DISSIPATIVE SOLUTIONS FOR THE CAMASSA–HOLM EQUATION

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Abstract. We show that the Camassa–Holm equation $u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0$ possesses a global continuous semigroup of weak dissipative solutions for initial data $u|_{t=0}$ in H^1. The result is obtained by introducing a coordinate transformation into Lagrangian coordinates. Stability in terms of H^1 and L^∞ norm is discussed.

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1. Introduction

The Camassa–Holm equation

$$u_t - u_{xxt} + 2\kappa u_x + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \quad u|_{t=0} = \bar{u}, \tag{1.1}$$

has been extensively studied since the first systematic analysis in [5, 6]. Part of the attraction is the surprising complexity of the equation and its deep and nontrivial properties. To list a few of its peculiarities: The Camassa–Holm equation has a bi-Hamiltonian structure [16], it is completely integrable [5], and it has infinitely many conserved quantities [5].

Here we study the equation with $\kappa = 0$ on the real line, that is,

$$u_t - u_{xxt} + 3uu_x - 2u_xu_{xx} - uu_{xxx} = 0, \tag{1.2}$$

and henceforth we refer to (1.2) as the Camassa–Holm equation.

The equation enjoys two distinct classes of solutions, and the dichotomy between the two classes is associated with wave breaking, which takes place in finite time in such a way that the H^1 and L^∞ norms of the solution remain finite while the spatial derivative u_x becomes unbounded pointwise. Classical solutions can only develop singularities in finite time in the form of wave breaking, cf. [11], and criteria for wave breaking are available, cf. e.g., [14]. More precisely, Constantin, Escher, and Molinet [12, 14, 15] showed the following result: If the initial data $u|_{t=0} = \bar{u} \in H^1(\mathbb{R})$ and

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$\bar u - \bar u''$ is a positive Radon measure, then equation (1.2) has a unique global weak solution $u \in C([0,T], H^1(\mathbb{R}))$, for any T positive, with initial data $\bar u$. However, any solution with odd initial data $\bar u$ in $H^3(\mathbb{R})$ such that $\bar u_x(0) < 0$ blows up in a finite time.

The dichotomy between the two classes of solutions can nicely be illustrated by studying multipeakon solutions of the Camassa–Holm of the form

$$u(t,x) = \sum_{i=1}^{n} p_i(t) e^{-|x-q_i(t)|}, \tag{1.3}$$

where the $(p_i(t), q_i(t))$ satisfy the explicit system of ordinary differential equations

$$\dot q_i = \sum_{j=1}^{n} p_j e^{-|q_i-q_j|}, \quad \dot p_i = \sum_{j=1}^{n} p_i p_j \operatorname{sgn}(q_i - q_j) e^{-|q_i-q_j|}.$$

Observe that the solution (1.3) is not smooth even with continuous functions $(p_i(t), q_i(t))$; one possible way to interpret (1.3) as a weak solution of (1.2) is to rewrite the equation (1.2) as

$$u_t + \Big(\frac{1}{2}u^2 + (1-\partial_x^2)^{-1}(u^2 + \frac{1}{2}u_x^2)\Big)_x = 0.$$

Wave breaking may appear when at least two of the q_i's coincide. If all the $p_i(0)$ have the same sign, the peakons move in the same direction. Higher peakons move faster than the smaller ones, and when a higher peakon overtakes a smaller, there is an exchange of mass, but no wave breaking, and the $q_i(t)$ remain distinct, and one has a global solution. However, if some of $p_i(0)$ have opposite sign, wave breaking may incur, see, e.g., [2, 24]. For simplicity, consider the case with $n = 2$ and one peakon $p_1(0) > 0$ (moving to the right) and one antipeakon $p_2(0) < 0$ (moving to the left). In the symmetric case ($p_1(0) = -p_2(0)$ and $q_1(0) = -q_2(0) < 0$) the solution will vanish pointwise at the collision time t^* when $q_1(t^*) = q_2(t^*)$, that is, $u(t^*, x) = 0$ for all $x \in \mathbb{R}$. Clearly, at least two scenarios are possible; one is to let $u(t,x)$ vanish identically for $t > t^*$, and the other possibility is to let the peakon and antipeakon "pass through" each other in a way that is consistent with the Camassa–Holm equation. In the first case the energy $\int (u^2 + u_x^2)\,dx$ decreases to zero at t^*, and remains equal to zero for $t \geq t^*$, while in the second case, the energy remains constant except at t^*. Clearly, the well-posedness of the equation is a delicate matter in this case. The first solution could be denoted a dissipative solution, while the second one could be called conservative. Other solutions are also possible.

Multipeakons play a fundamental role for the Camassa–Holm equation. Indeed, if the initial data $\bar u$ is in H^1 and $\bar u - \bar u''$ is a positive Radon measure, then one can construct a sequence of multipeakons that converges in $L^\infty_{\rm loc}(\mathbb{R}; H^1_{\rm loc}(\mathbb{R}))$ to the unique global solution of the Camassa–Holm equation, see [19]. Note that in this case there will be no wave breaking and dissipative and conservative solutions coincide.

The problem of continuation beyond wave breaking has been considered by Bressan and Constantin [3, 4] and Holden and Raynaud [22]. Bressan and Constantin reformulated the Camassa–Holm equation as a semilinear system of ordinary differential equations taking values in a Banach space. This formulation allowed them to continue the solution beyond collision time, giving either a global conservative solution where the energy is conserved for almost all times or a dissipative solution where energy may vanish from the system. Local existence of the semilinear system is obtained by a contraction argument. Furthermore, the clever reformulation allows for a global solution where all singularities disappear. Going back to the original function u, one obtains a global solution of the Camassa–Holm equation.

The well-posedness, i.e., the uniqueness and stability of the solution, is addressed as follows. In the conservative case one includes in addition to the solution u, a family of non-negative Radon measures μ_t with density $u_x^2\,dx$ with respect to the Lebesgue measure. The pair (u,μ_t) constitutes a continuous semigroup, in particular, one has uniqueness and stability. On the other hand, in the dissipative case, their solution is characterized by an entropy condition of the form

$$u_x(x,t) \le C(1+t^{-1}) \tag{1.4}$$

where the constant C only depends on the H^1 norm of the initial data. The uniqueness issue is considerably more delicate in the dissipative case. The procedure in [4] yields a unique semigroup of solutions in H^1. However, this does not exclude the possibility that there exist other solutions that satisfy the entropy condition (1.4).

In the present paper, as in Bressan and Constantin [4], we reformulate the equation using a different set of variables and obtain a semilinear system of ordinary differential equations. However, the change of variables we use is distinct from that of Bressan and Constantin and simply corresponds to the transformation between Eulerian and Lagrangian coordinates. Let $u=u(t,x)$ denote the solution (which corresponds to Eulerian coordinates), and $y(t,\xi)$ the corresponding characteristics (and we identify the variable ξ with a "particle"), thus $y_t(t,\xi)=u(t,y(t,\xi))$. Our new variables are $y(t,\xi)$ and

$$U(t,\xi)=u(t,y(t,\xi)), \quad h(t,\xi)=(u^2+u_x^2)\circ y\, y_\xi, \tag{1.5}$$

where U corresponds to the Lagrangian velocity while h is the change in Lagrangian energy distribution along the particle path. Furthermore, let

(1.6)
$$\begin{aligned}
Q(t,\xi) &= -\frac{1}{4}\int_{\mathbb{R}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta,\\
P(t,\xi) &= \frac{1}{4}\int_{\mathbb{R}} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta.
\end{aligned}$$

Then one can show that

$$\begin{cases} y_t = U,\\ U_t=-Q,\\ h_t=-2QUy_\xi+(3U^2-2P)U_\xi, \end{cases} \tag{1.7}$$

is equivalent to the Camassa–Holm equation (in Lagrangian coordinates). Indeed, a major part of the paper is to study the properties of the system, or rather a slightly extended system, and its relationship to the Eulerian variable. After differentiating (1.7) with respect to ξ, we obtain

$$\begin{cases} y_{\xi t}=U_\xi,\\ U_{\xi t}=\frac{1}{2}h+\left(\frac{1}{2}U^2-P\right)y_\xi,\\ h_t=-2Q\,Uy_\xi+\big(3U^2-2P\big)\,U_\xi, \end{cases} \tag{1.8}$$

which is semilinear in (y_ξ,U_ξ,h). In [4], the Lagrangian velocity U is used, and the second variable, $q=(1+u_x^2)\circ yy_\xi$ is equivalent to the Lagrangian energy density $h=(u^2+u_x^2)\circ yy_\xi$. The third variable $v=2\arctan(u_x\circ y)$, with no obvious physical interpretation, is necessary to close the system of ordinary differential equations so that a contraction argument can be applied. In this article, we rather use the characteristic $y(t,\xi)$ itself as the third variable.

Dissipative solutions differ from conservative solutions when particles collide, that is, where $y_\xi(t,\xi)=0$ for ξ in an interval of positive length. If we solve (1.7) and (1.8), we obtain the conservative solution. However, to obtain the dissipative solution, we impose that when particles collide, they lose their energy, that is, if $y_\xi(\tau,\xi)=0$ for some τ, then we set $h(\tau,\xi)=0$. One can show that $y_\xi(\tau,\xi)=0$ implies $U_\xi(\tau,\xi)=0$ so that the system (1.8) implies that, for $t\geq\tau$,

$$y_\xi(t,\xi)=U_\xi(t,\xi)=h(t,\xi)=0.$$

After defining

$$\tau(\xi)=\sup\{t\in\mathbb{R}^+\mid y_\xi(t',\xi)>0 \text{ for all } 0\leq t'<t\},$$

the modified system to be solved reads (where χ_B is the indicator function of the set B)

$$y_t=U,\quad U_t=-Q, \tag{1.9}$$

$$\begin{cases} y_{\xi t}=\chi_{\{\tau(\xi)>t\}}U_\xi,\\ U_{\xi t}=\chi_{\{\tau(\xi)>t\}}\big(\frac{1}{2}h+\left(\frac{1}{2}U^2-P\right)y_\xi\big),\\ h_t=\chi_{\{\tau(\xi)>t\}}\big(-2Q\,Uy_\xi+\big(3U^2-2P\big)\,U_\xi\big), \end{cases} \tag{1.10}$$

where Q and P are given by

$$Q(t,\xi)=-\frac{1}{4}\int_{\tau(\eta)>t}\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta,$$

$$P(t,\xi)=\frac{1}{4}\int_{\tau(\eta)>t}\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta.$$

Due to the singularity resulting from the use of the indicator function, existence of solutions is not evident. Global existence of solutions of (1.9)–(1.10), described in Theorem 2.11, is obtained starting from a contraction argument which offers short time existence, see Theorem 2.8. Part of the analysis is to identify an appropriate set, denoted $\mathcal{G}$, of initial data that is invariant under the flow, and that is consistent with the transition back to Eulerian variables. The proper set is introduced in Definition 2.2. The stability issue is considerably more subtle for dissipative solutions compared to conservative solutions. Let us give a verbal explanation here. Assuming a solution $X(t)=(y,U,h)$ of the system (1.9)–(1.10), we consider a given particle, that is, we fix ξ. The solution can be proven to be confined between two circles (see equations (3.1) and Figure 3). If the solution is below the horizontal axis y_ξ, the time evolution drives the solution into the origin. For conservative solutions, the solution should just continue past the origin, while for dissipative solutions, the solution should terminate at the origin. In the case of initial data that are close to the origin, but on opposite sides of the horizontal axis, time evolution will move the solutions apart in the dissipative case, and thus there is no stability in the Euclidean norm. However, one has to introduce a different measure of distance that separates points that are near the origin but on opposite sides of the horizontal axis. The new distance $d_\mathbb{R}$ is defined in (3.2) in terms of a function g (introduced in Definition 2.1) that treats the origin properly. The key result concerning global time stability states that (see Theorem 4.1 for exact conditions) for two solutions $X(t)$ and $\bar{X}(t)$ we have

$$\sup_{t\in[0,T]} d_\mathbb{R}(X(t),\bar{X}(t))\leq Kd_\mathbb{R}(X_0,\bar{X}_0)$$

for some constant K and any given T.

The next step is to transfer this stability into a similar statement in the Eulerian variable u. However, this is complicated by the redundancy in variables from three Lagrangian variables into the single Eulerian variable. This is associated with the notion of relabeling, which corresponds to the proper identification of the exact class of Lagrangian variables that corresponds to one and the same Eulerian variable. The definition of equivalence classes used in the conservative case, will not work for dissipative solutions. The relabeling is discussed in Section 5. More precisely (see Definition 5.3), $\bar{X} = (\bar{\zeta}, U, \bar{\zeta}_\xi, \bar{U}_\xi, \bar{h}) \in \mathcal{G}$ is a relabeling of $X = (\zeta, U, \zeta_\xi, U_\xi, h) \in \mathcal{G}$ if there exists a function ψ such that $\psi(\xi) - \xi \in L^\infty(\mathbb{R})$, $\psi_\xi - 1 \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$, $\psi_\xi \geq 0$, $\lim_{\xi\to-\infty}(\psi(\xi) - \xi) = 0$ such that $\bar{y} = y \circ \psi$ and $\bar{U} = U \circ \psi$ (recall that $y = \zeta + \xi$ and $\bar{y} = \bar{\zeta} + \bar{\xi}$). However, this is not an equivalence relation in the dissipative case, basically because ψ^{-1} is either not well-defined or sufficiently regular.

Section 6 discusses the transformation from Lagrangian to Eulerian variables. A key observation is that Eulerian variable u corresponds to a particular relabeling, namely where $y(\xi) = \xi$. The mapping introduces a metric d_{H^1}, defined indirectly. The existence of a metric is new. In terms of this metric the main result, Theorem 6.6, reads as follows:

There exists a semigroup T_t of weak dissipative solutions of the Camassa–Holm equation, that is, for any initial data u_0 in H^1, $u(t,x) = T_t(u_0)$ is a weak solution of (2.1). *The semigroup T_t is continuous with respect to the metric d_{H^1} on bounded sets of H^1, that is, for any $M > 0$ and any sequence $u_n \in H^1$ such that $\|u_n\|_{H^1} \leq M$, we have $\lim_{n\to\infty} d_{H^1}(u_n, u) = 0$ implies $\lim_{n\to\infty} d_{H^1}(T_t(u_n), T_t(u)) = 0$.*

Furthermore, in Proposition 6.7 we show the following entropy condition. Given any initial data $u_0 \in H^1$, the dissipative solution satisfies

$$u_x \leq \frac{2}{t} + \sqrt{2}\,\|u_0\|_{H^1} \tag{1.11}$$

for almost every x and all $t \geq 0$.

In Section 7 this metric is related to the standard metrics in H^1 and L^∞. Two results are proved: In Proposition 7.1 we prove that for $u_n \in H^1$ such that if $\|u_n - u\|_{H^1} \to 0$, then $d_{H^1}(u_n, u) \to 0$. In Proposition 7.2 the roles are interchanged: For given $u_n, u \in H^1$ such that $d_{H^1}(u_n, u) \to 0$, we show that $\|u_n - u\|_{L^\infty} \to 0$. These results are new.

Global dissipative solutions of a more general class of equations were derived by Coclite, Holden, and Karlsen [7, 8], improving upon [25, 26]. In their approach the solution was obtained by first regularizing the equation by adding a small diffusion term ϵu_{xx} to the equation, and subsequently analyzing the vanishing viscosity limit $\epsilon \to 0$. In this paper we take a completely different look at dissipative solutions by analyzing in great detail what happens as collisions in a particle, or Lagrangian formulation, applying methods that previously have been employed for conservative solutions [22]. Difference schemes that converge to dissipative solutions can be found in [9, 10], see also [18].

2. Existence of solutions

In [22], we showed that the Camassa–Holm equation formally is equivalent to a system of ordinary differential equations corresponding to a transformation from Eulerian to Lagrangian variables. We recall the reformulation here. Observe first that the equation (1.2) can be rewritten as the following system

$$u_t + uu_x + P_x = 0, \tag{2.1a}$$

$$P - P_{xx} = u^2 + \frac{1}{2}u_x^2. \tag{2.1b}$$

We have an explicit expression for P

$$P(t,x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-z|}\big(u^2 + \frac{1}{2}u_x^2\big)(t,z)\,dz. \tag{2.2}$$

For a smooth solution u we can reformulate (2.1) as

$$(u^2+u_x^2)_t+(u(u^2+u_x^2))_x=(u^3-2Pu)_x. \tag{2.3}$$

Define the characteristics $y(t,\xi)$ as the solutions of

$$y_t(t,\xi)=u(t,y(t,\xi)) \tag{2.4}$$

for a given $y(0,\xi)$. The Lagrangian velocity is given by $U(t,\xi)=u(t,y(t,\xi))$. Then we find

$$U_t(t,\xi)=u_t(t,y)+y_t(t,\xi)u_x(t,y)=-P_x\circ y\,(t,\xi), \tag{2.5}$$

using (2.1a). From (2.2) we infer (at one step changing variables $z=y(t,\eta)$)

$$\begin{aligned}P_x\circ y\,(t,\xi)&=-\frac12\int_{\mathbb{R}}\operatorname{sgn}(y(t,\xi)-z)e^{-|y(t,\xi)-z|}(u^2(t,z)+\frac12u_x^2(t,z))\,dz\\&=-\frac12\int_{\mathbb{R}}\operatorname{sgn}(y(t,\xi)-y(t,\eta))e^{-|y(t,\xi)-y(t,\eta)|}\\&\qquad\qquad\times\Big(u^2(t,y(t,\eta))+\frac12u_x^2(t,y(t,\eta))\Big)y_\xi(t,\eta)\,d\eta\\&=-\frac14\int_{\mathbb{R}}\operatorname{sgn}(y(\xi)-y(\eta))\exp(-|y(\xi)-y(\eta)|)\big(U^2y_\xi+h\big)(\eta)\,d\eta\end{aligned}$$

where the t variable has been dropped to simplify the notation, and where we have introduced

$$h(t,\xi)=(u^2+u_x^2)\circ y\,y_\xi. \tag{2.6}$$

Later we will prove that y is an increasing function for any fixed time t. If, for the moment, we take this for granted, then $P_x\circ y$ is equivalent to Q where

$$Q(t,\xi)=-\frac14\int_{\mathbb{R}}\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta, \tag{2.7}$$

which shows that (2.5) rewrites to

$$U_t=-Q. \tag{2.8}$$

To derive an equation for h_t we first introduce the cumulative Lagrangian energy

$$H(t,\xi)=\int_{-\infty}^{y(t,\xi)}(u^2+u_x^2)\,dx,\qquad H_\xi=h, \tag{2.9}$$

which implies

$$\begin{aligned}h_t=H_{\xi t}&=\frac{d}{d\xi}\Big(\frac{d}{dt}\int_{-\infty}^{y(t,\xi)}(u^2+u_x^2)\,dx\Big)\\&=\frac{d}{d\xi}\Big((u^2+u_x^2)\circ y\,y_t+\int_{-\infty}^{y(t,\xi)}(u^2+u_x^2)_t\,dx\Big)\\&=\frac{d}{d\xi}\Big((u^2+u_x^2)\circ y\,y_t+\int_{-\infty}^{y(t,\xi)}\big((u^3-2Pu)_x-(u(u^2+u_x^2))_x\big)\,dx\Big)\\&=\frac{d}{d\xi}\Big((u^2+u_x^2)\circ y\,y_t+\big[(u^3-2Pu)-u(u^2+u_x^2)\big]_{-\infty}^{y(t,\xi)}\Big)\\&=\frac{d}{d\xi}(u^3-2Pu)\circ y=\frac{d}{d\xi}(U^3-2P\circ y\,U)\\&=-2QUy_\xi+(3U^2-2P)U_\xi\end{aligned} \tag{2.10}$$

where we have used (2.3).

To summarize, we have obtained the following system

$$\begin{cases} y_t = U, \\ U_t = -Q, \\ h_t = -2QUy_\xi + (3U^2 - 2P)U_\xi, \end{cases} \tag{2.11}$$

where $Q(t,\xi)$ is given by (2.7), and

$$P(t,\xi) = \frac{1}{4}\int_{\mathbb{R}} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi + h\big)(\eta)\,d\eta. \tag{2.12}$$

After differentiating the first two equations in (2.11), we obtain

$$\begin{cases} y_{\xi t} = U_\xi, \\ U_{\xi t} = \dfrac{1}{2}h + \left(\dfrac{1}{2}U^2 - P\right)y_\xi, \\ h_t = -2Q\,Uy_\xi + \big(3U^2 - 2P\big)\,U_\xi, \end{cases} \tag{2.13}$$

that we can rewrite in compact form as

$$Z_t = F(X)Z \tag{2.14}$$

where X and Z are given by $X = (y, U, h)$ and $Z = (y_\xi, U_\xi, h)$ and the definition of the 3×3 matrix $F(X)$ follows from (2.13). From (2.14), one can see that the system (2.13) is semilinear with respect to Z. In [22], it is proven that the system (2.11) is a well-posed system of ordinary differential equations in a Banach space, and in this way we obtain that the existence, uniqueness and stability of solutions of this system. When going back to the Eulerian variables, we have to take into account a new variable μ which corresponds to the energy density and whose absolutely continuous part coincides with $(u^2 + u_x^2)\,dx$. We have that for a set of times which can only have zero measure, μ admits singular parts, which means that a strictly positive amount of energy is concentrated on a null set. This happens precisely when particles collide, that is, when $y_\xi(t,\xi) = 0$ for ξ in an interval of strictly positive length. Intuitively, if we allow particles to *rebound* with a strength which depends on the amount of energy transported by the colliding particles, we obtain the conservative solutions. If, at collision, we reset the energy transported by the colliding particles to zero, the particles will not rebound and remain stuck; some energy will be lost and the solutions we obtain are dissipative. The conservative solutions are obtained by solving (2.11) and (2.13).

