsymbol{w}^\varepsilon, \theta^\varepsilon\}$ ($\varepsilon^{-1} \in \mathbb{N}$) has the same limit $\{\boldsymbol{w}^*, \theta^*\}$. Hence the entire sequence $\{\boldsymbol{w}^\varepsilon, \theta^\varepsilon\}$ ($\varepsilon^{-1} \in \mathbb{N}$) is convergent. Theorem 1 is proved. □

5. Proof of theorem 2

1. We prove the assertion of item 1 for the tensor $\mathbb{A}_0$ only. For $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{C}_0$, $\mathbb{C}_1$, $\mathbb{E}_0$, and $\mathbb{E}_1$ the symmetry property is verified by quite analogous considerations.

Inserting $\boldsymbol{\varphi}(\boldsymbol{y}) = \boldsymbol{Z}_1^{qr}(\boldsymbol{y})$ as the test function into (4.5a), which is legal, we deduce

$$\begin{aligned} A_{01}^{ijqr} &\overset{def}{=} \big\langle \chi(\boldsymbol{y})\big[\alpha_\mu \mathbb{D}_{ij}(y, \boldsymbol{Z}_1^{qr}(\boldsymbol{y})) + \alpha_\nu \delta_{ij} \mathrm{div}_y \boldsymbol{Z}_1^{qr}(\boldsymbol{y})\big]\big\rangle_{\mathcal{Y}} \\ &= -\big\langle \chi(\boldsymbol{y})\big[\alpha_\mu \mathbb{D}(y, \boldsymbol{Z}_1^{ij}(\boldsymbol{y})) : \mathbb{D}(y, \boldsymbol{Z}_1^{qr}(\boldsymbol{y})) + \alpha_\nu \mathrm{div}_y \boldsymbol{Z}_1^{ij}(\boldsymbol{y}) \cdot \mathrm{div}_y \boldsymbol{Z}_1^{qr}(\boldsymbol{y})\big]\big\rangle_{\mathcal{Y}} \end{aligned}$$

($i, j, k, l = 1, 2, 3$). The right-hand side in this equality stays unchanged, if we interchange places of the pairs of indices (i, j) and (q, r), and the left-hand side stays unchanged, if we interchange places of the indices i and j. Combining these two features we conclude that the tensor $\mathbb{A}_{01} = \{A_{01}^{ijkl}\}$ is symmetric in the sense of the equalities (2.11). In turn, on the strength of (4.12), we have $A_0^{ijkl} = |\mathcal{Y}_f|(\alpha_\mu \delta_{il}\delta_{jk} + \alpha_\nu \delta_{ij}\delta_{kl}) + A_{01}^{klij}$, which immediately implies the symmetry property for $\mathbb{A}_0$.

2. We outline precisely justification of the assertion in item 2 of the theorem for the tensor $\mathcal{A}^\gamma$ only. The proof for $\mathcal{C}^\gamma$ is quite analogous and we skip it.

Introduce into consideration the stationary problem arising as the result of application of Laplace's transform in t to the integral equality (4.7a) (as usually, we consider that all the time-dependent functions in (4.7a) are extended by zero to the right outside the interval $(0, T)$):

For $i, j = 1, 2, 3$ find the vector-function $\boldsymbol{\Lambda}_\gamma^{ij} = \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})$ satisfying the system

$$\begin{aligned} &\int_{\mathcal{Y}_f} \Big\{\gamma\alpha_\mu \mathbb{D}\big(y, \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big) - \alpha_\mu \mathbb{D}\big(y, \boldsymbol{Z}_2^{ij}(\boldsymbol{y}, 0)\big) + (\alpha_p \mathrm{div}_y \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y}))\mathbb{I} \\ &\qquad + \gamma\big(\alpha_\nu \mathrm{div}_y \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\mathbb{I} - (\alpha_\nu \mathrm{div}_y \boldsymbol{Z}_2^{ij}(\boldsymbol{y}, 0))\Big\} : \mathbb{D}(y, \boldsymbol{\varphi}(\boldsymbol{y}))d\boldsymbol{y} \\ &+ \int_{\mathcal{Y}_s} \big\{\alpha_\lambda \mathbb{D}(y, \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})) + (\alpha_\eta \mathrm{div}_y \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y}))\mathbb{I}\big\} : \mathbb{D}(y, \boldsymbol{\varphi}(\boldsymbol{y}))d\boldsymbol{y} = 0, \\ &\qquad\qquad \forall\, \boldsymbol{\varphi} \in W_2^1(\mathcal{Y}), \quad \boldsymbol{\varphi} \text{ is 1-periodic,} \end{aligned} \tag{5.1a}$$

$$\boldsymbol{\Lambda}_\gamma^{ij} \in W_2^1(\mathcal{Y})/\mathbb{R}, \quad \boldsymbol{\Lambda}_\gamma^{ij} : \mathbb{R}^3 \mapsto \mathbb{R}^3 \text{ is 1-periodic.} \tag{5.1b}$$

The variable $\gamma > 0$ enters this problem parametrically.

As in the proof of proposition 3 we notice that this problem is a particular case of the simplest periodic elliptic problem [10, ch. 1] and therefore has a unique solution for all $\gamma > 0$. Clearly, $\boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})$ is Laplace's transform in t of $\boldsymbol{Z}_2^{ij}(\boldsymbol{y}, t)$. Thus the components of the tensor $\hat{\mathbb{A}}_2(\gamma)$ can be written in the form

$$\hat{A}_2^{ijkl}(\gamma) = \delta_{kl}\Big\langle \chi(\boldsymbol{y})\Big(\alpha_p \mathrm{div}_y \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y}) + \gamma\alpha_\nu \mathrm{div}_y \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y}) - \alpha_\nu \mathrm{div}_y \boldsymbol{Z}_2^{ij}(\boldsymbol{y},0)\Big\rangle_{\mathcal{Y}} \\ + \alpha_\mu\Big\langle \chi(\boldsymbol{y})\big[\gamma\mathbb{D}_{kl}\big(y,\boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big) - \mathbb{D}_{kl}\big(y,\boldsymbol{Z}_2^{ij}(\boldsymbol{y},0)\big)\big]\Big\rangle_{\mathcal{Y}} \\ + \alpha_\eta\delta_{kl}\big\langle\chi(\boldsymbol{y})\mathrm{div}_y\boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big\rangle_{\mathcal{Y}} + \alpha_\lambda\big\langle\chi(\boldsymbol{y})\mathbb{D}_{kl}(y,\boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y}))\big\rangle_{\mathcal{Y}}. \quad (5.2)$$

Let $\mathbb{X} = (X_{ij})$ be an arbitrary constant symmetric 3×3-matrix. Multiply integral equalities (4.5b), (4.6a), and (5.1a) by X_{ij}, and integral equality (4.5a) by γX_{ij}, sum up the resulting equations, assuming that all test functions are the same, in particular, that $\boldsymbol{\beta}(\boldsymbol{y}) = \boldsymbol{\varphi}(\boldsymbol{y})$, and then sum over i and j. As the result we deduce

$$\int_{\mathcal{Y}_f}\Big\{\gamma\alpha_\mu\Big[\mathbb{D}\Big(y,\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathbb{X}\Big] \\ + (\gamma\alpha_\nu + \alpha_p)\Big[\mathrm{div}_y\Big(\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathrm{tr}\,\mathbb{X}\Big]\mathbb{I}\Big\} : \mathbb{D}(y,\boldsymbol{\varphi}(\boldsymbol{y}))d\boldsymbol{y} \\ + \int_{\mathcal{Y}_s}\Big\{\alpha_\lambda\Big[\mathbb{D}\Big(y,\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathbb{X}\Big] \\ + \alpha_\eta\Big[\mathrm{div}_y\Big(\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathrm{tr}\,\mathbb{X}\Big]\mathbb{I}\Big\} : \mathbb{D}(y,\boldsymbol{\varphi}(\boldsymbol{y}))d\boldsymbol{y} = 0, \quad (5.3)$$

where $\boldsymbol{\varphi} \in W_2^1(\mathcal{Y})$ is an arbitrary 1-periodic function.