However, we here choose a different approach, as it is also possible to obtain the dissipative solutions by modifying (2.13) in the following way. Let $\tau(\xi)$ be the first time when $y_\xi(t,\xi)$ vanishes, i.e.,

$$\tau(\xi) = \sup\{t \in \mathbb{R}^+ \mid y_\xi(t',\xi) > 0 \text{ for all } 0 \le t' < t\}. \tag{2.15}$$

The dissipative solutions are obtained by imposing that

$$y_\xi(t,\xi) = 0 \text{ for all } t \ge \tau(\xi).$$

As explained earlier, the energy transported by colliding particles is reset to zero, and we require that $h(t,\xi) = 0$ for all $t > \tau(\xi)$. Thus, the expressions (2.7) and (2.12) for Q and P become

$$Q(t,\xi) = -\frac{1}{4}\int_{\tau(\eta)>t} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi + h\big)(\eta)\,d\eta, \tag{2.16}$$

and

$$P(t,\xi) = \frac{1}{4}\int_{\tau(\eta)>t} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi + h\big)(\eta)\,d\eta, \tag{2.17}$$

respectively. In this section we will prove the existence of solutions $X = (y, U, h)$ of

$$y_t = U, \quad U_t = -Q, \tag{2.18}$$

and

$$\begin{cases} y_{\xi t} = \chi_{\{\tau(\xi)>t\}} U_\xi, \\ U_{\xi t} = \chi_{\{\tau(\xi)>t\}} \big(\frac{1}{2}h + \left(\frac{1}{2}U^2 - P\right) y_\xi\big), \\ h_t = \chi_{\{\tau(\xi)>t\}} \big(-2Q\, U y_\xi + \left(3U^2 - 2P\right) U_\xi\big), \end{cases} \tag{2.19}$$

for Q and P given by (2.7) and (2.17), respectively, where τ is defined by (2.15). The system (2.19) can be rewritten in the compact form

$$Z_t = \chi_{\{\tau(\xi)>t\}} F(X) Z \tag{2.20}$$

where $Z = (y_\xi, U_\xi, h)$. Going back to the original variable u, we obtain the dissipative solution. However, this requires careful analysis. Note that in (2.19), we do not reset $h(t,\xi)$ to zero for $t \geq \tau(\xi)$ because it simplifies the analysis. The behavior of the system remains unchanged because in the new definition of P and Q given in (2.16) and (2.17) the regions where $t \geq \tau(\xi)$ are excluded from the domain of integration.

Notation: To ease the presentation we introduce the following notation for the Banach spaces we will often use. Let

$$E = L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$$

and

$$\begin{aligned} W &= L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^1(\mathbb{R}), \\ \bar{W} &= E \cap E \cap L^1(\mathbb{R}), \\ V &= L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}) \times L^1(\mathbb{R}), \\ \bar{V} &= L^\infty(\mathbb{R}) \times E \times E \times E \times L^1(\mathbb{R}). \end{aligned}$$

For subdomains Ω of $\mathbb{R}$, we denote

$$E(\Omega) = L^2(\Omega) \cap L^\infty(\Omega)$$

and similarly, $W(\Omega)$, $\bar{W}(\Omega)$, $V(\Omega)$ and $\bar{V}(\Omega)$. For any function $f \in C([0,T], B)$ for $T \geq 0$ and B a normed space, we denote

$$\|f\|_{L^1_T B} = \int_0^T \|f(t,\cdot)\|_B \, dt \text{ and } \|f\|_{L^\infty_T B} = \sup_{t\in[0,T]} \|f(t,\cdot)\|_B \,.$$

In the case $B = L^p(\mathbb{R})$, we write $\|f\|_{L^1_T B} = \|f\|_{L^1_T L^p_{\mathbb{R}}}$. For the existence results, the function g defined below does not play a particular role; nonetheless it will be discussed now as it will play an important role in the definition of the semigroup metric in the following sections.

Definition 2.1. *For $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^2$, we define*

$$\begin{aligned} g_1(\boldsymbol{x}) &= |x_4| + 2(1 + x_2^2) x_3, \\ g_2(\boldsymbol{x}) &= x_3 + x_5, \end{aligned}$$

and

$$g(\boldsymbol{x}) = \begin{cases} \min(g_1(\boldsymbol{x}), g_2(\boldsymbol{x})) & \text{if } x_4 \leq 0, \\ g_2(\boldsymbol{x}) & \text{if } x_4 > 0. \end{cases} \tag{2.21}$$

Equivalently, we have

$$g(\boldsymbol{x}) = \begin{cases} |x_4| + 2(1+x_2^2)x_3, & \text{if } \boldsymbol{x} \in \Omega, \\ x_3 + x_5, & \text{otherwise,} \end{cases}$$

where Ω *is the following subset of* $\mathbb{R}^5$

$$\Omega = \{\boldsymbol{x} \in \mathbb{R}^5 \mid |x_4| + 2(1+x_2^2)x_3 \le x_3 + x_5 \text{ and } x_4 \le 0\}, \tag{2.22}$$

see Figure 3.

We introduce the set $\mathcal{G}$ as follows.

Definition 2.2. *The set* $\mathcal{G}$ *consists of all triplet* (ζ, U, h) *such that*

$$X = (\zeta, U, \zeta_\xi, U_\xi, h) \in \bar{V}, \tag{2.23a}$$

$$g(y, U, y_\xi, U_\xi, h) - 1 \in E, \tag{2.23b}$$

$$y_\xi \ge 0, \quad h \ge 0, \text{ almost everywhere,} \tag{2.23c}$$

$$\lim_{\xi \to -\infty} \zeta(\xi) = 0, \tag{2.23d}$$

$$\frac{1}{y_\xi + h} \in L^\infty(\mathbb{R}), \tag{2.23e}$$

$$y_\xi h = y_\xi^2 U^2 + U_\xi^2 \text{ almost everywhere,} \tag{2.23f}$$

where we denote $y(\xi) = \zeta(\xi) + \xi$.

To simplify the notation, we will denote by X both (ζ, U, h) and $(\zeta, U, \zeta_\xi, U_\xi, h)$ and it should be clear from the context which of these functions X refers to. We want to prove that, for initial data in $X_0 \in \mathcal{G}$, there exists a solution $X(t)$ of (2.18) and (2.19). For this purpose we introduce the following system which follows directly from (2.18) and (2.19) after making the identification $q \leftrightarrow y_\xi$, $w \leftrightarrow U_\xi$, namely

$$\zeta_t = U \text{ and } U_t = -Q, \tag{2.24}$$

$$\begin{cases} q_t = \chi_{\{\tau(\xi)>t\}} w, \\ w_t = \chi_{\{\tau(\xi)>t\}} (\frac{1}{2}h + \left(\frac{1}{2}U^2 - P\right) q), \\ h_t = \chi_{\{\tau(\xi)>t\}} (-2QUq + \left(3U^2 - 2P\right) w), \end{cases} \tag{2.25}$$

where Q and P are given by

$$Q(t,\xi) = -\frac{1}{4} \int_{\tau(\eta)>t} \operatorname{sgn}(\xi - \eta) \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi) - y(\eta))\big)\big(U^2 q + h\big)(\eta)\, d\eta, \tag{2.26}$$

and

$$P(t,\xi) = \frac{1}{4} \int_{\tau(\eta)>t} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi) - y(\eta))\big)\big(U^2 q + h\big)(\eta)\, d\eta. \tag{2.27}$$

The definition of τ given by (2.15) (after replacing y_ξ by the corresponding variable q) is not appropriate for $q \in C([0,T], L^\infty(\mathbb{R}))$, and, in addition, it is not clear from this definition whether τ is measurable or not. That is why we replace this definition by the following one. Let $\{t_i\}$ be a dense countable subset of $[0,T]$. Define

$$A_t = \bigcup_{n \ge 1} \bigcap_{t_i \le t} \Big\{\xi \in \mathbb{R} \mid q(t,\xi) > \frac{1}{n}\Big\}.$$

The sets A_t are measurable for all t, and we have $A_{t'} \subset A_t$ for $t \leq t'$. We consider a dyadic partition of the interval $[0,T]$ (that is, for each n, we consider the set $\{2^{-n}iT\}_{i=0}^{2^n}$) and set

$$\tau^n(\xi) = \sum_{i=0}^{2^n} \frac{iT}{2^n}\chi_{i,n}(\xi)$$

where $\chi_{i,n}$ is the indicator function of the set $A_{2^{-n}iT} \setminus A_{2^{-n}(i+1)T}$. The function τ^n is by construction measurable. One can check that $\tau^n(\xi)$ is increasing with respect to n, it is also bounded by T. Hence, we can define

$$\tau(\xi) = \lim_{n\to\infty} \tau^n(\xi),$$

and τ is a measurable function. The next lemma gives the main property of τ.

Lemma 2.3. *If, for every $\xi \in \mathbb{R}$, $q(t,\xi)$ is positive and continuous with respect to time, then*

$$\tau(\xi) = \sup\{t \in \mathbb{R}^+ \mid q(t',\xi) > 0 \text{ for all } 0 \leq t' < t\}, \tag{2.28}$$

that is, we retrieve the definition (2.15).

Proof. We denote by $\bar\tau(\xi)$ the right-hand side of (2.28), and we want to prove that $\bar\tau = \tau$. We claim that

$$\text{for all } t < \bar\tau(\xi), \text{ we have } \xi \in A_t, \text{ and for all } t \geq \bar\tau(\xi), \text{ we have } \xi \notin A_t. \tag{2.29}$$

If $t < \bar\tau(\xi)$, then $\inf_{t'\in[0,t]} q(t',\xi) > 0$ because q is continuous in time and positive. Hence, there exists an n such that $\inf_{t'\in[0,t]} q(t',\xi) > \frac{1}{n}$ and $\xi \in \bigcap_{t_i\leq t}\left\{\xi \in \mathbb{R} \mid q(t_i,\xi) > \frac{1}{n}\right\}$ so that $\xi \in A_t$. If $t \geq \bar\tau(\xi)$, then there exists a sequence $t_{i(k)}$ of elements in the dense family $\{t_i\}$ of $[0,T]$ such that $t_{i(k)} \leq \bar\tau \leq t$ and $\lim_{k\to\infty} t_{i(k)} = \bar\tau$. Since $q(t,\xi)$ is continuous, $\lim_{k\to\infty} q(t_{i(k)},\xi) = q(\bar\tau(\xi),\xi) = 0$ and for any integer $n > 0$, there exists a k such $q(t_{i(k)},\xi) \leq \frac{1}{n}$ and $t_{i(k)} \leq t$. Hence, for any $n > 0$, $\xi \notin \bigcap_{t_i\leq t}\left\{\xi \in \mathbb{R} \mid q(t_i,\xi) > \frac{1}{n}\right\}$ and therefore $\xi \notin A_t$. When $\bar\tau(\xi) > 0$, for any $n > 0$, there exists $0 \leq i \leq 2^n - 1$ such that $2^{-n}iT < \bar\tau \leq 2^{-n}(i+1)T$. From (2.29), we infer that $\xi \in A_{2^{-n}iT} \setminus A_{2^{-n}(i+1)T}$. Hence, $\tau^n(\xi) = 2^{-n}iT$, so that

$$\bar\tau(\xi) - \frac{T}{2^n} \leq \tau^n(\xi) \leq \bar\tau(\xi) + \frac{T}{2^n}.$$

Letting n tend to infinity, we conclude that $\tau(\xi) = \bar\tau(\xi)$. If $\bar\tau(\xi) = 0$, then $\xi \notin A_t$ for all $t \geq 0$ and $\tau^n(\xi) = 0$ for all n. Hence, $\tau(\xi) = \bar\tau(\xi) = 0$. ◻

We denote generically (ζ, U, q, w, h) by X and the triplet (q, w, h) by Z. The existence of solutions is proved by a contraction argument in the Banach space $\bar V$. We define the mapping $\mathcal{P}\colon X \in C([0,T],\bar V) \mapsto \tilde X \in C([0,T],\bar V)$ as follows. Given X, we compute (P,Q) as defined in (2.26) and (2.27). Then, $\tilde X = \mathcal{P}(X)$ is given for each $\xi \in \mathbb{R}$, by the solutions of

$$\tilde U_t(t,\xi) = Q(t,\xi),\ \tilde\zeta_t(t,\xi) = \tilde U(t,\xi) \tag{2.30}$$

and, for $t \leq \tilde\tau(\xi)$, as the solution of the system of ordinary differential equations

$$\begin{cases} \tilde q_t(t,\xi) = \tilde w(t,\xi), \\ \tilde w_t(t,\xi) = \dfrac{1}{2}\tilde h(t,\xi) + \left(\dfrac{1}{2}\tilde U^2 - P\right)(t,\xi)\tilde q(t,\xi), \\ \tilde h_t(t,\xi) = -2(Q\tilde U)(t,\xi)\tilde q(t,\xi) + \left(3\tilde U^2 - 2P\right)(t,\xi)\tilde w(t,\xi), \end{cases} \tag{2.31}$$

with initial data $(\zeta_0, U_0, q_0, w_0, h_0) = (\zeta_0, U_0, \zeta_{0,\xi}, U_{0,\xi}, h_0)$ for a given $X_0 = (\zeta_0, U_0, h_0)$ in $\mathcal{G}$. For $t > \tilde\tau$, we set $\tilde Z(t,\xi) = \tilde Z(\tau(\xi),\xi)$. In a compact form we may write

$$\tilde Z_t(t,\xi) = \chi_{\{\tilde\tau(\xi)>t\}}(\xi)F(X,\tilde U)\tilde Z(t,\xi), \tag{2.32}$$

with a slight abuse of the notation of F when compared to (2.20). Note that, in this definition, we do not reset the energy density $\tilde{h}$ to zero after collision but keep the value it reached just before the collision. By direct integrations, from (2.30), we first obtain $\tilde{U}$ as Q is given and then $\tilde{\zeta}$. The equation $\tilde{Z}_t = F(X, \tilde{U})\tilde{Z}$ is linear and therefore it admits a unique global solution in $[0, T]$. Hence, we can define $\tilde{Z}(t, \xi)$ for $t \in [0, \tau(\xi)]$ and for $t \geq \tau(\xi)$, $\tilde{Z}(t, \xi) = \tilde{Z}(\tau(\xi), \xi)$. The mapping $\mathcal{P}$ is thus well-defined.

We have identified q with y_ξ. However, y_ξ does not decay at infinity and y_ξ does not belong to $L^2(\mathbb{R})$ but $y_\xi - 1 = \zeta_\xi$ belongs to $L^2(\mathbb{R})$. That is why we introduce the new variable $v = q - 1$. Let B_M denote the following subset of $\bar{V}$:

(2.33)
$$B_M = \big\{X = (\zeta, U, v, w, h) \in \bar{V} \mid \|X\|_V + \|g(X) - 1\|_E + \left\|\frac{1}{q+h}\right\|_{L^\infty_T L^\infty_{\mathbb{R}}} \leq M\big\}.$$

In the definition of B_M, we slightly abused the notation by denoting $g(y, U, q, w, h)$ by $g(X)$, and we will continue to do so in the remaining text.

The following lemma contains the estimate we will need for P and Q.

Lemma 2.4. *(i) For all $X \in C([0,T], B_M)$, we have*

$$\|Q\|_{L^\infty_T(L^2_{\mathbb{R}} \cap L^\infty_{\mathbb{R}})} + \|P\|_{L^\infty_T(L^2_{\mathbb{R}} \cap L^\infty_{\mathbb{R}})} \leq C(M) \tag{2.34}$$

for a constant $C(M)$ which only depends on M.

(ii) For all $X = (\zeta, U, v, w, h)$ and $\bar{X} = (\bar{\zeta}, \bar{U}, \bar{v}, \bar{w}, \bar{h})$ in $C([0,T], B_M)$, we have

$$\begin{aligned}
&\left\|Q - \bar{Q}\right\|_{L^1_T(L^2_{\mathbb{R}} \cap L^\infty_{\mathbb{R}})} + \left\|P - \bar{P}\right\|_{L^1_T(L^2_{\mathbb{R}} \cap L^\infty_{\mathbb{R}})} \\
&\leq C(M)\bigg(T\Big(\left\|\zeta - \bar{\zeta}\right\|_{L^\infty_T L^\infty_{\mathbb{R}}} + \left\|U - \bar{U}\right\|_{L^\infty_T L^2_{\mathbb{R}}} + \|v - \bar{v}\|_{L^\infty_T L^2_{\mathbb{R}}} + \left\|h - \bar{h}\right\|_{L^\infty_T L^1_{\mathbb{R}}}\Big) \\
&\qquad + \|\tau - \bar{\tau}\|_{L^1_{\mathbb{R}}} + \int_{\mathbb{R}} \left|\int_\tau^{\bar{\tau}} \left(\left|\bar{h}(t,\xi)\right| \chi_{\{\tau < \bar{\tau}\}}(\xi) + |h(t,\xi)|\, \chi_{\{\bar{\tau} < \tau\}}(\xi)\right) dt\right| d\xi\bigg)
\end{aligned}$$

for a constant $C(M)$ which only depends on M.

Proof. We establish only the estimates for Q as the estimates for P can be obtained in exactly the same way. Let $f(\xi) = \chi_{\{\xi > 0\}}(\xi) e^{-\xi}$. We can write Q as $Q = Q_1 + Q_2$ with

$$Q_1(t, \xi) = -\frac{e^{-\zeta(\xi)}}{4}(f \star [\chi_{\{\tau(\xi) > t\}} r])(t, \xi) \tag{2.35}$$

where

$$r = r_1 + r_2, \quad r_1 = e^{\zeta} U^2 (1 + v), \text{ and } r_2 = e^{\zeta} h, \tag{2.36}$$

and a similar expression for Q_2, see [22]. We recall Young's inequalities for convolution product of L^p functions: For any $f \in L^p$ and $g \in L^q$ with $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1 \geq 0$, we have

$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}. \tag{2.37}$$

This implies

$$\begin{aligned}
\|Q_1(t, \cdot)\|_E &\leq \frac{1}{4}\left\|e^{-\zeta}\right\|_{L^\infty_{\mathbb{R}}} \left(\left\|f \star [\chi_{\{\tau(\xi) > t\}} r_1]\right\|_E + \left\|f \star [\chi_{\{\tau(\xi) > t\}} r_2]\right\|_E\right) \\
&\leq C(M)\big((\|f\|_{L^1_{\mathbb{R}}} + \|f\|_{L^2_{\mathbb{R}}}) \|r_1\|_{L^2_{\mathbb{R}}} + (\|f\|_{L^1_{\mathbb{R}}} + \|f\|_{L^\infty_{\mathbb{R}}}) \|r_2\|_{L^1_{\mathbb{R}}}\big) \\
&\leq C(M)
\end{aligned}$$

where we have denoted by $C(M)$ a generic constant that only depends on M. We will continue to denote by $C(M)$ such generic constants. We have

(2.38)
$$\begin{aligned}f\star\big([\chi_{\{\tau(\xi)>t\}}r]-[\chi_{\{\bar\tau(\xi)>t\}}\bar r]\big)&=f\star[\chi_{\{\tau(\xi)>t\}}\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}(r-\bar r)]\\&\quad+f\star[(\chi_{\{\tau(\xi)>t\}}-\chi_{\{\bar\tau(\xi)>t\}})\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}\bar r]\\&\quad+f\star[\chi_{\{\bar\tau(\xi)>t\}}\chi_{\{\tau(\xi)\geq\bar\tau(\xi)\}}(r-\bar r)]\\&\quad+f\star[(\chi_{\{\tau(\xi)>t\}}-\chi_{\{\bar\tau(\xi)>t\}})\chi_{\{\tau(\xi)\geq\bar\tau(\xi)\}}r].\end{aligned}$$

We estimate each of these terms. The first and the third ones are similar, that is why we treat only the first one. After applying Young's inequalities, we obtain

$$\begin{aligned}\left\|f\star[\chi_{\{\tau(\xi)>t\}}\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}(r_1-\bar r_1)]\right\|_E&\leq(\|f\|_{L^1_{\mathbb{R}}}+\|f\|_{L^2_{\mathbb{R}}})\,\|r_1-\bar r_1\|_{L^2_{\mathbb{R}}}\\&\leq C(M)\big(\left\|\zeta-\bar\zeta\right\|_{L^\infty_{\mathbb{R}}}+\left\|U-\bar U\right\|_{L^2_{\mathbb{R}}}\\&\qquad+\|v-\bar v\|_{L^2_{\mathbb{R}}}\big),\end{aligned}$$

and

$$\begin{aligned}\left\|f\star[\chi_{\{\tau(\xi)>t\}}\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}(r_2-\bar r_2)]\right\|_E&\leq(\|f\|_{L^2_{\mathbb{R}}}+\|f\|_{L^\infty_{\mathbb{R}}})\,\|r_2-\bar r_2\|_{L^1_{\mathbb{R}}}\\&\leq C(M)\big(\left\|\zeta-\bar\zeta\right\|_{L^\infty_{\mathbb{R}}}+\left\|h-\bar h\right\|_{L^1_{\mathbb{R}}}\big).\end{aligned}$$

As far as the second term in (2.38) is concerned (the fourth term will be treated similarly), we have $(\chi_{\{\tau(\xi)>t\}}-\chi_{\{\bar\tau(\xi)>t\}})\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}=-\chi_{\{\tau(\xi)\leq t<\bar\tau(\xi)\}}$, and therefore

$$\begin{aligned}&\left\|f\star[(\chi_{\{\tau(\xi)>t\}}-\chi_{\{\bar\tau(\xi)>t\}})\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}\bar r_1]\right\|_{L^\infty_TE}\\&\qquad\leq(\|f\|_{L^2_{\mathbb{R}}}+\|f\|_{L^\infty_{\mathbb{R}}})\left\|\chi_{\{\tau(\xi)\leq t<\bar\tau(\xi)\}}e^{\bar\zeta}\bar U^2(1+\bar v)\right\|_{L^1_TL^1_{\mathbb{R}}}\\&\qquad\leq C(M)\int_{\mathbb{R}\times\mathbb{R}}\chi_{\{\tau(\xi)\leq t<\bar\tau(\xi)\}}(\xi)\,d\xi dt\leq C(M)\,\|\tau-\bar\tau\|_{L^1_{\mathbb{R}}},\end{aligned}$$

after applying Fubini's theorem. Similarly, we get

$$\begin{aligned}&\left\|f\star[(\chi_{\{\tau(\xi)>t\}}-\chi_{\{\bar\tau(\xi)>t\}})\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}\bar h]\right\|_{L^\infty_TE}\\&\qquad\leq(\|f\|_{L^2_{\mathbb{R}}}+\|f\|_{L^\infty_{\mathbb{R}}})\left\|\chi_{\{\tau(\xi)\leq t<\bar\tau(\xi)\}}e^{\bar\zeta}\bar h\right\|_{L^1_TL^1_{\mathbb{R}}}\\&\qquad\leq C(M)\int_{\mathbb{R}}\int_{\tau(\xi)}^{\bar\tau(\xi)}\left|\bar h(t,\xi)\right|\chi_{\{\tau(\xi)<\bar\tau(\xi)\}}(\xi)\,dtd\xi.\end{aligned}$$

□

We introduce the sets

$$\mathcal{A}_\varepsilon=\big\{\xi\in\mathbb{R}\mid 0<q_0(\xi)\leq\varepsilon\text{ and }-\varepsilon\leq w_0(\xi)<0\big\}\tag{2.39}$$

and

$$\mathcal{K}_\gamma=\big\{\xi\in\mathbb{R}\mid h_0(\xi)\geq\gamma\big\}.\tag{2.40}$$

As the next lemma shows, the domains $\mathcal{A}_\varepsilon$ and $\mathcal{K}_\gamma$ are domains for which we know a priori that the particles they contain are going to collide. This a priori control is essential when proving existence and stability. Since $\cap_{n>0}\mathcal{A}_{\frac{1}{n}}=\emptyset$, $\lim_{\varepsilon\to0}\operatorname{meas}(\mathcal{A}_\varepsilon)=0$. We have

$$\operatorname{meas}(K_\gamma)\leq\frac{1}{\gamma}\int_{\mathbb{R}}h_0(\xi)\,d\xi\leq\frac{1}{\gamma}\|h_0\|_{L^1_{\mathbb{R}}}$$

and therefore $\lim_{\gamma\to\infty}\operatorname{meas}(\mathcal{K}_\gamma)=0$. The structure of (2.31), and, in particular, the semilinearity plays a crucial role as the following lemma shows. We need to study two solutions X and $\tilde{X}$, and we denote the corresponding quantities associated with $\tilde{X}$ with a tilde, that is, $\tilde{\tau}$, etc.

Lemma 2.5. *Given $X_0 \in B_{M_0}$ for some constant M_0, given $X = (\zeta, U, v, w, h) \in C([0,T], B_M)$, we denote $\tilde{X} = (\tilde{\zeta}, \tilde{U}, \tilde{v}, \tilde{w}, \tilde{h})$ the solution of (2.30), (2.31), that is, $\tilde{X} = \mathcal{P}(X)$ with initial data X_0. Let $\bar{M} = \|P\|_{L^\infty_T L^\infty_\mathbb{R}} + \|Q\|_{L^\infty_T L^\infty_\mathbb{R}} + M_0$. The following holds:*
(i) For all t and almost all ξ

$$\tilde{q}(t,\xi) \geq 0, \quad \tilde{h}(t,\xi) \geq 0, \tag{2.41}$$

and

$$\tilde{q}\tilde{h} = \tilde{U}^2\tilde{q}^2 + \tilde{w}^2. \tag{2.42}$$

Thus, $\tilde{q}(t,\xi) = 0$ implies $\tilde{w}(t,\xi) = 0$. We recall the notation $\tilde{q} = \tilde{v} + 1$.

(ii) We have

$$\left\| \frac{1}{\tilde{q} + \tilde{h}}(t,\,\cdot\,) \right\|_{L^\infty_\mathbb{R}} \leq \frac{9}{2} e^{CT} \left\| \frac{1}{q_0 + h_0} \right\|_{L^\infty_\mathbb{R}} \tag{2.43}$$

for all $t \in [0,T]$ and a constant C which depends on $\bar{M}$ and T. In particular, $\tilde{q} + \tilde{h}$ remains bounded strictly away from zero.

(iii) There exists an ε depending only on T and $\bar{M}$ such that if $\xi \in \mathcal{A}_\varepsilon$, then $\tilde{X}(t,\xi) \in \Omega$ for all $t \in [0,T]$ (see (2.22) and Figure 3 for the definition of Ω), $\tilde{q}(t,\xi)$ is a decreasing function and $\tilde{w}(t,\xi)$ an increasing function of time, and therefore we have

$$-\varepsilon \leq \tilde{w}(t,\xi) \leq 0 \text{ and } 0 \leq \tilde{q}(t,\xi) \leq \varepsilon \tag{2.44}$$

for all $t \in [0,T]$. In addition, for ε sufficiently small, depending only on $\bar{M}$ and T, we have

$$\mathcal{A}_\varepsilon \subset \big\{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T\big\}. \tag{2.45}$$

(iv) There exists a γ depending only on T and $\bar{M}$ such that if $\xi \in \mathcal{K}_\gamma$, then $\tilde{X}(t,\xi) \in \Omega$ for all $t \in [0,T]$, $\tilde{q}(t,\xi)$ is a decreasing function and $\tilde{w}(t,\xi)$ an increasing function of time, and therefore

$$w_0(\xi) \leq \tilde{w}(t,\xi) \leq 0 \text{ and } 0 \leq \tilde{q}(t,\xi) \leq w_0(\xi).$$

In addition, for γ sufficiently large, depending only on $\bar{M}$ and T, we have

$$\mathcal{K}_\gamma \subset \big\{\xi \in \mathbb{R} \mid 0 \leq \tilde{\tau}(\xi) < T\big\}. \tag{2.46}$$

(v) For any $\varepsilon > 0$ and $\gamma > 0$, there exists $T > 0$ such that

$$\big\{\xi \in \mathbb{R} \mid 0 < \tilde{\tau}(\xi) < T\big\} \subset \mathcal{A}_\varepsilon \cup \mathcal{K}_\gamma. \tag{2.47}$$

Proof. (i) Since $X_0 \in \mathcal{G}$, the equations (2.41), (2.42) and the inequality (2.43) hold for almost every $\xi \in \mathbb{R}$ at $t = 0$. We consider such a ξ and will drop it in the notation. From (2.31), we have, on the one hand,

$$(\tilde{q}\tilde{h})_t = \tilde{q}_t\tilde{h} + \tilde{h}_t\tilde{q} = \tilde{w}\tilde{h} + 3\tilde{U}^2\tilde{w}\tilde{q} - 2Q\tilde{U}\tilde{q}^2 - 2P\tilde{w}\tilde{q},$$

and, on the other hand,

$$\begin{aligned}(\tilde{q}\tilde{U}^2 + \tilde{w}^2)_t &= 2\tilde{q}_t\tilde{q}\tilde{U}^2 + 2\tilde{q}^2\tilde{U}_t\tilde{U} + 2\tilde{w}_t\tilde{w} \\ &= 3\tilde{w}\tilde{U}^2\tilde{q} - 2P\tilde{w}\tilde{q} + \tilde{h}\tilde{w} - 2\tilde{q}^2 Q\tilde{U}.\end{aligned}$$

Thus, $(\tilde{q}\tilde{h} - \tilde{q}^2\tilde{U}^2 - \tilde{w}^2)_t = 0$, and since $\tilde{q}\tilde{w}(0) = (\tilde{q}^2\tilde{U}^2 + \tilde{w}^2)(0)$, we have $\tilde{q}\tilde{h}(t) = (\tilde{q}^2\tilde{U}^2 + \tilde{w}^2)(t)$ for all $t \in [0,T]$. We have proved (2.42). From the definition of τ, we have that $\tilde{q}(t) > 0$ on $[0, \tau(\xi))$ and by definition of $\tilde{q}$, we have $\tilde{q}(t) = 0$ for

$t \geq \tau(\xi)$. Hence, $\tilde{q}(t) \geq 0$ for $t \geq 0$. From (2.42), it follows that, for $t \in [0, \tau)$, $\tilde{h}(t) = \frac{\tilde{U}^2\tilde{q}^2 + \tilde{w}^2}{\tilde{q}}(t)$ and therefore $\tilde{h}(t) \geq 0$. By continuity (with respect to time) of $\tilde{h}$, we have $\tilde{h}(\tau(\xi)) \geq 0$ and, since the variable does not change for $t \geq \tau(\xi)$, we have $\tilde{h}(t) \geq 0$ for all $t \geq 0$.

(ii) We consider a fixed ξ that we ignore in the notation. We denote $\left|\tilde{Z}\right|_2 = (\tilde{q}^2 + \tilde{w}^2 + \tilde{h}^2)^{1/2}$ the Euclidean norm of $\tilde{Z} = (\tilde{q}, \tilde{w}, \tilde{h})$. Since $\tilde{Z}_t = F(X, \tilde{U})\tilde{Z}$, we have

$$\begin{aligned}\frac{d}{dt}|\tilde{Z}|_2^{-2} &= -2|\tilde{Z}|_2^{-4}\tilde{Z} \cdot \frac{d\tilde{Z}}{dt} = -2|\tilde{Z}|_2^{-4}\tilde{Z} \cdot F(X, \tilde{U})\tilde{Z} \\ &\leq \sup_{t\in[0,T]} |F(X(t), \tilde{U}(t))||\tilde{Z}|_2^{-2}.\end{aligned}$$

We have $|F(X(t), \tilde{U}(t))| \leq K(|P(t)| + |Q(t)| + |\tilde{U}(t)|) \leq K(\|P\|_{L_T^\infty L_{\mathbb{R}}^\infty} + \|Q\|_{L_T^\infty L_{\mathbb{R}}^\infty} + \|\tilde{U}\|_{L_T^\infty L_{\mathbb{R}}^\infty})$ for a constant K which depends on the norm chosen for the matrix F. From (2.30), we infer $\|\tilde{U}\|_{L_T^\infty L_{\mathbb{R}}^\infty} \leq M_0 + T\|Q\|_{L_T^\infty L_{\mathbb{R}}^\infty} \leq C(\bar{M}, T)$ for a constant depending only on $\bar{M}$ and T. We denote generically by $C(\bar{M}, T)$ such constant. Hence, $\left|F(X(t), \tilde{U}(t))\right| \leq C(\bar{M}, T)$. Applying Gronwall's lemma, we obtain $|Z(t)|_2^{-2} \leq e^{2C(\bar{M},T)T}|Z(0)|_2^{-2}$. Hence,

$$\frac{1}{|\tilde{q}| + |\tilde{w}| + |\tilde{h}|}(t) \leq \frac{3}{|\tilde{q}| + |\tilde{w}| + |\tilde{h}|}(0)e^{C(\bar{M},T)T}. \tag{2.48}$$

From (2.42), as $\tilde{q}$ and $\tilde{h}$ are positive, we have

$$|\tilde{w}(t)| \leq \sqrt{\tilde{q}(t)\tilde{h}(t)} \leq \frac{1}{2}(\tilde{q}(t) + \tilde{h}(t)),$$

and therefore $(|\tilde{q}| + |\tilde{w}| + |\tilde{h}|)(t) \leq \frac{3}{2}(\tilde{q} + \tilde{h})(t)$. Hence, (2.48) yields

$$\frac{1}{\tilde{q} + \tilde{h}}(t) \leq \frac{9}{2}e^{C(\bar{M},T)T}\frac{1}{q_0 + h_0}.$$

(iii) We claim that there exists a constant $\varepsilon(\bar{M}, T)$ depending only on $\bar{M}$ and T such that, for all $\varepsilon \leq \varepsilon(\bar{M}, T)$, $\xi \in \mathbb{R}$ and $t \in [0, T]$,

$$\tilde{q}(t, \xi) \leq \varepsilon \text{ and } \tilde{w}(t, \xi) = 0 \text{ implies } \tilde{q}(t, \xi) = 0 \tag{2.49}$$

and

$$\tilde{q}(t, \xi) \leq \varepsilon \text{ implies } \tilde{w}_t(t, \xi) \geq 0. \tag{2.50}$$

We consider a fixed $\xi \in \mathbb{R}$ and suppress it in the notation. If $\tilde{w}(t) = 0$, then (2.42) yields $\tilde{q}(t)\tilde{h}(t) = \tilde{q}(t)^2\tilde{U}^2(t)$. Assume that $\tilde{q}(t) \neq 0$, then $\tilde{h}(t) = \tilde{q}(t)\tilde{U}^2(t)$. We have seen in the proof of (ii) that $\|\tilde{U}\|_{L_T^\infty L_{\mathbb{R}}^\infty} \leq C_1(\bar{M}, T)$ for some constant $C_1(\bar{M}, T)$ depending only on $\bar{M}$ and T and therefore $\tilde{h}(t) = \tilde{q}(t)\tilde{U}^2(t) \leq \varepsilon C_1(\bar{M}, T)^2$. From Lemma (ii), we have $\tilde{h}(t) + \tilde{q}(t) \geq (C_2(\bar{M}, T))^{-1}$ for some constant $C_2(\bar{M}, T)$ depending only on $\bar{M}$ and T. Hence, $(C_2(\bar{M}, T))^{-1} \leq \tilde{h}(t) + \tilde{q}(t) \leq C_1(\bar{M}, T)^2\varepsilon$. By choosing $\varepsilon(\bar{M}, T) \leq (2C_1(\bar{M}, T)C_2(\bar{M}, T)^2)^{-1}$, we are led to a contradiction. Hence, $\tilde{q}(t) = 0$, and we have proved (2.49). If $\tilde{q}(t) \leq \varepsilon$, we have

$$\begin{aligned}\tilde{w}_t = \frac{1}{2}(\tilde{h} + \tilde{q}) + (\frac{1}{2}\tilde{U}^2 - P - \frac{1}{2})\tilde{q} &\geq (2C_2(\bar{M}, T))^{-1} - C_3(\bar{M}, T)\tilde{q} \\ &\geq (2C_2(\bar{M}, T))^{-1} - 2C_3(\bar{M}, T)\varepsilon.\end{aligned} \tag{2.51}$$

By choosing $\varepsilon(\bar{M}, T) \leq (4C_2(\bar{M}, T)C_3(\bar{M}, T))^{-1}$, we get $\tilde{w}_t \geq 0$, and we have proved (2.50). For any $\varepsilon \leq \varepsilon(\bar{M}, T)$, we consider a given ξ in $\mathcal{A}_\varepsilon$ and again suppress it in the notation. We define

$$t_0 = \sup\{t \in [0, \tilde{\tau}] \mid \tilde{q}(t') < 2\varepsilon \text{ and } \tilde{w}(t') < 0 \text{ for all } t' \leq t\}.$$

Let us prove that $t_0 = \tilde{\tau}$. Assume the opposite, that is, $t_0 < \tilde{\tau}$. Then, we have either $\tilde{q}(t_0) = 2\varepsilon$ or $\tilde{w}(t_0) = 0$. We have $\tilde{q}_t = \tilde{w} \leq 0$ on $[0, t_0]$ and $\tilde{q}(t)$ is decreasing on this interval. Hence, $\tilde{q}(t_0) \leq \tilde{q}(0) \leq \varepsilon$, and therefore we must have $\tilde{w}(t_0) = 0$. Then, (2.49) implies $\tilde{q}(t_0) = 0$, and therefore $t_0 = \tilde{\tau}$, which contradicts our assumption. From (2.51), we get, for ε sufficiently small,

$$0 = \tilde{w}(\tau) \geq \tilde{w}(0) - (4C_2(\bar{M}, T))^{-1}\tilde{\tau},$$

and therefore $\tilde{\tau} \leq 4\varepsilon C_2(\bar{M}, T)$. By taking ε small enough we can impose $\tilde{\tau} < T$, which proves (2.45). It is clear from (2.50) that $\tilde{w}$ is increasing. Assume that $\tilde{X}(t, \xi)$ leaves Ω for some t. Then, after using item (ii), we get

$$(C_1(\bar{M}, T))^{-1} \leq \tilde{q}(t) + \tilde{h}(t) \leq |\tilde{w}(t)| + 2(1 + \tilde{U}^2(t))\tilde{q}(t) \leq C_2(\bar{M}, t)\varepsilon$$

for two constants $C_1(\bar{M}, T)$ and $C_2(\bar{M}, T)$ depending only on $\bar{M}$ and T and, by taking ε small enough, we are led to a contradiction.

(iv) Let us consider a given ξ in $\mathcal{K}_\gamma$. We are going to determine a lower bound on γ depending only on $\bar{M}$ and T such that the conclusion of the lemma holds. For γ large enough, we have $X_0(\xi) \in \Omega$ as otherwise $g(X_0(\xi)) = q_0 + h_0$ and $M_0 + 1 \geq g(X_0(\xi)) = q_0 + h_0 \geq \gamma$ would lead to a contradiction. Hence, $w_0(\xi) \leq 0$. We now ignore the dependence on ξ in the notation. We define t_0 as

$$t_0 = \sup\{t \in [0, \tilde{\tau}] \mid \tilde{w}(0) \leq \tilde{w}(\bar{t}) \leq 0 \text{ for all } 0 \leq \bar{t} \leq t\}.$$

On $[0, t_0]$, we have $\tilde{q}_t = \tilde{w} \leq 0$ so that $\tilde{q}$ is decreasing and therefore $0 \leq \tilde{q}(t) \leq \tilde{q}(0)$ for $t \in [0, t_0]$. Hence, the functions $\tilde{w}(t)$ and $\tilde{q}(t)$ are bounded on $[0, t_0]$ by a their values at initial time and $|\tilde{w}(t)| \leq M_0$ and $|\tilde{q}(t)| \leq M_0 + 1$ on $[0, t_0]$. From the governing equation for $\tilde{h}$, i.e., $\tilde{h}_t = -2Q\tilde{q} + (3\tilde{U}^2 - 2P)\tilde{w}$, it follows that $\tilde{h}_t \geq -C(\bar{M})$, and we can choose γ large enough so that $h(t) \geq \frac{\gamma}{2}$ on $[0, t_0]$. We can also choose γ large enough so that, on $[0, t_0]$,

$$|\tilde{w}| + 2(1 + \tilde{U}^2)\tilde{q} \leq C(\bar{M}) \leq \frac{\gamma}{2} \leq \tilde{h}(t) + \tilde{q}(t),$$

and therefore $\tilde{X}(t)$ remains in Ω for $t \in [0, t_0]$. The function $\tilde{w}(t)$ is strictly increasing on $[0, t_0]$, indeed we have

$$\tilde{w}_t(t) = \frac{1}{2}\tilde{h}(t) + (\frac{1}{2}U^2(t) - P(t))\tilde{q}(t) \geq \frac{\gamma}{4} - C(\bar{M}) > \frac{\gamma}{8}, \tag{2.52}$$

for γ large enough. Thus we have proved that the conclusions of (iv) hold for $[0, t_0]$, and now we are going to prove that $t_0 = \tilde{\tau}$. From (2.52), we get that $\tilde{w}(t, \xi) \geq \tilde{w}(0, \xi) + \frac{\gamma}{8}t$, and there exists $t_1 \leq 8\gamma^{-1}|\tilde{w}(0, \xi)|\gamma^{-1} = \gamma^{-1}C(\bar{M})$ such that $\tilde{w}(t_1) = 0$. We choose γ large enough so that $t_1 \leq t_0$ (if $t_0 = 0$, we set $t_1 = t_0 = 0$) and $t_1 < T$. From item (i), we get $\tilde{q}(t_1)\tilde{h}(t_1) = \tilde{U}^2(t_1)\tilde{q}^2(t_1)$, so that either $\tilde{q}(t_1) = 0$ or $\frac{\gamma}{2} \leq \tilde{h}(t_1) = \tilde{U}^2(t_1)\tilde{q}(t_1) \leq C(\bar{M})$. By taking γ large enough, we can exclude the latter alternative and therefore $\tilde{q}(t_1) = 0$, and t_1 is a collision time for the particle ξ, i.e., $t_1 = \tilde{\tau}(\xi)$ and $t_1 = t_0 = \tilde{\tau}$. Since $t_1 < T$, we have also proved (2.46). The conclusions of point (iv) hold on $[0, \tilde{\tau}]$, but they also hold on $[\tilde{\tau}, T]$ as $\tilde{q}(t) = \tilde{w}(t) = 0$ on this interval.