Also introduce the quadratic expression

$$I_*(\mathbb{X},\mathbb{X}) \stackrel{def}{=} \int_{\mathcal{Y}}\big(\gamma\alpha_\mu\chi(\boldsymbol{y}) + \alpha_\lambda(1-\chi(\boldsymbol{y}))\big)\Big|\mathbb{D}\Big(y,\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathbb{X}\Big|^2 d\boldsymbol{y} \\ + \int_{\mathcal{Y}}\big((\alpha_p + \gamma\alpha_\nu)\chi(\boldsymbol{y}) + \alpha_\eta(1-\chi(\boldsymbol{y}))\big)\Big|\mathrm{div}_y\Big(\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathrm{tr}\,\mathbb{X}\Big|^2 d\boldsymbol{y}, \quad (5.4)$$

which is evidently nonegative. Substituting the test function $\boldsymbol{\varphi}(\boldsymbol{y}) = \sum_{i,j=1}^3 X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)$ into (5.3), which is legal, and then combining the resulting equation with (5.4), we arrive at the equality

$$I_*(\mathbb{X},\mathbb{X}) = \int_{\mathcal{Y}}\Big\{\big(\gamma\alpha_\mu\chi(\boldsymbol{y}) + \alpha_\lambda(1-\chi(\boldsymbol{y}))\big)\Big[\mathbb{D}\Big(y,\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathbb{X}\Big] \\ + \big((\alpha_p + \gamma\alpha_\nu)\chi(\boldsymbol{y}) + \alpha_\eta(1-\chi(\boldsymbol{y}))\big)\Big[\mathrm{div}_y\Big(\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) + \mathrm{tr}\,\mathbb{X}\Big]\mathbb{I}\Big\} : \mathbb{X}\,d\boldsymbol{y}. \quad (5.5)$$

On the strength of the representations (4.12), (4.13), and (5.2), the right-hand side of (5.5) coincides with $(\mathcal{A}^\gamma : \mathbb{X}) : \mathbb{X}$. Hence $(\mathcal{A}^\gamma : \mathbb{X}) : \mathbb{X} \geq 0$ for any symmetric matrix

$\mathbb{X}$ for all $\gamma > 0$. In turn, the demand of symmetry of $\mathbb{X}$ can be easily removed in this inequality due to the symmetry properties of the tensors $\mathbb{A}_0$, $\mathbb{A}_1$, $\mathbb{A}_2(t)$, and, consequently, $\mathcal{A}^\gamma$. Hence non-negativeness of the tensor $\mathcal{A}^\gamma$ is proved.

The strict positive definiteness of $\mathcal{A}^\gamma$ is justified by the contradiction method, following the lines of [9, lemma 8]. Suppose that for some nontrivial matrix $\mathbb{X}$, i.e., $\mathbb{X} \neq 0$, the equality $(\mathcal{A}^\gamma : \mathbb{X}) : \mathbb{X} = 0$ take place. Hence $I_*(\mathbb{X}, \mathbb{X}) = 0$ and due to (5.4) one has

$$\mathbb{D}\Big(y, \sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)\Big) = -\mathbb{X}, \quad \boldsymbol{y} \in \mathcal{Y}. \tag{5.6}$$