(v) Without loss of generality we assume $T \leq 1$. From item (ii), there exists ε' depending only on $\bar{M}$ such that $(\tilde{q}(t, \xi) + \tilde{h}(t, \xi))(2(\tilde{U}^2(t, \xi) + 1))^{-1} \geq \varepsilon'$ for all ξ and $t \in [0, T]$. Let $\bar{\varepsilon} = \min(\varepsilon, \varepsilon')$. We consider a fixed ξ, which we will drop in the notation, such that $\tilde{\tau}(\xi) < T$ and $\xi \in \mathcal{K}_\gamma^c$. After applying Gronwall's lemma on (2.32), we get that $|\tilde{Z}(t)| \leq C(\bar{M}, T)|\tilde{Z}(0)| \leq C(\bar{M}, \gamma)$ for a constant $C(\bar{M}, \gamma)$ depending only on $\bar{M}$ and γ as we assumed that $T \leq 1$. Let us introduce

$$t_0 = \inf\{t \in [0, \tilde{\tau}) \mid 0 < \tilde{q}(\bar{t}) \leq \bar{\varepsilon},\ -\bar{\varepsilon} < \tilde{w}(\bar{t}, \xi) \leq 0 \text{ for all } \bar{t} \in [t, \tilde{\tau})\}.$$

Since $\tilde{q}_{tt}(\tilde{\tau}) = \tilde{w}_t(\tilde{\tau}) = \frac{1}{2}\tilde{h}(\tilde{\tau}) > 0$ and $\tilde{q}_t(\tilde{\tau}) = \tilde{w}(\tilde{\tau}) = \tilde{q}(\tilde{\tau}) = 0$, the definition of t_0 is well-posed when $\tilde{\tau} > 0$, and we have $t_0 < \tilde{\tau}$. Assume that $t_0 > 0$, then $q(t_0) = 0$ or $\tilde{q}(t_0) = \bar{\varepsilon}$ or $\tilde{w}(t_0) = -\bar{\varepsilon}$ or $\tilde{w}(t_0) = 0$. We cannot have $\tilde{q}(t_0) = 0$ by the definition of $\tilde{\tau}$. If $\tilde{q}(t_0) = \bar{\varepsilon}$, then $0 = \tilde{q}(\tilde{\tau}) = \tilde{q}(t_0) + \int_{t_0}^{\tilde{\tau}} \tilde{w}(t)\,dt$ implies $0 \geq \bar{\varepsilon} - \bar{\varepsilon}(\tilde{\tau} - t_0)$ which leads to a contradiction if we choose $T < 1$. If $\tilde{w}(t_0) = -\bar{\varepsilon}$, then the second equation in (2.31) implies $0 \leq -\bar{\varepsilon} + C(\bar{M}, \gamma)(\tilde{\tau} - t_0)$ which leads to a contradiction if T is chosen small enough. At last, if $\tilde{w}(t_0) = 0$ then, by (2.42), $\tilde{q}(t_0)\tilde{U}(t_0)^2 = \tilde{h}(t_0)$ and it implies $\bar{\varepsilon} \geq q(t_0) \geq \frac{\tilde{q}(t_0) + \tilde{h}(t_0)}{\tilde{U}^2(t_0) + 1} = 2\varepsilon'$ which contradicts the definition of $\bar{\varepsilon}$. Hence, $t_0 = 0$ and then $\xi \in \mathcal{A}_{\bar{\varepsilon}} \subset \mathcal{A}_{\varepsilon}$, and we have proved (iv). ☐

For any $X_0 \in \mathcal{G}$, we have, by definition, that $\|X_0\|_{\bar{V}} + \|g(X_0) - 1\|_E + \left\|\frac{1}{q_0 + h_0}\right\|_{L^\infty} < \infty$. The following lemma holds.

Lemma 2.6. *Given $M \geq 0$, there a time $\bar{T}$ and $\bar{M} > 0$ such that, for all $T \leq \bar{T}$ and any initial data $X_0 \in \mathcal{G} \cap B_M$, $\mathcal{P}$ is a mapping from $C([0, T], B_{\bar{M}})$ to $C([0, T], B_{\bar{M}})$.*

Proof. To simplify the notation, we denote generically by $K(M)$ and $C(\bar{M})$ increasing functions of M and $\bar{M}$, respectively. The functions $K(M)$ and $C(\bar{M})$ — that may change from line to line — can always be computed explicitly but in order to ease the notation and because the exact expressions of these functions do not matter in the end, we do not compute their detailed form. Without loss of generality, we assume $T \leq 1$ and that there exists γ such that the conclusions of Lemma 2.5 hold for $t \in [0, T]$. We consider the following sets:

$$\mathcal{B}_1 = \{\xi \in \mathbb{R} \mid 0 \leq h_0(\xi) \leq M\}, \quad \mathcal{B}_2 = \{\xi \in \mathbb{R} \mid M < h_0(\xi) < \gamma\},$$
$$\text{and } \mathcal{B}_3 = \mathcal{K}_\gamma = \{\xi \in \mathbb{R} \mid \gamma \leq h_0(\xi)\}.$$

We have $\operatorname{meas}(\mathcal{B}_2 \cup \mathcal{B}_3) \leq \frac{1}{M}\|h_0\|_{L^1(\mathbb{R})} \leq 1$. Let $X \in C([0, T], B_{\bar{M}})$ for a value of $\bar{M}$ that will be determined at the end as a function of M. We can assume without loss of generality that $\bar{M} \geq M$. Let $\tilde{X} = \mathcal{P}(X)$. From Lemma 2.5, we have

$$\|Q\|_{L^\infty_T E} \leq C(\bar{M}) \text{ and } \|P\|_{L^\infty_T E} \leq C(\bar{M}). \tag{2.53}$$

Since $\tilde{U}_t = -Q$, we get

$$\|\tilde{U}\|_{L^\infty_T E} \leq \|U_0\|_E + T\|Q\|_{L^\infty_T E} \leq M + C(\bar{M})T, \tag{2.54}$$

and, since $\tilde{\zeta}_t = \tilde{U}$, we get

$$\|\tilde{\zeta}\|_{L^\infty_T L^\infty_{\mathbb{R}}} \leq M + \|\tilde{U}\|_{L^\infty_T L^\infty_{\mathbb{R}}} T \leq M + C(\bar{M})T. \tag{2.55}$$

On $\mathcal{B}_1$, we have $\|h_0\|_{L^2(\mathcal{B}_1)} \leq M\|h_0\|_{L^1(\mathcal{B}_1)}$ and therefore $\|h_0\|_{E(\mathcal{B}_1)} \leq K(M)$. From (2.31), by the Minkowsky inequality for integrals (see [23, Theorem 5.60]), we get

$$\|\tilde{v}(t,\cdot)\| \leq \|v_0\| + \int_0^t \|\tilde{w}(t',\cdot)\|\,dt', \tag{2.56a}$$

$$\begin{aligned}\|\tilde{w}(t,\cdot)\| \leq \|w_0\| &+ \|\frac{1}{2}\tilde{U}^2 - P\|T \\ &+ \int_0^t \big(\frac{1}{2}\|\tilde{h}(t',\cdot)\| + \|\frac{1}{2}\tilde{U}^2 - P\|_{L^\infty}\|\tilde{v}(t',\cdot)\|\big)\,dt',\end{aligned} \tag{2.56b}$$

$$\|\tilde{h}(t,\cdot)\| \leq \|h_0\| + \|2Q\tilde{U}\|T \tag{2.56c}$$

$$+\int_0^t \left(\|2Q\tilde{U}\|_{L^\infty}\|\tilde{v}(t',\,\cdot\,)\| + \|3\tilde{U}^2 - 2P\|_{L^\infty}\|\tilde{w}(t',\,\cdot\,)\|\right) dt',$$

where $\|\cdot\|$ stands for either the L^1, L^2, or the L^∞ norms. On $\mathcal{B}_1$, the inequalities (2.56) imply

$$\|(\tilde{v},\tilde{w},\tilde{h})(t,\,\cdot\,)\|_{E(\mathcal{B}_1)} \leq K(M) + C(\bar{M})T + C(\bar{M})\int_0^t \|(\tilde{v},\tilde{w},\tilde{h})(t',\,\cdot\,)\|_{E(\mathcal{B}_1)}\,dt'$$

and, by Gronwall's lemma,

$$\|(\tilde{v},\tilde{w},\tilde{h})\|_{L_T^\infty E(\mathcal{B}_1)} \leq (K(M) + C(\bar{M})T)e^{C(\bar{M})T}. \tag{2.57}$$

From (2.42), we have $\tilde{h} = \tilde{U}^2(1+\tilde{v})^2 - \tilde{h}\tilde{v} + \tilde{w}^2$. Hence,

$$\begin{aligned}
\|\tilde{h}\|_{L_T^\infty L^1(\mathcal{B}_1)} &\leq \|\tilde{U}\|^2_{L_T^\infty E(\mathcal{B}_1)}(1 + 2\|\tilde{v}\|_{L_T^\infty E(\mathcal{B}_1)} + \|\tilde{v}\|^2_{L_T^\infty E(\mathcal{B}_1)}) \\
&\quad + \|\tilde{h}\|^2_{L_T^\infty E(\mathcal{B}_1)}\|\tilde{v}\|^2_{L_T^\infty E(\mathcal{B}_1)} + \|\tilde{w}\|^2_{L_T^\infty E(\mathcal{B}_1)} \qquad (2.58)\\
&\leq (K(M) + C(\bar{M})T)e^{C(\bar{M})T}, \qquad (2.59)
\end{aligned}$$

from (2.57). On $\mathcal{B}_2$, we have $\|h_0\|_{L^2(\mathcal{B}_2)} \leq \gamma M \leq C(\bar{M})$ if we assume without loss of generality that $\bar{M} \geq \gamma$. The inequality (2.56c) yields

$$\|\tilde{h}\|_{L_T^\infty E(\mathcal{B}_2)} \leq C(\bar{M})(1 + \|(\tilde{v},\tilde{w})\|_{L_T^\infty E(\mathcal{B}_2)}). \tag{2.60}$$

The inequalities (2.56a) and (2.56b) give

$$\|(\tilde{v},\tilde{w})(t,\,\cdot\,)\|_{E(\mathcal{B}_2)} \leq K(M) + C(\bar{M})T + T\|\tilde{h}\|_{L_T^\infty E(\mathcal{B}_2)} + \int_0^t \|(\tilde{v},\tilde{w})(t',\,\cdot\,)\|_{E(\mathcal{B}_2)}\,dt'$$

and, by Gronwall's lemma,

$$\|(\tilde{v},\tilde{w})\|_{L_T^\infty E(\mathcal{B}_2)} \leq (K(M) + C(\bar{M})T + T\|\tilde{h}\|_{L_T^\infty E(\mathcal{B}_2)})e^{C(\bar{M})T}. \tag{2.61}$$

After inserting (2.60) into (2.61), we get

$$\|(\tilde{v},\tilde{w})\|_{L_T^\infty E(\mathcal{B}_2)} \leq (K(M) + C(\bar{M})T + \|(\tilde{v},\tilde{w})\|_{L_T^\infty E(\mathcal{B}_2)}T)e^{C(\bar{M})T}, \tag{2.62}$$

which after choosing T small enough, implies

$$\|(\tilde{v},\tilde{w})\|_{L_T^\infty E(\mathcal{B}_2)} \leq (K(M) + C(\bar{M})T)e^{C(\bar{M})T}. \tag{2.63}$$

On $\mathcal{B}_3 = \mathcal{K}_\gamma$, we can apply Lemma 2.5 and get $\|\tilde{q}\|_{L_T^\infty L^\infty(\mathcal{B}_3)} \leq \|q_0\|_{L^\infty(\mathcal{B}_3)}$, $\|\tilde{w}\|_{L_T^\infty L^\infty(\mathcal{B}_3)} \leq \|w_0\|_{L^\infty(\mathcal{B}_3)}$ so that, since $\operatorname{meas}(\mathcal{B}_3) \leq 1$,

$$\|(\tilde{v},\tilde{w})\|_{L_T^\infty E(\mathcal{B}_3)} \leq K(M). \tag{2.64}$$

From (2.56c), we get

$$\begin{aligned}
\|\tilde{h}\|_{L_T^\infty L^1(\mathcal{B}_2\cup\mathcal{B}_3)} &\leq M + C(\bar{M})\|(\tilde{v},\tilde{w})\|_{L_T^\infty L^1(\mathcal{B}_2\cup\mathcal{B}_3)} \\
&\leq M + C(\bar{M})\|(\tilde{v},\tilde{w})\|_{L_T^\infty L^\infty(\mathcal{B}_2\cup\mathcal{B}_3)}
\end{aligned}$$

because $\operatorname{meas}(\mathcal{B}_2\cup\mathcal{B}_3) \leq 1$. By (2.63) and (2.64), it follows that

$$\|\tilde{h}\|_{L_T^\infty L^1(\mathcal{B}_2\cup\mathcal{B}_3)} \leq (K(M) + C(\bar{M})T)e^{C(\bar{M})T}. \tag{2.65}$$

Gathering (2.54), (2.55), (2.57), (2.59), (2.63), (2.64) and (2.65), we get

$$\|\tilde{X}\|_{L_T^\infty \bar{V}} \leq (K(M) + C(\bar{M})T)e^{C_1(\bar{M})T}. \tag{2.66}$$

We claim that

$$g(\tilde{X}(t,\xi)) \leq g(\tilde{X}_0(\xi)) + C\int_0^t z(t',\xi)\,dt' \tag{2.67}$$

where $z(t,\xi)$ denotes $(|\tilde{U}| + |\tilde{v}| + |\tilde{w}| + |\tilde{h}| + |P| + |Q|)(t,\xi)$, and where the constant C in (2.67) depends on $\|\tilde{q}\|_{L_T^\infty L^\infty(\mathbb{R})}$, $\|\tilde{U}\|_{L_T^\infty L^\infty(\mathbb{R})}$, $\|\tilde{P}\|_{L_T^\infty L^\infty(\mathbb{R})}$ and $\|\tilde{Q}\|_{L_T^\infty L^\infty(\mathbb{R})}$.

In particular, by (2.66), C is bounded by some constant depending only on $\bar M$. Let us prove this claim. We consider a fixed ξ and drop it in the notation. We have $\frac{d}{dt}g_2(\tilde X(t)) = \nabla g_2(\tilde X)\cdot\frac{d\tilde X}{dt}$. Hence, since $\nabla g_2(\tilde X) = (0,0,1,0,1)$, we get that for any $\bar t$ such that $g(\tilde X(\bar t)) = g_2(\tilde X(\bar t))$,

$$
\begin{aligned}
g(\tilde X(t)) \le g_2(\tilde X(t)) &\le g_2(\tilde X(\bar t)) + C(\bar M)\int_{\bar t}^{t} z(t')\,dt' \\
&= g(\tilde X(\bar t)) + C(\bar M)\int_{\bar t}^{t} z(t')\,dt' \qquad (2.68)
\end{aligned}
$$

for any $t\in[\bar t, T]$. In particular, (2.68) implies the claim (2.67) if $X_0\in\Omega^c$. If $X_0\in\Omega$, since $|\nabla g_1(\tilde X)| = |[0, 2\tilde U\tilde q, 2(1+\tilde U^2), -1, 0]| \le C(\bar M)$, we get

$$
\begin{aligned}
g_1(\tilde X(t)) &\le g_1(X_0) + C(\bar M)\int_0^t z(t',\xi)\,dt' \\
&= g(X_0) + C(\bar M)\int_0^t z(t',\xi)\,dt'. \qquad (2.69)
\end{aligned}
$$

If $\tilde X(t)\in\Omega$, then the claim (2.67) follows from (2.69). If $\tilde X(t)\in\Omega^c$, then we consider the first time $\bar t$ when $\tilde X$ leaves Ω. We have either $\tilde w(\bar t) = 0$ or $g_1(\tilde X(\bar t)) = g_2(\tilde X(\bar t))$. In the latter case, the claim follows from combining (2.69) and (2.68). If $\tilde w(\bar t) = 0$, then, from (2.42), either $\tilde q(\bar t) = 0$ so that $\tilde\tau(\xi) = \bar t$ and $\tilde X$ does not leave Ω or $\tilde h(\bar t) = \tilde U^2(\bar t)\tilde q(\bar t)$. In this case, $g_1(\tilde X(\bar t)) = 2(1+\tilde U^2)\tilde q(\bar t) > (1+\tilde U^2)\tilde q(\bar t) = g_2(\tilde X(\bar t))$ which cannot hold as, by continuity, we must have $g_1(\tilde X(\bar t)) \le g_2(\tilde X(\bar t))$. Hence, we have proved the claim (2.67). From (2.67), we get that

$$
\|g(\tilde X) - 1\|_{L^\infty_T E(B_1\cup B_2)} \le M + C(\bar M)T\,\|z\|_{L^\infty_T E(B_1\cup B_2)} \le M + C(\bar M)T \qquad (2.70)
$$

because $\|z\|_{L^\infty_T E(B_1\cup B_2)} \le C(\bar M)$ by (2.63), (2.66), and (2.53). For $\xi\in\mathcal B_3$, we can apply Lemma 2.5 which tells us that $\tilde X(t,\xi)\in\Omega$ for all t so that

$$
g(\tilde X(t,\xi)) = g_1(\tilde X(t,\xi)) = (|\tilde w| + 2(1+U^2)\tilde q)(t,\xi) \le (K(M) + C(\bar M))e^{C(\bar M)T}
$$

from (2.66). Since $\operatorname{meas}(\mathcal B_3)\le 1$, it implies

$$
\|g(\tilde X) - 1\|_{L^\infty_T E(\mathcal B_3)} \le (K(M) + C(\bar M)T)e^{C(\bar M)T}. \qquad (2.71)
$$

Gathering (2.70) and (2.71), we get

$$
\|g(\tilde X) - 1\|_{L^\infty_T E} \le (K(M) + C(\bar M)T)e^{C(\bar M)T}. \qquad (2.72)
$$

From (2.43), we get

$$
\left\|\frac{1}{\tilde q + \tilde h}\right\|_{L^\infty_T L^\infty_{\mathbb R}} \le K(M)e^{C(\bar M)T}. \qquad (2.73)
$$

Gathering (2.66), (2.72) and (2.73), we finally obtain

$$
\|\tilde X\|_{L^\infty_T \bar V} + \|g(\tilde X) - 1\|_{L^\infty_T E} + \left\|\frac{1}{\tilde q + \tilde h}\right\|_{L^\infty_T L^\infty} \le (K_1(M) + C_1(\bar M))e^{C(\bar M)T}
$$

for some constant $K_1(M)$ and $C_1(\bar M)$ that only depends on M and $\bar M$, respectively. We now set $\bar M = 2K_1(M)$, we can choose T small enough such that $(K_1(M) + C_1(\bar M)T)e^{C_1(\bar M)T} \le 2K_1(M) = \bar M$ and therefore $\|\tilde X\|_{L^\infty_T \bar V} + \|g(\tilde X) - 1\|_{L^\infty_T E} + \left\|\frac{1}{\tilde q+\tilde h}\right\|_{L^\infty_T L^\infty} \le \bar M$. □

Given $X_0 \in B_M$, by Lemma 2.6, there exists an $\bar{M}$ which depends only on M such that $\mathcal{P}$ is a mapping from $C([0,T], B_{\bar{M}})$ to $C([0,T], B_{\bar{M}})$ for T small enough. We denote by $\mathrm{Im}(\mathcal{P})$ the image by $\mathcal{P}$ of $C([0,T], B_{\bar{M}})$, i.e.,

$$\mathrm{Im}(\mathcal{P}) = \left\{\mathcal{P}(X) \mid X \in C([0,T], B_{\bar{M}})\right\}.$$

The following lemma holds.

Lemma 2.7. *There exists $\bar{T} > 0$ such that, for all $T < \bar{T}$, the mapping $\mathcal{P}\colon \mathrm{Im}(\mathcal{P}) \to \mathrm{Im}(\mathcal{P})$ is a contraction in $\bar{V}$.*

Proof. Let X, $\bar{X}$ be two elements of $\mathrm{Im}(\mathcal{P})$. We denote $\boldsymbol{X} = (\boldsymbol{\zeta}, \boldsymbol{U}, \boldsymbol{q}, \boldsymbol{w}, \boldsymbol{h}, \boldsymbol{U}) = \mathcal{P}(X)$, $\boldsymbol{Z} = (\boldsymbol{q}, \boldsymbol{w}, \boldsymbol{h})$, $\bar{\boldsymbol{X}} = (\bar{\boldsymbol{\zeta}}, \bar{\boldsymbol{U}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{h}}) = \mathcal{P}(\bar{X})$ and $\bar{\boldsymbol{Z}} = (\bar{\boldsymbol{q}}, \bar{\boldsymbol{w}}, \bar{\boldsymbol{h}})$. Given $\varepsilon > 0$, let $G_\varepsilon = \mathcal{A}_\varepsilon \cap \mathcal{K}_{\frac{1}{\varepsilon}}$. From Lemma 2.5, we know that, for T small enough, $\{\xi \in \mathbb{R} \mid \boldsymbol{\tau}(\xi) < T \text{ or } \bar{\boldsymbol{\tau}}(\xi) < T\} \subset G_\varepsilon$ and we consider such T. Without loss of generality we also assume $T \leq 1$.

***Step 1**: Estimates for $\int_\tau^{\bar{\tau}} \bar{h}(t,\xi)\,dt$, $\int_{\bar{\tau}}^{\tau} h(t,\xi)\,dt$ and $\|\tau - \bar{\tau}\|_{L^1_{\mathbb{R}}}$.*

Recall that we denote by $C(M)$ a generic constant that may change from line to line but only depend on M. Let us now consider ξ such that $\tau(\xi) \neq \bar{\tau}(\xi)$. Without loss of generality we assume $\bar{\tau}(\xi) < \tau(\xi)$. At time $t = 0$, X and $\bar{X}$ coincide, and therefore we cannot have $\tau(\xi) = 0$ because it would imply $\bar{\tau}(\xi) = 0$. Hence, $0 < \bar{\tau}(\xi) < \tau(\xi) \leq T$. Since X belongs to the image of $\mathcal{P}$, we can apply Lemma 2.5 with $\tilde{X} = X$ and get that there exists $\bar{\varepsilon}$ such that for any $\varepsilon \leq \bar{\varepsilon}$, $w(t,\xi) \in \Omega$ and, in particular, $w(t,\xi) \leq 0$. Hence, from (2.31), we get that for $t \in [\bar{\tau}(\xi), \tau(\xi)]$

$$0 \geq w(t,\xi) = w(\bar{\tau},\xi) + \frac{1}{2}\int_{\bar{\tau}}^{t} h(t',\xi)\,dt' + \int_{\bar{\tau}}^{t} (\frac{1}{2}U - P)q(t',\xi)\,dt.$$

Hence,

$$\begin{aligned}
\int_{\bar{\tau}}^{\tau} h(t,\xi)\,dt &\leq -w(\bar{\tau},\xi) + C(M)\int_{\bar{\tau}}^{\tau} q(t',\xi)\,dt' \\
&\leq \bar{w}(\bar{\tau},\xi) - w(\bar{\tau},\xi) + C(M)\int_{\bar{\tau}}^{\tau} q(t',\xi) - \bar{q}(t',\xi)\,dt' \\
&\leq C(M)(\|w - \bar{w}\|_{L^\infty_T E} + \|q - \bar{q}\|_{L^\infty_T E}) \qquad (2.74)
\end{aligned}$$

where we have used the fact that $\bar{w}(\bar{\tau},\xi) = 0$ and $\bar{q}(t,\xi) = 0$ for $t \in [\bar{\tau}(\xi), \tau(\xi)]$. Since $X(t) \in B_{\bar{M}}$, $1/(q(t,\xi) + h(t,\xi)) \leq \bar{M}$, we have

$$\begin{aligned}
\tau(\xi) - \bar{\tau}(\xi) &\leq \frac{1}{\bar{M}}\int_{\bar{\tau}}^{\tau} (h + q)(t,\xi)\,dt \\
&\leq C(M)(\|w - \bar{w}\|_{L^\infty_T E} + \|q - \bar{q}\|_{L^\infty_T E}) + \frac{1}{\bar{M}}\int_{\bar{\tau}}^{\tau} (q - \bar{q})(t,\xi)\,dt \\
&\leq C(M)(\|w - \bar{w}\|_{L^\infty_T E} + \|q - \bar{q}\|_{L^\infty_T E}). \qquad (2.75)
\end{aligned}$$

From (2.74) and (2.75), we get

$$\|\tau - \bar{\tau}\|_{L^1_{\mathbb{R}}} + \int_{\mathbb{R}} \left|\int_\tau^{\bar{\tau}} \left(\left|\bar{h}(t,\xi)\right| \chi_{\tau<\bar{\tau}}(\xi) + |h(t,\xi)|\,\chi_{\bar{\tau}<\tau}(\xi)\right) dt\right| d\xi \leq C(M)\,\mathrm{meas}(G_\varepsilon)\left\|X - \bar{X}\right\|_{L^\infty_T E}. \qquad (2.76)$$

***Step 2**: Estimate for $\left\|\boldsymbol{Z} - \bar{\boldsymbol{Z}}\right\|_{L^\infty_T \bar{W}(G^c_\varepsilon)}$.*

For $\xi \in G^c_\varepsilon$, we have $\boldsymbol{Z}_t = F(X, \boldsymbol{U})\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}_t = F(X, \bar{\boldsymbol{U}})\bar{\boldsymbol{Z}}$ for all $t \in [0,T]$. Hence,

(2.77)

$$\left\|(\boldsymbol{Z} - \bar{\boldsymbol{Z}})(t,\,\cdot\,)\right\|_{E(G^c_\varepsilon)} \leq \int_0^t \left\|\left(F(X,\boldsymbol{U}) - F(\bar{X},\bar{\boldsymbol{U}})\right)(t',\,\cdot\,)\boldsymbol{Z}(t',\,\cdot\,)\right\|_{E(G^c_\varepsilon)} dt'$$

$$+\int_0^t \left\| F(\bar{X},\bar{\boldsymbol{U}})(\boldsymbol{Z}-\bar{\boldsymbol{Z}})(t',\cdot)\right\|_{E(G_\varepsilon^c)}\,dt'.$$

We have

$$\big(F(X,\boldsymbol{U})-F(\bar{X},\bar{\boldsymbol{U}})\big)\,\boldsymbol{Z}=\Big[0,\big(\frac{1}{2}(\boldsymbol{U}^2-\bar{\boldsymbol{U}}^2)-(P-\bar{P})\big)\boldsymbol{q},\\ -2(Q\boldsymbol{U}-\bar{Q}\bar{\boldsymbol{U}})\boldsymbol{q}+\big(3(\boldsymbol{U}^2-\bar{\boldsymbol{U}}^2)-2(P-\bar{P})\big)\boldsymbol{w}\Big],$$

and therefore

$$\left\|(F(X,\boldsymbol{U})-F(\bar{X},\bar{\boldsymbol{U}}))\boldsymbol{Z}\right\|_{L^1_TE}\\ \leq C(M)\big(\left\|Q-\bar{Q}\right\|_{L^1_TE}+\left\|P-\bar{P}\right\|_{L^1_TE}+\left\|\boldsymbol{U}-\bar{\boldsymbol{U}}\right\|_{L^1_TE}\big). \tag{2.78}$$

Applying Gronwall's lemma to (2.77), as $\left\|F(\bar{X},\bar{\boldsymbol{U}})\right\|_{L^\infty_TL^\infty(\mathbb{R})}\leq C(M)$, we get

$$\left\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\right\|_{E(G_\varepsilon^c)}\leq C(M)\left\|(F(X,\boldsymbol{U})-F(\bar{X},\bar{\boldsymbol{U}}))\boldsymbol{Z}\right\|_{L^1_TE}. \tag{2.79}$$

Hence, since $\left\|\boldsymbol{U}-\bar{\boldsymbol{U}}\right\|_{L^1_TE}\leq T\left\|Q-\bar{Q}\right\|_{L^1_TE}$ from (2.30), we get by (2.78) that

$$\left\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\right\|_{L^\infty_TE(G_\varepsilon^c)}\leq C(\bar{M},T)\big(\left\|Q-\bar{Q}\right\|_{L^1_TE}+\left\|P-\bar{P}\right\|_{L^1_TE}\big). \tag{2.80}$$

Lemma 2.4 and (2.76) then give us

$$\left\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\right\|_{L^\infty_TE(G_\varepsilon^c)}\leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}. \tag{2.81}$$

From (2.42), we have $\boldsymbol{h}=\boldsymbol{U}^2(1+\boldsymbol{v})^2+\boldsymbol{w}^2-\boldsymbol{v}\boldsymbol{h}$ and a similar expression for $\bar{\boldsymbol{h}}$. It follows that

$$\left\|\boldsymbol{h}-\bar{\boldsymbol{h}}\right\|_{L^1(G_\varepsilon^c)}\leq C(M)\left\|X-\bar{X}\right\|_{\bar{V}(G_\varepsilon^c)}+\left\|\boldsymbol{v}\boldsymbol{h}-\bar{\boldsymbol{v}}\bar{\boldsymbol{h}}\right\|_{L^1(G_\varepsilon^c)}\\ \leq C(M)(\left\|X-\bar{X}\right\|_{\bar{V}(G_\varepsilon^c)}+\left\|\boldsymbol{h}-\bar{\boldsymbol{h}}\right\|_{L^2(G_\varepsilon^c)}), \tag{2.82}$$

and (2.81) gives us

$$\left\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\right\|_{L^\infty_T\bar{W}(G_\varepsilon^c)}\leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}. \tag{2.83}$$

Step 3: *Estimate for* $\left\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\right\|_{L^\infty_T\bar{W}(G_\varepsilon)}$.
We assume without loss of generality that $\bar{\boldsymbol{\tau}}(\xi)<\boldsymbol{\tau}(\xi)\leq T$. From Lemma 2.5, we have that $\boldsymbol{q}$ is positive decreasing and $\boldsymbol{w}$ is negative decreasing so that

$$|\boldsymbol{q}(t,\xi)|\leq|\boldsymbol{q}(\bar{\tau},\xi)|\ \text{ and }\ |\boldsymbol{w}(t,\xi)|\leq|\boldsymbol{w}(\bar{\tau},\xi)| \tag{2.84}$$

for $t\in[\bar{\tau}(\xi),T]$, and therefore

$$|\boldsymbol{q}(t,\xi)-\bar{\boldsymbol{q}}(t,\xi)|\leq|\boldsymbol{q}(\bar{\tau},\xi)-\bar{\boldsymbol{q}}(\bar{\tau},\xi)|\ \text{ and }\ |\boldsymbol{w}(t,\xi)-\bar{\boldsymbol{w}}(t,\xi)|\leq|\boldsymbol{w}(\bar{\tau},\xi)-\bar{\boldsymbol{w}}(\bar{\tau},\xi)| \tag{2.85}$$

for $t\in[\bar{\tau}(\xi),T]$ because $\bar{\boldsymbol{q}}(t,\xi)=\bar{\boldsymbol{w}}(t,\xi)=0$ for $t\in[\bar{\tau}(\xi),T]$. For $t\in[\bar{\tau}(\xi),T]$, we have $\boldsymbol{h}_t=-2(Q\boldsymbol{U})\boldsymbol{q}+(3\boldsymbol{U}^2-2P)\boldsymbol{w}$ and $\bar{\boldsymbol{h}}_t=0$. Hence,

$$\left|(\boldsymbol{h}-\bar{\boldsymbol{h}})(t,\xi)\right|\leq\left|(\boldsymbol{h}-\bar{\boldsymbol{h}})(\tau,\xi)\right|+C(M)(|(\boldsymbol{q}-\bar{\boldsymbol{q}})(\tau,\xi)|+|(\boldsymbol{w}-\bar{\boldsymbol{w}})(\tau,\xi)|), \tag{2.86}$$

from (2.84) and (2.85). For $t\in[0,\bar{\tau}]$, we have $\boldsymbol{Z}_t=F(X,\boldsymbol{U})\boldsymbol{Z}$ and $\bar{\boldsymbol{Z}}_t=F(X,\bar{\boldsymbol{U}})\bar{\boldsymbol{Z}}$. We proceed as in the previous step and in the same way as we obtained (2.79), we now obtain

$$|(\boldsymbol{Z}-\bar{\boldsymbol{Z}})(\bar{\tau},\xi)|\leq C(M)\left\|(F(X,\boldsymbol{U})-F(\bar{X},\bar{\boldsymbol{U}}))\boldsymbol{Z}\right\|_{L^1_TL^\infty_{\mathbb{R}}},$$

and, after using (2.78) and (2.80), we get

$$|(\boldsymbol{Z}-\bar{\boldsymbol{Z}})(\bar{\tau},\xi)|\leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}. \tag{2.87}$$

Combining (2.85), (2.86) and (2.87), we get

$$|(\boldsymbol{Z}-\bar{\boldsymbol{Z}})(t,\xi)| \leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}} \tag{2.88}$$

for all $t\in[0,T]$. We take ε small enough so that $\operatorname{meas}(G_\varepsilon)\leq 1$, and then (2.88) implies

$$\|(\boldsymbol{Z}-\bar{\boldsymbol{Z}})\|_{L^\infty_T\bar{W}(G_\varepsilon)} \leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}. \tag{2.89}$$

***Step 4**: Conclusion.*
Combining (2.83) and (2.89), we get

$$\left\|\boldsymbol{Z}-\bar{\boldsymbol{Z}}\right\|_{L^\infty_T\bar{W}} \leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}. \tag{2.90}$$

From (2.30), we obtain

$$\begin{aligned}\left\|\boldsymbol{U}-\bar{\boldsymbol{U}}\right\|_{L^\infty_T E} &\leq T\left\|Q-\bar{Q}\right\|_{L^\infty_T E}\\ &\leq C(\bar{M},T)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}\end{aligned} \tag{2.91}$$

by Lemma 2.4 and (2.76). We have also

$$\left\|\boldsymbol{\zeta}-\bar{\boldsymbol{\zeta}}\right\|_{L^\infty_T L^\infty(\mathbb{R})} \leq T\left\|\boldsymbol{U}-\bar{\boldsymbol{U}}\right\|_{L^\infty_T L^1(\mathbb{R})} \leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}} \tag{2.92}$$

by (2.91). Combining (2.90), (2.91) and (2.92), we get

$$\left\|\boldsymbol{X}-\bar{\boldsymbol{X}}\right\|_{L^\infty_T\bar{V}} \leq C(M)(T+\operatorname{meas}(G_\varepsilon))\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}}.$$

Finally, as $\lim_{\varepsilon\to 0}\operatorname{meas}(G_\varepsilon)=0$, we can choose first $\varepsilon>0$ and then $T>0$ small enough such that

$$\left\|\boldsymbol{X}-\bar{\boldsymbol{X}}\right\|_{L^\infty_T\bar{V}} \leq \frac{1}{2}\left\|X-\bar{X}\right\|_{L^\infty_T\bar{V}},$$

and thus $\mathcal{P}$ is a contraction. □

The physical or Lagrangian variables are (y,U,h). However, the analysis is carried out with the variables $(\zeta,U,\zeta_\xi,U_\xi,h)$ which belong to the Banach space $\bar{V}$. In the remaining part of the paper, we will for the sake of clarity intentionally abuse the notation and denote generically (y,U,h), (ζ,U,h) or $(\zeta,U,\zeta_\xi,U_\xi,h)$ by the same symbol X. We denote (ζ_ξ,U_ξ,h) generically by Z.

Theorem 2.8 (Short time solution)**.** *For any initial data in $X_0=(y_0,U_0,h)\in\mathcal{G}$, there exists a time $T>0$ such that there exists a unique solution $X=(\zeta,U,h)\in C([0,T],\bar{V})$ of* (2.18) *and* (2.19)*. Moreover $X(t)\in\mathcal{G}$ for all $t\in[0,T]$.*

Proof. For a short enough time T, we obtain from Lemma 2.7 that $\mathcal{P}$ is a contraction on $\operatorname{Im}(\mathcal{P})$, and therefore, there exists a unique fixed point $(\zeta,U,v,w,h)\in C([0,T],\bar{V})$ which is solution to (2.31). We have to prove that $U_\xi=w$ and $y_\xi=q$. We can rewrite Q as

$$\begin{aligned}Q=&-\frac{1}{4}e^{-y(\xi)}\int_{-\infty}^{\xi}(U^2q+h)(\eta)\chi_{\{\tau(\eta)>t\}}(\eta)e^{y(\eta)}\,d\eta\\ &+\frac{1}{4}e^{y(\xi)}\int_{\xi}^{\infty}(U^2q+h)(\eta)\chi_{\{\tau(\eta)>t\}}(\eta)e^{-y(\eta)}\,d\eta,\end{aligned}$$

and we can see that Q is differentiable. A direct computation gives us that

$$Q_\xi=-\frac{1}{2}(U^2q+h)(\xi)\chi_{\{\tau(\xi)>t\}}(\xi)+qP, \tag{2.93}$$

and $Q_\xi\in L^1_{\mathrm{loc}}([0,1]\times\mathbb{R})$. In addition, as $q(t,\xi)=\chi_{\{\tau(\xi)>t\}}(\xi)q(t,\xi)$, we have $w_t(t,\xi)=-Q_\xi(t,\xi)$, that is,

$$w(t,\xi)=U_{0\xi}(\xi)-\int_0^t Q_\xi(t',\xi)\,dt'. \tag{2.94}$$

On the other hand, we have

$$U(t,\xi) = U_0(\xi) - \int_0^t Q(t',\xi)\,d\xi.$$

Using Fubini's theorem and integrating by parts, one can prove that

$$\int_{\mathbb{R}} \psi(\xi) w_\xi(t,\xi)\,d\xi + \int_{\mathbb{R}} \psi_\xi(\xi) w(t,\xi)\,d\xi = 0$$

for any ψ smooth with compact support, and therefore $w = U_\xi$. In the same way, as

$$\zeta(t,\xi) = \zeta_0 + \int_0^t U(t',\xi)\,dt' \text{ and } v(t,\xi) = \zeta_{0\xi} + \int_0^t U_\xi(t',\xi)\,dt'$$

because $U_\xi = w$, we get $v = \zeta_\xi$. Let us prove that $X(t) \in \mathcal{G}$ for all t. From (2.41) and (2.42), we get $q(t,\xi) \geq 0$, $h(t,\xi) \geq 0$ and $qh = U^2q^2 + w^2$ for all t and almost all ξ and therefore, since $U_\xi = w$ and $y_\xi = q$, the conditions (2.23c) and (2.23f) are fulfilled. Since $\zeta(t,\xi) = \zeta(\xi,0) + \int_0^t U(t,\xi)\,dt$, by the Lebesgue dominated convergence theorem, we obtain $\lim_{\xi\to-\infty} \zeta(t,\xi) = 0$ because $U \in H^1(\mathbb{R})$. Hence, since in addition $X(t) \in B_{\bar{M}}$, $X(t)$ fulfills all the conditions (2.23) and $X(t) \in \mathcal{G}$. □

Note that the set $\mathcal{G} \cap B_M$ is closed with respect to the topology of $\bar{V}$. We have

$$h_t = (-2QUy_\xi + (3U^2 - 2P)U_\xi)\chi_{\{\tau(\xi)>t\}}(\xi),$$

and, since $y_\xi(t,\xi) = U_\xi(t,\xi) = 0$ for $t \leq \tau(\xi)$ and $P_\xi = y_\xi Q$, we get that, for all time,

$$h_t = -2QUy_\xi + (3U^2 - 2P)U_\xi = (U^3 - 2PU)_\xi. \tag{2.95}$$

The time derivative of h is an exact derivative in ξ. We have that $(\zeta, U, \zeta_\xi, U_\xi, h)$ is a fixed point of $\mathcal{P}$, and the results of Lemma 2.5 hold for $X = \tilde{X} = (\zeta, U, \zeta_\xi, U_\xi, h)$. Since this lemma is going to be used extensively we rewrite it for the fixed point solution X. For this purpose, we redefine B_M, $\mathcal{A}_\varepsilon$ and K_γ, see (2.33), (2.39) and (2.40), as

$$B_M = \left\{X = (\zeta, U, \zeta_\xi, U_\xi, h) \in \bar{V} \mid \|X\|_{\bar{V}} + \|g(X) - 1\|_E + \left\|\frac{1}{y_\xi + h}\right\|_{L^\infty(\mathbb{R})} \leq M\right\}$$

with $X = (\zeta, U, \zeta_\xi, U_\xi, h)$,

$$\mathcal{A}_\varepsilon = \left\{\xi \in \mathbb{R} \mid 0 < y_\xi(0,\xi) \leq \varepsilon \text{ and } -\varepsilon \leq U_\xi(0,\xi) < 0\right\}, \tag{2.96}$$

and

$$\mathcal{K}_\gamma = \left\{\xi \in \mathbb{R} \mid h_0(\xi) \geq \gamma\right\}. \tag{2.97}$$

Recall that $g(X)$ denotes $g(y, U, y_\xi, U_\xi, h)$. Lemma 2.5 rewrites as follows.

Lemma 2.9. *Given a constant M_0, initial data $X_0 \in B_{M_0}$, let $X = (\zeta, U, \zeta_\xi, U_\xi, h) \in C([0,T], B_M)$ denote the solution of* (2.18) *and* (2.19) *with initial data X_0. Introduce $\bar{M} = \|P\|_{L^\infty_T L^\infty(\mathbb{R})} + \|Q\|_{L^\infty_T L^\infty(\mathbb{R})} + M_0$. The following holds:*

(i) We have

$$\left\|\frac{1}{y_\xi + h}(t,\,\cdot\,)\right\|_{L^\infty_{\mathbb{R}}} \leq \frac{9}{2} e^{CT} \left\|\frac{1}{y_{0\xi} + h_0}\right\|_{L^\infty_{\mathbb{R}}} \tag{2.98}$$

for all $t \in [0,T]$ and a constant C which depends on $\bar{M}$ and T.

(ii) There exists an ε depending only on T and $\bar{M}$ such that if $\xi \in \mathcal{A}_\varepsilon$, then $y_\xi(t,\xi)$ is a decreasing function and $U_\xi(t,\xi)$ an increasing function of time and therefore we have

$$-\varepsilon \leq U_\xi(t,\xi) \leq 0 \text{ and } 0 \leq y_\xi(t,\xi) \leq \varepsilon \tag{2.99}$$

for all $t \in [0,T]$. In addition, for ε sufficiently small, depending only on $\bar{M}$ and T, we have

$$\mathcal{A}_\varepsilon \subset \big\{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T\big\}. \tag{2.100}$$

(iii) There exists γ depending only on $\bar{M}$ and T such that, if $\xi \in \mathcal{K}_\gamma$ then $(y, U, y_\xi, U_\xi, h)(t,\xi) \in \Omega$ for all $t \in [0,T]$, $y_\xi(t,\xi)$ is a decreasing function and $U_\xi(t,\xi)$ an increasing function of time and therefore

$$U_\xi(0,\xi) \le U_\xi(t,\xi) \le 0 \text{ and } 0 \le y_\xi(t,\xi) \le y_\xi(0,\xi).$$

In addition, for γ sufficiently large, depending only on $\bar{M}$ and T, we have

$$\mathcal{K}_\gamma \subset \big\{\xi \in \mathbb{R} \mid 0 \le \tau(\xi) < T\big\}. \tag{2.101}$$

(iv) For any $\varepsilon > 0$ and $\gamma > 0$, there exists $T > 0$ such that

$$\big\{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T\big\} \subset \mathcal{A}_\varepsilon \cup \mathcal{K}_\gamma. \tag{2.102}$$

To prove global existence of the solution we will use the estimate contained in the following lemma.

Lemma 2.10. *Given $M_0 > 0$ and $T_0 > 0$, there exists a constant M which only depends on M_0 and T_0 such that, for any $X_0 = (y_0, U_0, h_0) \in B_{M_0}$, the following holds:*
(i) if $h_0 \in E(\mathcal{B})$ for some set $\mathcal{B}$, then $\|Z\|_{L^\infty_T E(\mathcal{B})}$ is bounded by a constant that depends only on M and $\|h_0\|_{E(\mathcal{B})}$;
(ii) $X(t) \in B_M$ for all $t \in [0,T]$, where $X(t)$ denotes the short time solution on $[0,T]$ with $T \le T_0$ given by Theorem 2.8 for initial data X_0.