This equality immediately implies that the sum $\sum_{i,j=1}^{3} X_{ij}\big(\boldsymbol{Z}_1^{ij}(\boldsymbol{y}) + \boldsymbol{\Lambda}_\gamma^{ij}(\boldsymbol{y})\big)$ is linear, that is, has the form $\boldsymbol{c}_0 + \sum_{k=1}^{3} \boldsymbol{c}_k y_k$, where $\boldsymbol{c}_k$ $(k = 0, 1, 2, 3)$ are some constant vectors. However, on the strength of 1-periodicity of $\boldsymbol{Z}_1^{ij}$ and $\boldsymbol{\Lambda}_\gamma^{ij}$, this is possible only if $\boldsymbol{c}_k = 0$ for $k = 1, 2, 3$. From this and equality (5.6) it follows that $\mathbb{X} = 0$, which contradicts the initial assumption $\mathbb{X} \neq 0$. Thus, there exists a constant $c(\gamma) > 0$ such that $(\mathcal{A}^\gamma : \mathbb{X}) : \mathbb{X} \geq c(\gamma)|\mathbb{X}|^2$ for all 3×3-matrices $\mathbb{X}$. The strict positive definiteness of $\mathcal{A}^\gamma$ for $\gamma > 0$ is proved.

3. The proof is similar to the proof of item 2 immediately above. It only worth to notice that the connectivity of $\mathcal{Y}_f$ and E_f is used in the justification of the strict positive definiteness of $\mathbb{A}_0$ just like the connectivity of $\mathcal{Y}_s$ and E_s was used above in the justification of the strict positive definiteness of $\mathcal{A}^\gamma$.

4. By the direct substitution we verify that whenever $\partial\mathcal{Y} \cap \partial\mathcal{Y}_f = \emptyset$, the solution $\boldsymbol{Z}_1^{ij}$ of problem (4.5) is linear in $\mathcal{Y}_f$ and satisfies the equality

$$\alpha_\mu \mathbb{D}(y, \boldsymbol{Z}_1^{ij}) + (\alpha_\nu \mathrm{div}_y \boldsymbol{Z}_1^{ij})\mathbb{I} + \alpha_\mu \mathbb{J}^{ij} + \alpha_\nu \delta_{ij}\mathbb{I} = 0, \quad \boldsymbol{y} \in \mathcal{Y}_f \quad (i, j = 1, 2, 3).$$

Combining this equality with (4.12) we immediately deduce that $\mathbb{A}_0 = 0$.

The proof of the strict positive definiteness of $\mathbb{A}_1$ is similar to the proofs in items 2 and 3.

5. Consider

$$\begin{aligned} I_0 &\overset{def}{=} \sum_{i,j=1}^{3} \big\langle \big(\chi(\boldsymbol{y})\varkappa_f + (1 - \chi(\boldsymbol{y}))\varkappa_s\big)(\nabla_y g_1^j(\boldsymbol{y}) + \boldsymbol{e}^j)\xi_j (\nabla_y g_1^i(\boldsymbol{y}) + \boldsymbol{e}^i)\xi_i \big\rangle_{\mathcal{Y}} \\ &= \left\langle \big(\chi(\boldsymbol{y})\varkappa_f + (1 - \chi(\boldsymbol{y}))\varkappa_s\big)\Big(\sum_{i=1}^{3} (\nabla_y g_1^i(\boldsymbol{y}) + \boldsymbol{e}^i)\xi_i\Big)^2 \right\rangle_{\mathcal{Y}} \geq 0 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3. \end{aligned} \tag{5.7}$$

Substituting the test function $\psi = g_1^i(\boldsymbol{y})\xi_i\xi_j$ into (4.10a), which is legal, summing over i and j, and using representation (4.17) we arrive at the equality $I_0 = \mathbb{E}_0\boldsymbol{\xi} \cdot \boldsymbol{\xi}$. Combining it with estimate (5.7) we conclude that $\mathbb{E}_0$ is nonnegative definite. The strict positive definiteness of $\mathbb{E}_0$ follows from the connectivity of the set E_s due to the arguments, similar to those from the justification of the assertion in item 2.

Next consider

$$I_1^{ij} \overset{def}{=} \Big\langle \big\{ \chi(\boldsymbol{y}) \big[\varkappa_f \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^j \big]$$

$$+ (1-\chi(\boldsymbol{y})) \big[\varkappa_s \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^j \big] \big\} \cdot \big(\varkappa_f \nabla_y g_2^i(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^i + \varkappa_s \nabla_y g_2^i(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^i \big) \Big\rangle_{\mathcal{Y}}$$

$$= -\Big\langle \big\{ \chi(\boldsymbol{y}) \big(\varkappa_f \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^j \big) + (1-\chi(\boldsymbol{y})) \big(\varkappa_s \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^j \big) \big\} \cdot (\alpha_{\theta f} + \alpha_{\theta s}) \boldsymbol{e}^i \Big\rangle_{\mathcal{Y}}$$

$$= (\alpha_{\theta f} + \alpha_{\theta s}) E_1^{ij} \quad (i,j = 1,2,3). \quad (5.8)$$

In this chain of two equalities the former holds true due to (4.11a) and the latter is valid on the strength of (4.18). On the other hand,

$$\sum_{i,j=1}^{3} I_1^{ij} \xi_i \xi_j = \sum_{i,j=1}^{3} \Big\langle \chi(\boldsymbol{y}) \big(\varkappa_f \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^j \big) \xi_j \cdot (\varkappa_f \nabla_y g_2^i(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^i) \xi_i \Big\rangle_{\mathcal{Y}}$$

$$+ \sum_{i,j=1}^{3} \Big\langle (1-\chi(\boldsymbol{y})) \big(\varkappa_s \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^j \big) \xi_j \cdot \big(\varkappa_s \nabla_y g_2^i(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^i \big) \xi_i \Big\rangle_{\mathcal{Y}}$$

$$+ \sum_{i,j=1}^{3} \Big\langle \chi(\boldsymbol{y}) \big(\varkappa_f \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^j \big) \xi_j \cdot \big(\varkappa_s \nabla_y g_2^i(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^i \big) \xi_i \Big\rangle_{\mathcal{Y}}$$

$$+ \sum_{i,j=1}^{3} \Big\langle (1-\chi(\boldsymbol{y})) \big(\varkappa_s \nabla_y g_2^j(\boldsymbol{y}) - \alpha_{\theta s} \boldsymbol{e}^j \big) \xi_j \cdot \big(\varkappa_f \nabla_y g_2^i(\boldsymbol{y}) - \alpha_{\theta f} \boldsymbol{e}^i \big) \xi_i \Big\rangle_{\mathcal{Y}}$$

$$\overset{def}{=} I_2 + I_3 + I_4 + I_5 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3. \quad (5.9)$$

Here clearly $I_2 + I_3 \geq 0$, and interchanging indices i and j within I_4 we also have

$$I_4 + I_5 = \sum_{i,j=1}^{3} \Big\langle (\varkappa_f \nabla_y g_2^i - \alpha_{\theta f} \boldsymbol{e}^i) \cdot (\varkappa_s \nabla_y g_2^j - \alpha_{\theta s} \boldsymbol{e}^j) \xi_i \xi_j \Big\rangle_{\mathcal{Y}}$$

$$= \sum_{i,j=1}^{3} \varkappa_f \varkappa_s \Big\langle \nabla_y g_2^i \xi_i \cdot \nabla_y g_2^j \xi_j \Big\rangle_{\mathcal{Y}} + \alpha_{\theta s} \alpha_{\theta s} |\boldsymbol{\xi}|^2$$

$$- \sum_{i,j=1}^{3} \alpha_{\theta s} \varkappa_f \xi_i \xi_j \left\langle \frac{\partial g_2^i}{\partial y_j} \right\rangle_{\mathcal{Y}} - \sum_{i,j=1}^{3} \alpha_{\theta f} \varkappa_s \xi_i \xi_j \left\langle \frac{\partial g_2^j}{\partial y_i} \right\rangle_{\mathcal{Y}}$$

$$\overset{def}{=} I_6 + \alpha_{\theta s} \alpha_{\theta s} |\boldsymbol{\xi}|^2 - I_7 - I_8. \quad (5.10)$$

It is easy to see that $I_6 \geq 0$ and that $I_7 = I_8 = 0$ due to 1-periodicity of $g_2^i(\boldsymbol{y})$ and the integration by parts formula. Hence, combining expressions (5.8)–(5.10), we establish the estimate

$$\mathbb{E}_1 \boldsymbol{\xi} \cdot \boldsymbol{\xi} \geq \frac{\alpha_{\theta f} \alpha_{\theta s}}{\alpha_{\theta f} + \alpha_{\theta s}} |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3$$

and thus conclude that $\mathbb{E}_1$ is strictly positively definite.