Proof. From (2.95), we get

$$\int_{-N}^{N} h(t,\xi)\,d\xi = \int_{-N}^{N} h(0,\xi)\,d\xi + \int_0^t \big((U^3 - 2PU)(t',N) - (U^3-2PU)(t',-N)\big)\,dt'. \tag{2.103}$$

Since $U(t,\,\cdot\,) \in H^1(\mathbb{R})$, $\lim_{\xi\to\pm\infty} U(t,\xi) = 0$. By the Lebesgue dominated convergence theorem, and since h is positive, by letting N tend to infinity in (2.103), we get that

$$\|h(t,\,\cdot\,)\|_{L^1} = \|h_0\|_{L^1} \tag{2.104}$$

for all $t \in [0,T]$. We denote generically by M a constant that depends only on M_0 and T_0. To simplify the notation we suppress the dependence in t for the moment. We have

$$\begin{aligned} U^2(\xi) &= \int_{-\infty}^{\xi} U(\eta)U_\xi(\eta)\,d\eta - \int_\xi^\infty U(\eta)U_\xi(\eta)\,d\eta \\ &\le \int_{\mathbb{R}} U(\eta)U_\xi(\eta)\,d\eta = \int_{\{y_\xi(\eta)>0\}} U(\eta)U_\xi(\eta)\,d\eta \end{aligned} \tag{2.105}$$

since, from (2.23e), $U_\xi(\xi) = 0$ when $y_\xi(\xi) = 0$. For almost every ξ such that $y_\xi(\xi) > 0$, we have

$$|U(\xi)U_\xi(\xi)| = \left|\sqrt{y_\xi}U(\xi)\frac{U_\xi(\xi)}{\sqrt{y_\xi(\xi)}}\right| \le \frac{1}{2}\left(U(\xi)^2 y_\xi(\xi) + \frac{U_\xi^2(\xi)}{y_\xi(\xi)}\right) = \frac{1}{2}h(\xi),$$

from (2.23e). Inserting this inequality in (2.105), we obtain

$$\|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \le \frac{1}{\sqrt{2}}\|h(t,\,\cdot\,)\|_{L^1(\mathbb{R})}^{1/2} = \frac{1}{\sqrt{2}}\|h_0\|_{L^1(\mathbb{R})}^{1/2} \le M. \tag{2.106}$$

From (2.11), we obtain that

$$|\zeta(t,\xi)| \le |\zeta(0,\xi)| + \|U(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} T \le M, \tag{2.107}$$

that is, $\|\zeta(t,\cdot)\|_{L^\infty_{\mathbb{R}}} \le M$. Since h and y_ξ are positive, we have

$$|Q(t,\xi)| \le \frac{1}{4}\|U(t,\cdot)\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}} e^{-|y(t,\xi)-y(t,\eta)|}y_\xi(t,\eta)\,d\eta + \frac{1}{4}\int_{\mathbb{R}} h(t,\eta)\,d\eta, \tag{2.108}$$

and, after a change of variables, we obtain

$$|Q(t,\xi)| \le M\int_{\mathbb{R}} e^{-|y(t,\xi)-\eta|}\,d\eta + \frac{1}{4}\|h(t,\cdot)\|_{L^1(\mathbb{R})} = 2M + \frac{1}{4}\|h_0\|_{L^1(\mathbb{R})} \le M. \tag{2.109}$$

A similar bound holds for P. To summarize, we have established that

$$\begin{aligned}\|\zeta(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|U(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|h(t,\cdot)\|_{L^1(\mathbb{R})}\\ + \|P(t,\cdot)\|_{L^\infty(\mathbb{R})} + \|Q(t,\cdot)\|_{L^\infty(\mathbb{R})} \le M.\end{aligned} \tag{2.110}$$

Let us consider a set $\mathcal{B}$ such that $\|h_0\|_{E(\mathcal{B})}$ is finite. We have, from (2.35), that

$$\begin{aligned}\|Q_1(t,\cdot)\|_{L^2(\mathcal{B})} &\le M\left\|f\star\chi_{\{\tau(\xi)>t\}}r\right\|_{L^2(\mathcal{B})}\\ &\le M\|f\|_{L^\infty(\mathbb{R})}\left\|\chi_{\{\tau(\xi)>t\}}r\right\|_{L^2(\mathcal{B})}\\ &\le M\|r(t,\cdot)\|_{L^2(\mathcal{B})}\\ &\le M(\|U(t,\cdot)\|_{L^2(\mathcal{B})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathcal{B})} + \|h(t,\cdot)\|_{L^2(\mathcal{B})}).\end{aligned}$$

The same bound holds for Q_2, and therefore

$$\|Q(t,\cdot)\|_{L^2(\mathcal{B})} \le M(\|U(t,\cdot)\|_{L^2(\mathcal{B})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathcal{B})} + \|h(t,\cdot)\|_{L^2(\mathcal{B})}). \tag{2.111}$$

Similarly, one proves

$$\|P(t,\cdot)\|_{L^2(\mathcal{B})} \le M(\|U(t,\cdot)\|_{L^2(\mathcal{B})} + \|\zeta_\xi(t,\cdot)\|_{L^2(\mathcal{B})} + \|h(t,\cdot)\|_{L^2(\mathcal{B})}). \tag{2.112}$$

Let $\alpha(t) = \|U(t,\cdot)\|_{E(\mathcal{B})} + \|\zeta_\xi(t,\cdot)\|_{E(\mathcal{B})} + \|U_\xi(t,\cdot)\|_{E(\mathcal{B})} + \|h(t,\cdot)\|_{E(\mathcal{B})}$. From the integrated versions of (2.18) and (2.19), after taking the $E(\mathcal{B})$-norm on both sides, adding the four norms and using (2.112) and (2.111), we obtain

$$\alpha(t) \le \alpha(0) + M + M\int_0^t \alpha(t')\,dt'.$$

Hence, Gronwall's lemma gives us that

$$\alpha(t) \le (\alpha(0)+M)e^{MT} \le C \tag{2.113}$$

for some constant C which depends only on M and $\|h_0\|_{E(\mathcal{B})}$. We have thus proved *(i)*.

We consider the constant γ given by Lemma 2.9 (*iii*). By (2.110) and Lemma 2.9, the constant γ only depends on M_0 and T_0 and therefore $\|h_0\|_{E(\mathcal{K}_\gamma^c)} \le M$. Then (2.104), (2.107) and item *(i)* of the present lemma with $\mathcal{B} = \mathcal{K}_\gamma^c$ imply that $\|X(t,\cdot)\|_{\bar{V}(\mathcal{K}_\gamma^c)} \le M$. For $\xi \in \mathcal{K}_\gamma$, by Lemma 2.9, $y_\xi(t,\xi)$ is positive decreasing in time while $U_\xi(t,\xi)$ is negative and increasing. Hence,

$$\|y_\xi(t,\cdot)\|_{L^\infty(\mathcal{K}_\gamma)} \le M \text{ and } \|U_\xi(t,\cdot)\|_{L^\infty(\mathcal{K}_\gamma)} \le M. \tag{2.114}$$

We have $\operatorname{meas}(K_\gamma) \le \frac{1}{\gamma}\int_{\mathbb{R}} h_0(\xi)\,d\xi \le M$ and therefore (2.114) implies together with (2.104), (2.107) and (2.106) that $\|X\|_{\bar{V}(\mathcal{K}_\gamma)} \le M$. Hence, $\|X\|_{\bar{V}} \le M$. To estimate $\|g(X)-1\|_E$, we use the claim (2.67) which in the present context rewrites

$$g(X(t,\xi)) \le g(X_0(\xi)) + M\int_0^t z(t',\xi)\,dt' \tag{2.115}$$

where $z(t,\xi)$ denotes $(|U|+|\zeta_\xi|+|U_\xi|+|h|+|P|+|Q|)(t,\xi)$. We use the Minkowski inequality on (2.115) and get

$$\|g(X(t,\,\cdot\,))-1\|_{E(\mathcal{K}_\gamma^c)} \leq \|g(X_0(\xi))-1\|_{E(\mathcal{K}_\gamma^c)} + M$$

by (2.113), (2.111), and (2.112). For $\xi \in \mathcal{K}_\gamma$, we have $X(t,\xi) \in \Omega$ and therefore $g(X(t,\xi)) = -U_\xi + 2(1+U^2)y_\xi$. Hence, $\|g(X(t,\,\cdot\,))-1\|_{E(\mathcal{K}_\gamma)} \leq M$ because $\|X\|_{\bar{V}} \leq M$ and $\operatorname{meas}(\mathcal{K}_\gamma) \leq M$. Thus we have proved that $\|g(X(t,\,\cdot\,))-1\|_E \leq M$. It remains to prove that $\left\|\frac{1}{y_\xi+h}\right\|_{L_T^\infty L_{\mathbb{R}}^\infty} \leq M$, but this follows directly from (2.98) now that we have established that $\|P\|_{L_T^\infty L_{\mathbb{R}}^\infty} + \|Q\|_{L_T^\infty L_{\mathbb{R}}^\infty} \leq M$. □

We can now prove global existence of solutions.

Theorem 2.11 (Global solution). *For any initial data in $(y_0, U_0, h_0) \in \mathcal{G}$, there exists a unique global solution $(y,U,h) \in C(\mathbb{R}_+,\mathcal{G})$ of (2.18) and (2.19).*

Proof. The argument is somewhat classical but is complicated here by the fact that the short term solution described in Theorem 2.8 does not provide any lower bound on the time of existence of the solution. Let us introduce the maximum time of existence $T_{\max}$ defined as follows

$$T_{\max} = \sup\{t \in \mathbb{R}_+ \mid \text{the solution } X(t) \text{ to (2.11) exists in } [0,t]\}.$$

Let us assume that $T_{\max} < \infty$. From the previous lemma, there exists an M such that $X(t) \in B_M$ for all $t \in [0, T_{\max})$. From the local existence theorem, there exists T such that the solution is defined on $[0,T]$. We then set the constant γ given by Lemma 2.9 (*iii*) so that (2.101) holds. From (2.18), we get
(2.116)

$$\|\zeta(t',\,\cdot\,)-\zeta(t,\,\cdot\,)\|_{L^\infty(\mathbb{R})} \leq M\,|t'-t| \text{ and } \|U(t',\,\cdot\,)-U(t,\,\cdot\,)\|_E \leq C(M)\,|t'-t|$$

for a constant $C(M)$ depending only on M. We denote generically by $C(M)$ such constants. From Lemma 2.10 *(i)*, we have $\|h_0\|_{E(\mathcal{K}_\gamma^c)} \leq C(M)$ and $\|h(t,\,\cdot\,)\|_{E(\mathcal{K}_\gamma^c)} \leq C(M)$. From (2.20), we get

$$\|Z(t',\,\cdot\,)-Z(t,\,\cdot\,)\|_{E(\mathcal{K}_\gamma^c)} \leq \int_t^{t'} \|F(X)Z\|_{E(\mathcal{K}_\gamma^c)} \leq C(M)\,|t'-t|\,. \tag{2.117}$$

After using (2.23f) we obtain

$$\begin{aligned}
&\|h(t',\,\cdot\,)-h(t,\,\cdot\,)\|_{L^1(\mathcal{K}_\gamma^c)}\\
&\quad= \left\|[U^2(1+\zeta_\xi)^+U_\xi^2-h\zeta_\xi](t',\,\cdot\,)-[U^2(1+\zeta_\xi)^2+U_\xi^2-h\zeta_\xi](t,\,\cdot\,)\right\|_{L^1(\mathcal{K}_\gamma^c)}\\
&\quad\leq C(M)(\|U(t',\,\cdot\,)-U(t,\,\cdot\,)\|_{L^2(\mathcal{K}_\gamma^c)} + \|Z(t',\,\cdot\,)-Z(t,\,\cdot\,)\|_{L^2(\mathcal{K}_\gamma^c)})\\
&\quad\leq C(M)\,|t'-t|\,.
\end{aligned}$$

Hence,

$$\left\|X(t',\,\cdot\,)-\bar{X}(t,\,\cdot\,)\right\|_{\bar{V}(\mathcal{K}_\gamma^c)} \leq C(M)\,|t'-t|\,.$$

For $\xi \in \mathcal{K}_\gamma$, from (2.101), we have $\zeta_\xi(t',\xi) = \zeta_\xi(t,\xi)$, $U_\xi(t',\xi) = U_\xi(t,\xi)$ and $h(t',\xi) = h(t,\xi)$ for $t' \geq t \geq T$. Therefore,

$$\left\|X(t',\,\cdot\,)-\bar{X}(t,\,\cdot\,)\right\|_{\bar{V}} \leq C(M)\,|t'-t|\,. \tag{2.118}$$

Since $\bar{V}$ is a Banach space, (2.118) implies that the limit of $X(t)$ exists as t tends to $T_{\max}$ and we denote it by $\bar{X}$. We claim that $\bar{X}$ belongs to $\mathcal{G}$. The conditions (2.23c), (2.23d), (2.23f) hold because L^∞ or L^1 convergence implies almost everywhere convergence up to a subsequence. Since $X(t) \in B_M$ for all $t \in [0, T_{\max})$, we have $\left\|\frac{1}{y_\xi+h}(t,\,\cdot\,)\right\| \leq M$ and therefore $\left\|\frac{1}{\bar{y}_\xi+h}\right\| \leq M$ and the condition (2.23e) is fulfilled. It remains to check (2.23b). First we prove that the mapping g defined in (2.21) is lower-semicontinuous. Indeed, let us consider a sequence $\boldsymbol{x}^n$ in $\mathbb{R}^5$ which converges to $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)$. We denote $g_3 = \min(g_1, g_2)$. If $x_4 < 0$, or $x_4 > 0$, then we use the continuity of g_3 and g_2 to conclude that $\lim_{n\to\infty} g(\boldsymbol{x}^n) = g(\boldsymbol{x})$. If $x_4 = 0$, we have that $g_3(\boldsymbol{x}^n) \leq g(\boldsymbol{x}^n)$ and therefore

$$g(\boldsymbol{x}) = g_3(\boldsymbol{x}) = \liminf g_3(\boldsymbol{x}^n) \leq \liminf g(\boldsymbol{x}^n)$$

and g is lower semi-continuous. We consider a sequence $t_n \to T_{\max}$ such that $X(t_n, \xi)$ converges to $\bar{X}(\xi)$ for almost every ξ. Since y_ξ and h are positive, we can check from the definition of g that $g(X) - y_\xi$ is also positive. Hence, by the lower semicontinuity of g, we get that

$$0 \leq g(\bar{X}(\xi)) - \bar{y}_\xi(\xi) \leq \liminf \big(g(X(t_n, \cdot\,)) - y_\xi(t_n, \xi)\big) \leq 2M$$

and $g(\bar{X}) - 1$ belongs to $L^\infty(\mathbb{R})$ because $\left\|g(\bar{X}) - 1\right\|_{L^\infty(\mathbb{R})} \leq \left\|g(\bar{X}) - \bar{y}_\xi\right\|_{L^\infty(\mathbb{R})} + \left\|\bar{\zeta}_\xi\right\|_{L^\infty(\mathbb{R})} \leq 3M$. The composition of an increasing lower semicontinuous function with a lower semicontinuous function is also lower semincontinuous. Hence, since $z \mapsto z^2$ is increasing for $z \geq 0$ and $g(X) - y_\xi$ is positive, we have that

$$(g(\bar{X}(\xi)) - \bar{y}_\xi(\xi))^2 \leq \liminf (g(X(t_n, \xi)) - y_\xi(t_n, \xi))^2$$

and

$$\begin{aligned} \int_{\mathbb{R}} \left|g(\bar{X}(\xi)) - \bar{y}_\xi(\xi)\right|^2 d\xi &\leq \int_{\mathbb{R}} \liminf \left|g(X(t_n, \xi)) - y_\xi(t_n, \xi)\right|^2 d\xi \\ &\leq \liminf \int_{\mathbb{R}} \left|g(X(t_n, \xi)) - y_\xi(t_n, \xi)\right|^2 d\xi \end{aligned} \tag{2.119}$$

by Fatou's Lemma. We have $\|g(X(t_n, \cdot\,)) - y_\xi(t_n, \cdot\,)\|_{L^2(\mathbb{R})} \leq \|g(X(t_n, \cdot\,)) - 1\|_{L^2(\mathbb{R})} + \|\zeta_\xi(t_n, \cdot\,)\|_{L^2(\mathbb{R})} \leq 2M$ and (2.119) implies

$$\left\|g(\bar{X}) - 1\right\|_{L^2(\mathbb{R})} \leq \left\|g(\bar{X}) - \bar{y}_\xi\right\|_{L^2(\mathbb{R})} + \left\|\bar{\zeta}_\xi\right\|_{L^2(\mathbb{R})} \leq 3M.$$

Hence, $\bar{X}$ fulfills (2.23b) and $\bar{X}$ belongs to $\mathcal{G}$. We can then apply Theorem 2.8 and get the existence of a short time solution with initial data $\bar{X}$ which, combined with X on $[0, T_{\max})$, gives a solution on $[0, T_{\max} + \delta]$ for some $\delta > 0$. The assumption regarding $T_{\max}$ is contradicted, and we have proved the global existence of solutions. □

3. Short time stability

First we give a short description of the dynamics of the system (2.19) for a given particle ξ. We consider a solution $X(t) = (y, U, h) \in C(\mathbb{R}_+, \mathcal{G})$ of (2.18) and (2.19). By Lemma 2.10, there exists an M depending on T and the initial data such that $X(t) \in B_M$ for all $t \in [0, T]$. For a given $\xi \in \mathbb{R}$ (that we drop in the notation), we obtain, from (2.23e) and (2.23f), that

$$U_\xi^2 + y_\xi^2 \leq C_1 y_\xi \quad \text{and} \quad C_2 y_\xi \leq U_\xi^2 + y_\xi^2 \tag{3.1}$$

for two strictly positive constants C_1 and C_2 that depend on T and M. The projection of the trajectory $X(t)$ in the plane (y_ξ, U_ξ) lies between the two circles defined by the equations (3.1), see Figure 3. The two circles intersect at the origin. The origin is an attractive point when $U_\xi < 0$ and repulsive one when $U_\xi > 0$, since we have, from (2.19), that, close to the origin (that is, $y_\xi \approx U_\xi \approx 0$),

$$(y_{\xi,t}, U_{\xi,t}) \approx (0, \frac{1}{2}h)$$

and $h \approx y_\xi + h \geq \frac{1}{M} > 0$ as $y_\xi + h$ is always bounded strictly away from zero. When dealing with conservative solutions, $(y_\xi(t), U_\xi(t))$ always follows the vector field defined by (2.14) and can go through the origin. With dissipative solutions, we

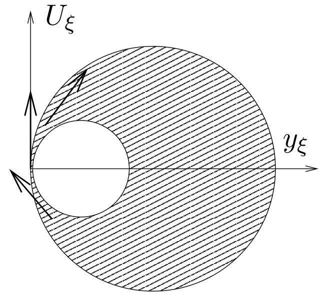

FIGURE 1. For a solution $X = (y, U, h)$, we have that $y_\xi(t, \xi)$ and $U_\xi(t, \xi)$ belong to the dashed region contained between the two circles, see (3.1). The origin is a point of attraction for $U_\xi < 0$ and a repulsive point for $U_\xi > 0$.

terminate the process at the origin so that when $(y_\xi(t), U_\xi(t))$ reaches the origin, it does not go any further.

As far as stability is concerned in the case of dissipative solution, we face the following problem: If we consider two initial data X_0 and $\bar{X}_0$ close to the origin and close to each other for some given ξ, the first one being below the horizontal axis, that is, $U_{0\xi}(\xi) < 0$, and the other above, that is, $\bar{U}_{0\xi}(\xi) > 0$ and look at their trajectory, we will observe that the first point will reach the origin and stop while the second one will travel away from the origin. Thus, these two points which may be very close (with respect to the Euclidean distance) at the beginning, that is, $\left|U - \bar{U}\right|$, $|y_\xi - \bar{y}_\xi|$, $\left|U_\xi - \bar{U}_\xi\right|$ and $\left|h - \bar{h}\right|$ may be as small as we want, will stray apart very quickly. Therefore, in order to obtain stability, we need a new distance that separates points that have negative and positive U_ξ and that are close to the origin.

We introduce the mapping $d\colon \mathbb{R}^5 \times \mathbb{R}^5 \to \mathbb{R}_+$ defined as

$$d(\boldsymbol{x}, \bar{\boldsymbol{x}}) = \sum_{i=2}^{5} |x_i - \bar{x}_i| + |g(\boldsymbol{x}) - g(\bar{\boldsymbol{x}})|$$

for $\boldsymbol{x}, \bar{\boldsymbol{x}}$ in $\mathbb{R}^5$. For a subset Ω of $\mathbb{R}$, we define the mapping $d_\Omega\colon \mathcal{G} \times \mathcal{G} \to \mathbb{R}_+$ as

$$d_\Omega(X, \bar{X}) = \left\|X - \bar{X}\right\|_{V(\Omega)} + \left\|g(X) - g(\bar{X})\right\|_{L^2(\Omega)} \tag{3.2}$$

for X, $\bar{X}$ in $\mathcal{G}$. Then, $d_{\mathbb{R}}$ defines a distance on $\mathcal{G}$. For two points $X_0(\xi)$ and $\bar{X}_0(\xi)$ close to the origin such that $U_{0,\xi}(\xi) < 0$ and $U_{0,\xi}(\xi) > 0$, we have $d(X_0(\xi), \bar{X}_0(\xi)) = \left|Z_0(\xi) - \bar{Z}_0(\xi)\right| + \left|g(X_0(\xi)) - g(\bar{X}_0(\xi))\right| \approx \left|g(X_0(\xi)) - g(\bar{X}_0(\xi))\right| \approx h_0(\xi) \geq \frac{1}{M}$ which is bounded away from zero, see Figure 3, so that $X_0(\xi)$ and $\bar{X}_0(\xi)$ which are close to each other in the Euclidian distance are no longer closer with respect to the metric d. Of course, there is a degree of arbitrariness in the choice of g which we will not discuss here. However, two properties of g are essential:
(i) g splits the collision point (as we just explained, see Figure 3).
(ii) g is positive homogenous in the three last variables, that is, $g(x_1, x_2, \lambda x_3, \lambda x_4, \lambda x_5) = \lambda g(x_1, x_2, x_3, x_4, x_5)$ for all $\lambda \geq 0$ and $\boldsymbol{x} \in \mathbb{R}^5$.