The strict positive definiteness of $\mathbb{C}_0$ is proved similarly.

Theorem 2 is proved. □

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References

[1] *G. Allaire,* Homogenization and two-scale convergence, SIAM J. Math. Anal., **23** (1992), 1482–1518.

[2] *H. Bateman, A. Erdélyi,* Tables of integral transforms, Vol. 1, New York, Toronto, McGraw-Hill, 1954.

[3] *M.J. Buckingham,* Seismic wave propagation in rocks and marine sediments: a new theoretical approach, in Underwater Acoustics, Volume I (E.A. Alippi, G.B. Cannelli, eds.) CNR-IDAC, Rome, 1998, 299–300.

[4] *R. Burridge, J.B. Keller,* Poroelasticity equations derived from microstructure, J. Acoust. Soc. Am., **70**, No 4 (1981), 1140–1146.

[5] *T. Clopeau, J.L. Ferrin, R.P. Gilbert, A. Mikelić,* Homogenizing the acoustic properties of the seabed: Part II, Math. Comp. Modelling **33** (2001), 821–841.

[6] *Coussy O.* Poromechanics. Chichester: John Wiley & Sons, 2004.

[7] *R. Dautray, J.-L. Lions,* Mathematical analysis and numerical methods for science and technology, Vol. 5: Evolution problems I, Berlin, Springer, 2000.

[8] *P. Germain,* Course of mechanics of continuous media. Vol. 1, Moscow, Vysshaia Shkola, 1983. (Russian translation of Cours de Mécanique des Milieux Continus. Tome 1: Théorie générale, Masson et Cie, Paris, 1973 (in French).)

[9] *R.P. Gilbert, A. Mikelić,* Homogenizing the acoustic properties of the seabed: Part I, Nonlinear Anal., **40** (2000), 185–212.

[10] *V.V. Jikov, S.M. Kozlov, O.A. Oleinik,* Homogenization of differential operators and integral functionals, New York, Springer, 1994.

[11] *O.A. Ladyzhenskaya,* The boundary value problems of mathematical physics, Applied Mathematical Sciences **49**, New York, Springer, 1985.

[12] *A.M. Meirmanov, S.A. Sazhenkov,* Generalized solutions to linearized equations of thermoelastic solid and viscous thermofluid, Electronic J. Diff. Equat., 30 pages (Submitted.)

[13] *G. Nguetseng,* A general convergence result for a functional related to the theory of homogenization, SIAM J. Math. Anal., **20** (1989), 608–623.

[14] *P.V. O'Neil,* Advanced Engineering Mathematics, Pacific Grove, USA, Brooks/Cole Publ. Co., 1995.

[15] *P.Ya. Polubarinova-Kochina, S.N. Numerov, I.A. Charniy, V.M. Entov (eds.),* Development of studies in filtration theory in the USSR (1917–1967), Moscow, Nauka, 1969. (In Russian.)

[16] *Yu.G. Reshetnyak,* Stability theorems in geometry and analysis, Mathematics and Its Appl., **304**, Dordrecht, Kluwer, 1994.

[17] *E. Sanchez-Palencia,* Non-homogeneous media and vibration theory, Lecture Notes in Physics, **127**, Springer, Berlin, 1980.

[18] *R.M. Temam, A.M. Miranville,* Mathematical modelling in continuum mechanics, Cambridge, Cambridge University Press, 2005.

[19] *V.S. Zarubin, G.N. Kuvyrkin,* Mathematical models of thermomechanics, Moscow, Fizmatlit, 2002. (In Russian.)

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