The last property is going to be used in Sections 5–6 when we map back the solutions into Eulerian coordinates.

Our goal in this section is to prove the following short-time stability theorem.

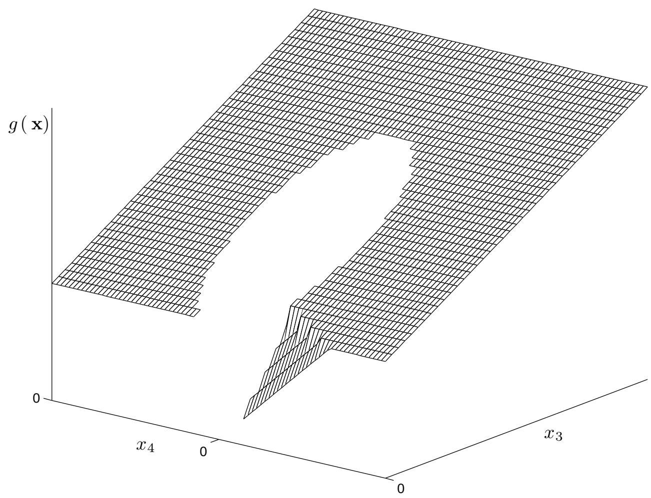

FIGURE 2. Plot of the function g for x_1, x_2, x_5 fixed. The function g separates the origin for positive and negative values of x_4. The hole in the middle corresponds to the inner circle in Figure 3, a region where the solutions cannot enter.

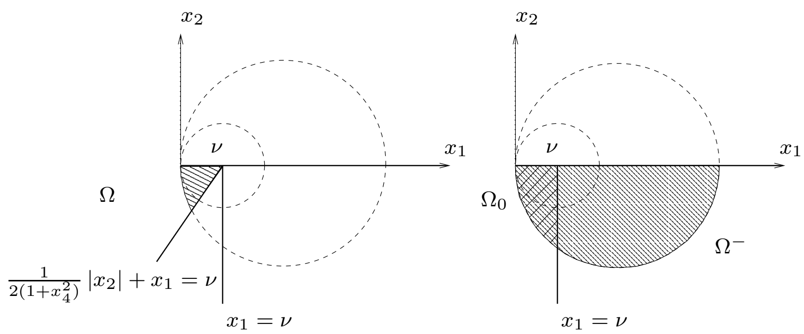

FIGURE 3. Plots of the region Ω, Ω^- and Ω_0. We denote $\nu = \frac{x_1+x_3}{2(1+x_4^2)}$. We plot the projection of the domains in the (x_1, x_2) plane for ν and x_4 fixed. Note that the region inside the inner circle is excluded for the solutions, see Figure 3.

Theorem 3.1 (short-time stability)**.** *Given $M > 0$, there exist constants K, ε and $\bar{T}$ all depending only on M such that, for any initial data X_0, $\bar{X}_0$ in B_M, we have*

$$\sup_{t\in[0,T]} d_{\mathbb{R}}(X(t), \bar{X}(t)) \leq K d_{\mathbb{R}}(X_0, \bar{X}_0) \tag{3.3}$$

for any $T \leq \bar{T}$ such that

$$\operatorname{meas}(\{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T\}) + \operatorname{meas}(\{\xi \in \mathbb{R} \mid 0 < \bar{\tau}(\xi) < T\}) \leq \varepsilon.$$

In this short-time stability theorem we impose that the measure of the set of the particles that will collide before T is bounded by a small constant ε.

The existence of the solutions has been established in $C(\mathbb{R}, \bar{V})$, and, in particular, for a solution $X = (y, U, h)$, $\zeta_\xi(t, \cdot)$, and $U_\xi(t, \cdot)$ belong to L^∞ for all time.

However, we establish stability only for L^2 norms, see the definition (3.2) of $d_{\mathbb{R}}$. We need this stronger result in order to prove Proposition 5.2 which allows us to go back to Eulerian coordinates. For h, we use the L^1 norm because it is the only norm which is invariant with respect to relabeling, a property which is needed in Section 5, see (5.8).

In order to prove this result, we need to establish several auxiliary results. We introduce the sets

$$\Omega_- = \{\boldsymbol{x} \in \mathbb{R}^5 \mid x_4 \leq 0\},$$
$$\Omega_M = \{\boldsymbol{x} \in \mathbb{R}^5 \mid |x_2| + |x_3| + |x_4| \leq M\},$$

see Figure 3. The mapping g is of course not Lipschitz on $\mathbb{R}^5$ (otherwise it would be have been useless to introduce it), but we have following result.

Lemma 3.2. *The restrictions of g to $\Omega^c \cap \Omega_M$ and $\Omega_- \cap \Omega_M$ are Lipschitz. More precisely, for any $M > 0$, there exists a constant $\kappa > 0$ only depending on M such that*

$$|g(\boldsymbol{x}) - g(\bar{\boldsymbol{x}})| \leq \kappa \sum_{i=2}^{5} |x_i - \bar{x}_i|$$

for any $\boldsymbol{x}, \bar{\boldsymbol{x}} \in \Omega^c \cap \Omega_M$ or any $\boldsymbol{x}, \bar{\boldsymbol{x}} \in \Omega_- \cap \Omega_M$.

Proof. The case where both $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ belong to Ω^c is straightforward as

$$|g(\bar{\boldsymbol{x}}) - g(\boldsymbol{x})| = |\bar{x}_3 + \bar{x}_5 - x_3 - x_5| \leq |\bar{x}_3 - x_3| + |\bar{x}_5 - x_5| \leq |\bar{\boldsymbol{x}} - \boldsymbol{x}|.$$

Let us consider the case where $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ belong to Ω_-, that is, $x_4 \leq 0$ and $\bar{x}_4 \leq 0$. If $-x_4+2(1+x_2)^2x_3 \leq x_3+x_5$, then $g(\boldsymbol{x}) = -x_4+2(1+x_2)^2x_3$. If $-\bar{x}_4+2(1+\bar{x}_2)^2\bar{x}_3 \leq \bar{x}_3 + \bar{x}_5$, $g(\bar{\boldsymbol{x}}) = -\bar{x}_4 + 2(1+\bar{x}_2)^2\bar{x}_3$, and the result follows directly from the fact that the mapping $(x_2, x_3, x_4) \mapsto -x_4 + 2(1+x_2)^2x_3$ is Lipschitz on bounded sets and therefore when $\boldsymbol{x}$ and $\bar{\boldsymbol{x}}$ belong Ω_M. We denote the Lipschitz constant of the previous mapping by $\tilde{\kappa}$. If $-\bar{x}_4 + 2(1+\bar{x}_2)^2\bar{x}_3 \geq \bar{x}_3 + \bar{x}_5$, $g(\bar{\boldsymbol{x}}) = \bar{x}_3 + \bar{x}_5$ and we have

$$\begin{aligned}
|g(\bar{\boldsymbol{x}}) - g(\boldsymbol{x})| &\leq \left|\bar{x}_3 + \bar{x}_5 + x_4 - 2(1+x_2^2)x_3\right| \\
&\leq |\bar{x}_3 - x_3| + |\bar{x}_5 - x_5| + \left|x_3 + x_5 + x_4 - 2(1+x_2^2)x_3\right| \\
&= |\bar{x}_3 - x_3| + |\bar{x}_5 - x_5| + x_3 + x_5 + x_4 - 2(1+x_2^2)x_3 \\
&\leq 2\,|\bar{x}_3 - x_3| + 2\,|\bar{x}_5 - x_5| + x_4 - 2(1+x_2^2)x_3 - (\bar{x}_4 - 2(1+\bar{x}_2^2)\bar{x}_3) \\
&\leq (2+\tilde{\kappa}) \sum_{i=2}^{5} |x_i - \bar{x}_i|.
\end{aligned}$$

□

The following lemma describes the situation when for two solutions of (2.18) and (2.19), the mapping g behaves like a Lipchitz function.

Lemma 3.3. *Given $M \geq 0$, there exist $T > 0$, $\kappa > 0$ and $\delta > 0$ which depend only on M such that for any $\xi \in \mathbb{R}$ satisfying $d(X_0(\xi), \bar{X}_0(\xi)) < \delta$ we have*

$$\left|g(X(t,\xi)) - g(\bar{X}(t,\xi))\right| \leq C(\left|U(t,\xi) - \bar{U}(t,\xi)\right| + \left|Z(t,\xi) - \bar{Z}(t,\xi)\right|) \tag{3.4}$$

for all $t \in [0,T]$ and the solutions $X(t)$ and $\bar{X}(t)$ with any initial data X_0 and $\bar{X}_0$ in B_M.

Proof. Without loss of generality, we assume $T \leq 1$. From Lemma 2.10, there exists $\bar{M}$ depending only on M such that $X(t)$ and $\bar{X}(t)$ belong to $B_{\bar{M}}$ for all $t \in [0,T]$. We consider $\xi \in \mathbb{R}$ such that $d(X(0,\xi), \bar{X}(0,\xi)) < \delta$ for a δ that we are going to determine. We drop ξ in the notation. Since $X(t)$ and $\bar{X}(t)$ belong to $\Omega_{\bar{M}}$, by

Lemma 3.2, we can see that the lemma will be proved if we can prove that $X(t)$ and $\bar{X}(t)$ belong to the same set, either Ω^c or Ω_-. Let Ω_0 denote the set

$$\Omega_0 = \left\{ \boldsymbol{x} \in \mathbb{R}^5 \mid x_3 \leq \frac{x_3 + x_5}{2(1 + x_2^2)} \text{ and } x_4 \leq 0 \right\}, \tag{3.5}$$

see Figure 3. We consider the following three cases: (i) $X_0, \bar{X}_0 \in \Omega_0$; (ii) $X_0 \in \Omega_0, \bar{X}_0 \in \Omega_0^c$; and (iii) $X_0, X_0 \in \Omega_0^c$.

Case (i). Since $\Omega_0 \subset \Omega_-$, X and $\bar{X}$ also belong to Ω_-. Let t_0 be the first time when X exits Ω_-. By continuity, we have $U_\xi(t_0) = 0$. From (2.23f) we infer that $(1+U^2)y_\xi(t_0) = (y_\xi + h)(t_0)$ because we cannot have $y_\xi(t_0) = 0$ as the points that reach the origin remain there. Let $z(t) = 2(1 + U^2(t))y_\xi(t) - (y_\xi + h)(t)$, we have $z(0) \leq 0$ because $X(0) \in \Omega_0$ and $z(t_0) = (y_\xi + h)(t_0) \geq \frac{1}{\bar{M}}$. On the other hand, from (2.19), z_t can be computed, and we obtain $z_t \leq C_1(\bar{M})$ for some constant $C_1(\bar{M})$ only depending on $\bar{M}$ and therefore on M, so that $z(t) \leq z(0) + C_1(\bar{M})T$. Hence, if we choose T small enough so that $T < (\bar{M}C_1(\bar{M}))^{-1}$, we obtain $z(t_0) < \frac{1}{\bar{M}}$, which is a contradiction, and we have proved that X remains in Ω_-. Similarly one proves that $\bar{X}$ remains in Ω_-.

Case (ii). We have already seen that we can choose T small enough so that $X(t)$ remains in Ω_- for $t \in [0, T]$. Let us denote $\bar{z}(t) = 2(1 + \bar{U}^2(t))\bar{y}_\xi(t) - (\bar{y}_\xi + \bar{h})(t)$. For z as defined above, we have $z(0) \leq 0$. Hence, $\bar{z}(0) \leq z(0) + |\bar{z}(0) - z(0)| \leq C(\bar{M})(\left|\bar{U}(0) - U(0)\right| + |\bar{y}_\xi(0) - y_\xi(0)| + \left|\bar{h}(0) - h(0)\right|)$ for some constant only depending on $\bar{M}$ and therefore $\bar{z}(0) \leq C(\bar{M})\delta$. We claim that, for δ small enough, $\bar{X}_0 \in \Omega_-$. Let us assume the opposite. Then, $|U_{0,\xi} - \bar{U}_{0,\xi}| \leq \delta$ implies $|\bar{U}_{0,\xi}| \leq \delta$ and $|U_{0,\xi}| \leq \delta$ because $U_{0,\xi} \leq 0$ and $\bar{U}_{0,\xi} \geq 0$. Since $X_0 \in \Omega_0$, we have $2(1 + U_0^2)y_{0,\xi}^2 \leq y_{0,\xi}^2 + h_0 y_{0,\xi}$ and it implies, by (2.23f), that $(1 + U_0^2)y_\xi^2 \leq U_\xi^2$ and therefoer $y_\xi \leq \delta$. We have

$$\delta \geq g(\bar{X}_0) - g(X_0) \geq \bar{y}_{0,\xi} + \bar{h}_0 - |U_{0,\xi}| - 2(1 + U_0^2)y_{0,\xi} \geq \frac{1}{M} - C(M)\delta$$

and, by taking δ small enough, we are led to a contradiction. Hence, $\bar{X}_0 \in \Omega_-$. Again, let us consider the first time t_0 when $\bar{X}$ leaves Ω_-. We have $\bar{z}(t_0) = (\bar{y}_\xi + \bar{h})(t_0) \geq \frac{1}{\bar{M}}$ and $\bar{z}(t_0) \leq \bar{z}(0) + C(\bar{M})T$ implies $\frac{1}{\bar{M}} \leq C(\bar{M})(\delta + T)$ which leads to a contradiction if we choose T and δ small enough. Hence, $\bar{X}$ remains in Ω_-.

Case (iii). In this case, since $\Omega_0^c \subset \Omega^c$, X and $\bar{X}$ also belong to Ω^c. We consider $z(t)$ as defined before. Since $X \in \Omega_0^c$, we have $z(0) \geq 0$. Let us denote by t_0 the first time when X leaves Ω^c. Since the origin is a repulsive point in Ω^c, see Figure 3, we must have $|U_\xi(t_0)| + 2(1 + U^2)(t_0)y_\xi(t_0) = (y_\xi + h)(t_0)$ which gives $U_\xi(t_0) = z(t_0)$. Inserting this into (2.23f), we obtain after some computation that $z(t_0) = -\frac{y_\xi + h}{\sqrt{4U^2+5}}(t_0)$. Thus, $-z(t_0) \leq -z(0) + C_1(\bar{M})T$ implies $(C_2(\bar{M}))^{-1} \leq \frac{y_\xi + h}{\sqrt{4U^2+5}}(t_0) \leq C_1(\bar{M})T$, which leads to a contradiction if T is chosen small enough. We have thus proven that X remains in Ω^c, and the same result holds by the same argument for $\bar{X}$. $\square$

When dealing with discontinuous systems of ordinary differential equations in finite dimensions, it is essential to have some control over the time when the solutions hit the discontinuity line and change behavior. Consider the situation with two solutions with initial data that are close and such that one solution hits the discontinuity line soon after the other. If the time between the two solutions hit the discontinuity line goes to zero, then, for stability reasons, the difference between the initial data should also go to zero. This is precisely the content of the next lemma. Note that this property has already been used when proving the existence of solutions, see (2.75).

Lemma 3.4. *Given $M \geq 0$, there exist T and δ which depend only on M such that, for any $\xi \in \mathbb{R}$ that satisfies $d(X_0(\xi), \bar{X}_0(\xi)) \leq \delta$, we have*

$$(\bar{\tau} - \tau)(\xi) + \int_{\tau}^{\bar{\tau}} \bar{h}(t,\xi)\,dt \leq C(M)\left|Z(\tau(\xi),\xi) - \bar{Z}(\tau(\xi),\xi)\right| \tag{3.6}$$

if $\tau(\xi) < \bar{\tau}(\xi) \leq T$, and

$$(\tau - \bar{\tau})(\xi) + \int_{\bar{\tau}}^{\tau} h(t,\xi)\,dt \leq C(M)\left|Z(\bar{\tau}(\xi),\xi) - \bar{Z}(\bar{\tau}(\xi),\xi)\right| \tag{3.7}$$

if $\bar{\tau}(\xi) < \tau(\xi) \leq T$, for a constant $C(M)$ depending only on M and any solutions $X(t)$ and $\bar{X}(t)$ with initial data X_0 and $\bar{X}_0$ in B_M.

Proof. We assume without loss of generality that $T \leq 1$. There exist ε and γ depending only on M such that for any $\xi \in \bar{\mathcal{A}}_\varepsilon \cup \bar{\mathcal{K}}_\gamma$ and $\xi \in \mathcal{A}_\varepsilon \cup \mathcal{K}_\gamma$, the conclusions of items *(ii)* and *(iii)* in Lemma 2.9 hold, that is, $\bar{X}(t,\xi) \in \Omega$, $\bar{y}_\xi(t,\xi)$ decreasing and $\bar{U}_\xi(t,\xi)$ negative and increasing for $t \in [0,1]$ and the corresponding properties for $X(t,\xi)$. From Lemma 2.9 item *(iv)*, there exists T depending only on M, ε and γ and therefore only on M such that $\{0 < \tau(\xi) < T\} \subset \mathcal{A}_{\varepsilon/2} \cup \mathcal{K}_{2\gamma}$. Let us consider $\xi \in \mathbb{R}$ such that $\tau(\xi) < \bar{\tau}(\xi) \leq T$. Then, $\xi \in \mathcal{A}_{\varepsilon/2} \cup \mathcal{K}_{2\gamma}$ or $\tau(\xi) = 0$. We set $\delta = \min(\varepsilon/2, \gamma)$ and δ depends only on M. If $\xi \in \mathcal{A}_{\varepsilon/2}$ or $\tau(\xi) = 0$, we have $\left|\bar{U}_{0\xi}(\xi)\right| \leq \left|U_{0\xi}(\xi) - \bar{U}_{0\xi}(\xi)\right| + \left|U_{0\xi}(\xi)\right| \leq \varepsilon$ and $\left|\bar{y}_{0\xi}(\xi)\right| \leq \left|y_{0\xi}(\xi) - \bar{y}_{0\xi}(\xi)\right| + \left|y_{0\xi}(\xi)\right| \leq \varepsilon$. Since $X(t,\xi) \in \Omega$, $g(X_0(\xi)) = -U_{0\xi}(\xi) + 2(1+U^2)y_\xi(\xi)$ for some constant $C(M)$ depending only on M, and we have

$$g(\bar{X}_0(\xi)) \leq \delta + |g(X_0(\xi))| \leq \frac{\varepsilon}{2} + C(M)\varepsilon < \frac{1}{M} \leq \bar{y}_{0\xi}(\xi) + \bar{U}_{0\xi}(\xi)$$

for ε small enough and depending only on M. Hence, $g(\bar{X}_0(\xi)) < \bar{y}_{0\xi}(\xi) + \bar{U}_{0\xi}(\xi)$, which implies that $\bar{X}_0(\xi) \in \Omega$ so that $U_{0\xi}(\xi) \leq 0$. Thus we have proved that $\xi \in \mathcal{A}_\varepsilon$. If $\xi \in \mathcal{K}_{2\gamma}$, we have $\bar{h}_0(\xi) \geq |h_0(\xi)| - \delta \geq \gamma$ and $\xi \in \mathcal{K}_\gamma$. From the governing equation (2.19), we obtain

$$\bar{U}_{\xi,t}(t,\xi) \geq \bar{h}(t,\xi) - C(M)\bar{y}_\xi(t,\xi)$$

for some constant $C(M)$ depending only on M. Hence, after integrating over the time interval $[\tau, \bar{\tau}]$,

$$\bar{U}_\xi(\bar{\tau}(\xi),\xi) \geq \bar{U}_\xi(\tau(\xi),\xi) + \frac{1}{2}\int_\tau^{\bar{\tau}} \bar{h}(t,\xi)\,dt - C(M)T\bar{y}_\xi(\tau,\xi) \tag{3.8}$$

because $\bar{y}_\xi(t,\xi)$ is decreasing so that $\bar{y}_\xi(t,\xi) \leq \bar{y}_\xi(\tau(\xi),\xi)$ for $t \geq \tau$. Since $\bar{X}(t,\xi) \in \Omega$, we have $\bar{U}_\xi(t,\xi) \leq 0$ and (3.8) yields, for T small enough,

$$\begin{aligned}\frac{1}{2}\int_\tau^{\bar{\tau}} \bar{h}(t,\xi)\,d\xi &\leq 2(\left|\bar{U}_\xi(\tau(\xi),\xi)\right| + \left|\bar{y}_\xi(\tau(\xi),\xi)\right|)\\ &= 2(\left|\bar{U}_\xi(\tau(\xi),\xi) - U_\xi(\tau(\xi),\xi)\right| + \left|\bar{y}_\xi(\tau(\xi),\xi) - y_\xi(\tau(\xi),\xi)\right|).\end{aligned} \tag{3.9}$$

By Lemma 2.10, there exists $\bar{M}$ depending only on M such that $\bar{X}(t) \in B_{\bar{M}}$ for $t \in [0,1]$. We have, as $\frac{1}{\bar{M}} \leq \bar{h} + \bar{y}_\xi$,

$$\frac{1}{\bar{M}}(\bar{\tau} - \tau)(\xi) \leq \int_\tau^{\bar{\tau}} (\bar{h} + \bar{y}_\xi)(t,\xi)\,dt \leq 4\left|Z(\tau(\xi),\xi) - \bar{Z}(\tau(\xi),\xi)\right| + y_\xi(\tau(\xi),\xi),$$

by (3.9) and because $y_\xi(t,\xi) \leq y_\xi(\tau,\xi)$ for $t \geq \tau$. Hence, $(\bar{\tau} - \tau)(\xi) \leq C(M)\left|Z(\tau(\xi),\xi) - \bar{Z}(\tau(\xi),\xi)\right|$ because $y_\xi(t,\xi) = 0$ for $t \geq \tau(\xi)$ and we have proved (3.6). The inequality (3.7) is proved in the same way. □

We can now start the proof of the short time stability theorem, Theorem 3.1.

Proof of Theorem 3.1. We divide the proof in several steps. Without loss of generality, we assume $T \leq 1$.

***Step 1**: Estimates for* $\|P - \bar{P}\|_{L^1_T(L^\infty_{\mathbb{R}} \cap L^2_{\mathbb{R}})}$ *and* $\|Q - \bar{Q}\|_{L^1_T(L^\infty_{\mathbb{R}} \cap L^2_{\mathbb{R}})}$.
For ξ such that $\tau(\xi) < \bar{\tau}(\xi)$, we obtain after using Gronwall's lemma in (2.19), that

$$|(Z - \bar{Z})(\tau, \xi)| \leq C(M)\big(\,|(Z - \bar{Z})(0, \xi)| + \|(F(X) - F(\bar{X}))Z\|_{L^1_T L^\infty_{\mathbb{R}}}\,\big). \tag{3.10}$$

We have

$$\begin{aligned}(F(X) - \bar{F}(\bar{X}))Z = \Big(0, \big(\frac{1}{2}(U^2 - \bar{U}^2) - (P - \bar{P})\big)y_\xi,\\ -2(QU - \bar{Q}\bar{U})y_\xi + \big(3(U^2 - \bar{U}^2) - 2(P - \bar{P})\big)U_\xi\Big).\end{aligned} \tag{3.11}$$

Hence, as $\|P\|_{L^\infty_T L^\infty_{\mathbb{R}}} + \|Q\|_{L^\infty_T L^\infty_{\mathbb{R}}} \leq C(M)$, see (2.34), we have

$$\begin{aligned}\|(F(X) - F(\bar{X}))Z\|_{L^1_T L^\infty_{\mathbb{R}}} \leq C(M)\big(\,\|U - \bar{U}\|_{L^1_T L^\infty_{\mathbb{R}}}\\ + \|P - \bar{P}\|_{L^1_T L^\infty_{\mathbb{R}}} + \|Q - \bar{Q}\|_{L^1_T L^\infty_{\mathbb{R}}}\,\big)\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\|(F(X) - F(\bar{X}))Z\|_{L^1_T L^2_{\mathbb{R}}} \leq C(M)\big(\,\|U - \bar{U}\|_{L^1_T L^2_{\mathbb{R}}}\\ + \|P - \bar{P}\|_{L^1_T L^2_{\mathbb{R}}} + \|Q - \bar{Q}\|_{L^1_T L^2_{\mathbb{R}}}\,\big).\end{aligned} \tag{3.13}$$

From (3.10) and (3.12), we get

$$\begin{aligned}|(Z - \bar{Z})(\tau, \xi)| \leq C(M)\big(\,|(Z - \bar{Z})(0, \xi)| + \|U - \bar{U}\|_{L^1_T L^\infty_{\mathbb{R}}}\\ + \|Q - \bar{Q}\|_{L^1_T L^\infty_{\mathbb{R}}} + \|P - \bar{P}\|_{L^1_T L^\infty_{\mathbb{R}}}\,\big).\end{aligned} \tag{3.14}$$

From (2.18), we get

$$\|U - \bar{U}\|_{L^\infty_T L^\infty_{\mathbb{R}}} \leq \|U_0 - \bar{U}_0\|_{L^\infty_{\mathbb{R}}} + \|Q - \bar{Q}\|_{L^1_T L^\infty_{\mathbb{R}}}$$

and (3.14) rewrites

$$|(Z - \bar{Z})(\tau, \xi)| \leq C(M)\big(\,|(Z - \bar{Z})(0, \xi)| + \|Q - \bar{Q}\|_{L^1_T L^\infty_{\mathbb{R}}} + \|P - \bar{P}\|_{L^1_T L^\infty_{\mathbb{R}}}\,\big). \tag{3.15}$$

To simplify the notation, let us introduce $z(\xi)$ and $\bar{z}(\xi)$ defined by

$$z(\xi) = \chi_{\{\tau(\xi) < \bar{\tau}(\xi)\}}(\xi)\Big((\bar{\tau} - \tau)(\xi) + \int_\tau^{\bar{\tau}} h(t, \xi)\,dt\Big)$$

and

$$\bar{z}(\xi) = \chi_{\{\bar{\tau}(\xi) < \tau(\xi)\}}(\xi)\Big((\tau - \bar{\tau})(\xi) + \int_{\bar{\tau}}^{\tau} \bar{h}(t, \xi)\,dt\Big).$$

Then, Lemma 2.4 rewrites as

$$\begin{aligned}\|Q - \bar{Q}\|_{L^1_T(L^2_{\mathbb{R}} \cap L^\infty_{\mathbb{R}})} + \|P - \bar{P}\|_{L^1_T(L^2_{\mathbb{R}} \cap L^\infty_{\mathbb{R}})}\\ \leq C(M)\Big(T\,\|X - \bar{X}\|_{L^\infty_T V} + \|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}}\Big).\end{aligned} \tag{3.16}$$

***Step 2**: Estimates for* $\|z\|_{L^1_{\mathbb{R}}}$ *and* $\|\bar{z}\|_{L^1_{\mathbb{R}}}$.
Let us introduce the set $A = \{\xi \in \mathbb{R} \mid d(X(0, \xi), X(0, \xi)) \leq \delta\}$ where δ, which

depends only on M, is given by Lemma 3.4. From (3.15), (3.16) and Lemma 3.4, we infer that, for $\xi \in A$,

$$z(\xi) \leq C(M)\Big(\left|(Z-\bar{Z})(0,\xi)\right| + \left\|U_0 - \bar{U}_0\right\|_{L^\infty_{\mathbb{R}}} + T\left\|X - \bar{X}\right\|_{L^\infty_T V} + \|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}}\Big).$$

Hence,

$$\begin{aligned}(3.17)\quad z(\xi) \leq C(M) d(X_0(\xi),\bar{X}_0(\xi)) &+ C(M)\left\|U_0-\bar{U}_0\right\|_{L^\infty_{\mathbb{R}}}\\ &+ C(M)\Big(T\left\|X-\bar{X}\right\|_{L^\infty_T V} + \|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}}\Big).\end{aligned}$$

The same estimate holds for $\bar{z}(\xi)$. From (2.19), we get

$$(3.18)\qquad \frac{1}{2}\int_0^T h(t)\,dt = U(T,\xi) - U(0,\xi) - \int_0^T (\frac{1}{2}U^2(t,\xi) - P(t,\xi))y_\xi(t,\xi)\,dt$$

so that $\int_0^T h(t,\xi)\,dt \leq C(M)$. For the same reasons, we have $\int_0^T \bar{h}(t,\xi)\,dt \leq C(M)$ and therefore $z(\xi) + \bar{z}(\xi) \leq C(M)$. It implies that the inequality (3.17) holds also for $\xi \in A^c$, that is, those ξ which satisfy $d(X_0(\xi),\bar{X}_0(\xi)) \geq \delta$, as

$$z(\xi) \leq C(M)\frac{d(X_0(\xi),\bar{X}_0(\xi))}{\delta} \leq C(M)\, d(X_0(\xi),\bar{X}_0(\xi)),$$

and therefore inequality (3.17) holds for every $\xi \in \mathbb{R}$. Let us introduce

$$\mathcal{C}_\tau^T = \{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T\} \text{ and } \mathcal{C}_{\bar{\tau}}^T = \{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T\}.$$

By the assumptions of the theorem we have $\operatorname{meas}(\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T) \leq \varepsilon$ for some ε we have to determine, and we may assume without loss of generality that $\varepsilon \leq 1$. We have, after using Cauchy–Schwarz, that

$$(3.19)\quad \int_{\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T} d(X_0(\xi),\bar{X}_0(\xi))\,d\xi \leq (\operatorname{meas}(\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T)^{1/2} + 1) d_{\mathbb{R}}(X_0,\bar{X}_0) \leq 2 d_{\mathbb{R}}(X_0,\bar{X}_0)$$

and

$$(3.20)\qquad \int_{\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T} \left\|U_0-\bar{U}_0\right\|_{L^\infty_{\mathbb{R}}} \leq C(M)\left\|U_0-\bar{U}_0\right\|_{H^1(\mathbb{R})} \leq C(M) d_{\mathbb{R}}(X_0,\bar{X}_0)$$

by the Sobolev embedding $L^\infty(\mathbb{R}) \subset H^1(\mathbb{R})$. We integrate (3.17) and its counterpart with $\bar{z}$ over $\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T$. Since $\bar{\tau}(\xi) - \tau(\xi) = 0$ on the complement of $\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T$ we obtain, after using (3.19) and (3.20), that

$$\begin{aligned}(3.21)\quad \|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}} \leq C(M) d_{\mathbb{R}}(X_0,\bar{X}_0) &+ C(M) T\left\|X-\bar{X}\right\|_{L^\infty_T V}\\ &+ C_0(M)\operatorname{meas}(\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T)(\|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}})\end{aligned}$$

for some constant $C_0(M)$ only depending on M. Let $\varepsilon = (2C_0(M))^{-1}$, if T is such that $\operatorname{meas}(\mathcal{C}_\tau^T \cup \mathcal{C}_{\bar{\tau}}^T) \leq \varepsilon$, then (3.21) implies

$$(3.22)\qquad \|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}} \leq C(M) d_{\mathbb{R}}(X_0,\bar{X}_0) + C(M) T\left\|X-\bar{X}\right\|_{L^\infty_T V}.$$

Step 3: *Estimates for* $\left\|Z-\bar{Z}\right\|_{L^\infty_T W}$.

With the estimate (3.22) on $\|z\|_{L^1_{\mathbb{R}}} + \|\bar{z}\|_{L^1_{\mathbb{R}}}$, we can rewrite (3.16) as follows

$$\begin{aligned}(3.23)\quad \left\|Q-\bar{Q}\right\|_{L^1_T(L^2_{\mathbb{R}}\cap L^\infty_{\mathbb{R}})} &+ \left\|P-\bar{P}\right\|_{L^1_T(L^2_{\mathbb{R}}\cap L^\infty_{\mathbb{R}})}\\ &\leq C(M) d_{\mathbb{R}}(X_0,\bar{X}_0) + C(M) T\left\|X-\bar{X}\right\|_{L^\infty_T V}.\end{aligned}$$

Moreover, by integrating $(\zeta-\bar{\zeta})_t = U - \bar{U}$, we obtain that

$$\left\|\zeta-\bar{\zeta}\right\|_{L^\infty_T L^\infty_{\mathbb{R}}} \leq \left\|\zeta(0,\cdot\,)-\bar{\zeta}(0,\cdot\,)\right\|_{L^\infty_{\mathbb{R}}} + T\left\|U-\bar{U}\right\|_{L^\infty_T L^\infty_{\mathbb{R}}}$$

$$\leq d_{\mathbb{R}}(X_0, \bar{X}_0) + CT(\|U - \bar{U}\|_{L^\infty_T L^2_{\mathbb{R}}} + \|U_\xi - \bar{U}_\xi\|_{L^\infty_T L^2_{\mathbb{R}}}) \tag{3.24}$$

where C denotes the constant of the Sobolev embedding $L^\infty \subset H^1$. By integrating $(U - \bar{U})_t = Q - \bar{Q}$, we obtain

$$\|U - \bar{U}\|_{L^\infty_T L^2_{\mathbb{R}}} \leq \|U(0,\cdot) - U(0,\cdot)\|_{L^2_{\mathbb{R}}} + \|Q - \bar{Q}\|_{L^1_T L^2_{\mathbb{R}}} \leq d_{\mathbb{R}}(X_0, \bar{X}_0) + \|Q - \bar{Q}\|_{L^1_T L^2_{\mathbb{R}}} \tag{3.25}$$

and, by (3.25),

$$\|U - \bar{U}\|_{L^\infty_T L^2_{\mathbb{R}}} \leq C(M)\big(d_{\mathbb{R}}(X_0, \bar{X}_0) + T\|X - \bar{X}\|_{L^\infty_T V}\big). \tag{3.26}$$

Let us denote $G = \mathbb{R} \setminus (\mathcal{C}^T_\tau \cup \mathcal{C}^T_{\bar{\tau}} \cup \mathcal{K}_1 \cup \bar{\mathcal{K}}_1)$, see (2.97) for the definition of $\mathcal{K}_1$ and $\bar{\mathcal{K}}_1$. By Lemma 2.10, we have that $\|h\|_{L^\infty_T E(G)} \leq C(M)$ and $\|\bar{h}\|_{L^\infty_T E(G)} \leq C(M)$ as $\|h_0\|_{E(G)}$ and $\|\bar{h}_0\|_{E(G)}$ are bounded. For $\xi \in G$, we have $Z_t = F(X)Z$ and $\bar{Z}_t = F(\bar{X})\bar{Z}$ for all $t \in [0, T]$, and, after applying Gronwall's lemma, we obtain

$$\begin{aligned}\|Z(t,\cdot) - \bar{Z}(t,\cdot)\|_{L^2(G)} \leq C(M)\Big(&\|Z(0,\cdot) - \bar{Z}(0,\cdot)\|_{L^2(G)} \\ &+ \|(F(X) - F(\bar{X}))Z\|_{L^1_T L^2(G)}\Big).\end{aligned} \tag{3.27}$$

From (3.13), it follows that

$$\begin{aligned}\|Z - \bar{Z}\|_{L^\infty_T L^2(G)} \leq C(M)\Big(&\|Z(0,\cdot) - \bar{Z}(0,\cdot)\|_{L^2_{\mathbb{R}}} \\ &+ \|U - \bar{U}\|_{L^1_T L^2_{\mathbb{R}}} + \|P - \bar{P}\|_{L^1_T L^2_{\mathbb{R}}} + \|Q - \bar{Q}\|_{L^1_T L^2_{\mathbb{R}}}\Big)\end{aligned}$$

which implies, after using (3.26) and because $\|h - \bar{h}\|_{L^2(G)} \leq C(M)\|h - \bar{h}\|_{L^1(G)}$, that

$$\|Z - \bar{Z}\|_{L^\infty_T L^2(G)} \leq C(M)(d_{\mathbb{R}}(X_0, \bar{X}_0) + T\|X - \bar{X}\|_{L^\infty_T V}). \tag{3.28}$$

After using (2.23f) we obtain

$$\begin{aligned}\|h - \bar{h}\|_{L^\infty_T L^1(G)} &= \|U^2(1 + \zeta_\xi)^2 + U_\xi^2 - h\zeta_\xi - \bar{U}^2(1 + \bar{\zeta}_\xi)^2 + \bar{U}_\xi^2 + \bar{h}\bar{\zeta}_\xi\|_{L^\infty_T L^1(G)} \\ &\leq C(M)(\|U - \bar{U}\|_{L^\infty_T L^2(G)} + \|Z - \bar{Z}\|_{L^\infty_T L^2(G)}).\end{aligned}$$

Hence, by (3.28), (3.26), we get

$$\|Z - \bar{Z}\|_{L^\infty_T W(G)} \leq C(M)(d_{\mathbb{R}}(X_0, \bar{X}_0) + T\|X - \bar{X}\|_{L^\infty_T V}). \tag{3.29}$$

Let us consider $\xi \in G^c = \mathcal{C}^T_\tau \cup \mathcal{C}^T_{\bar{\tau}} \cup \mathcal{K}_1 \cup \bar{\mathcal{K}}_1$, we assume without loss of generality that $\tau(\xi) \leq \bar{\tau}(\xi) \leq T$. We have $\operatorname{meas}(\mathcal{K}_1) \leq \int_{\mathbb{R}} h_0 \leq M$ and a similar inequality for $\bar{\mathcal{K}}_1$ so that $\operatorname{meas}(G^c) \leq C(M)$. For $t \in [\tau, \bar{\tau}]$, we have $Z(t,\xi) = Z(\tau,\xi)$ and $\bar{Z}_t = F(\bar{X})\bar{Z}$. Hence,

$$\frac{d}{dt}(\bar{Z}(t,\xi) - Z(t,\xi)) = F(\bar{X})(\bar{Z}(t,\xi) - Z(t,\xi)) + F(\bar{X})Z(\tau,\xi)$$

and, after applying Gronwall's lemma, we obtain

$$|\bar{Z}(t,\xi) - Z(t,\xi)| \leq C(M)\big(|\bar{Z}(\tau,\xi) - Z(\tau,\xi)| + \int_\tau^{\bar{\tau}} |F(\bar{X})Z(\tau,\xi)|\,dt\big) \tag{3.30}$$

because $\|F(\bar{X})\|_{L^1_T L^\infty_{\mathbb{R}}} \leq C(M)$. We have $F(\bar{X})Z(\tau,\xi) = (0, \frac{1}{2}h(\tau,\xi), 0)$ because $y_\xi(\tau,\xi) = U_\xi(\tau,\xi) = 0$. Hence,

$$\int_\tau^{\bar{\tau}} |F(\bar{X})Z(\tau,\xi)|\,dt = \frac{1}{2}\int_\tau^{\bar{\tau}} h(\tau,\xi)\,dt \leq \frac{1}{2}\int_\tau^{\bar{\tau}} |h(t,\xi) - \bar{h}(t,\xi)|\,dt + \int_\tau^{\bar{\tau}} \bar{h}(t,\xi)\,dt$$

and (3.30) becomes, after using (3.6),

$$\left|\bar{Z}(t,\xi)-Z(t,\xi)\right| \leq C(M)\big(\left|\bar{Z}(\tau,\xi)-Z(\tau,\xi)\right|+\frac{1}{2}T\sup_{t\in[\tau,\bar{\tau}]}\left|\bar{h}(t,\xi)-h(t,\xi)\right|\big).$$

Hence, for T small enough, we obtain

$$\left|\bar{Z}(t,\xi)-Z(t,\xi)\right| \leq C(M)\left|\bar{Z}(\tau,\xi)-Z(\tau,\xi)\right|. \tag{3.31}$$

For $t\leq\tau(\xi)$, we have $Z_t=F(X)Z$ and $\bar{Z}_t=F(\bar{X})\bar{Z}$ and, after applying Gronwall's lemma, we obtain

$$\left|\bar{Z}(t,\xi)-Z(t,\xi)\right| \leq C(M)\big(\left|\bar{Z}(0,\xi)-Z(0,\xi)\right|+\left\|(F(X)-F(\bar{X}))Z\right\|_{L^1_TL^\infty_\mathbb{R}}\big),$$

which, by (3.12), gives

$$\left|\bar{Z}(t,\xi)-Z(t,\xi)\right| \leq C(M)\big(\left|\bar{Z}(0,\xi)-Z(0,\xi)\right|+\left\|P-\bar{P}\right\|_{L^1_TL^\infty_\mathbb{R}}+\left\|Q-\bar{Q}\right\|_{L^1_TL^\infty_\mathbb{R}}\big)$$

and, using (3.23),

$$\left|\bar{Z}(t,\xi)-Z(t,\xi)\right| \leq C(M)\big(\left|\bar{Z}(0,\xi)-Z(0,\xi)\right|+d_\mathbb{R}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_TV}\big). \tag{3.32}$$

Due to (3.31), the inequality (3.32) holds not only for $t\in[0,\tau]$ but also for $t\in[0,T]$. We integrate (3.32) and get

$$\begin{aligned}\left\|\bar{Z}(t,\cdot)-Z(t,\cdot)\right\|_{L^1(G^c)} &\leq C(M)\left\|\bar{Z}(0,\cdot)-Z(0,\cdot)\right\|_{L^1(G^c)}\\ &\quad+C(M)\operatorname{meas}(G^c)(d_\mathbb{R}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_TV})\\ &\leq C(M)(d_\mathbb{R}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_TV}) \end{aligned}\tag{3.33}$$

because $\operatorname{meas}(G^c)\leq 1$. Since $\|\zeta_\xi\|_{L^\infty_TL^\infty_\mathbb{R}}$, $\|U_\xi\|_{L^\infty_TL^\infty(\mathbb{R})}$, $\left\|\bar{\zeta}_\xi\right\|_{L^\infty_TL^\infty_\mathbb{R}}$ and $\left\|\bar{U}_\xi\right\|_{L^\infty_TL^\infty_\mathbb{R}}$ are bounded by some constant depending only on M, we have

$$\left\|Z(t,\cdot)-\bar{Z}(t,\cdot)\right\|_{W(G^c)} \leq C(M)\left\|\bar{Z}(t,\cdot)-\bar{Z}(t,\cdot)\right\|_{L^1(G^c)},$$

and (3.33) yields

$$\left\|\bar{Z}(t,\cdot)-Z(t,\cdot)\right\|_{W(G^c)} \leq C(M)(d_\mathbb{R}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_TV}).$$

Hence, combining (3.34), (3.29), we get

$$\left\|\bar{Z}-Z\right\|_{L^\infty_TW} \leq C(M)(d_\mathbb{R}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_TV}). \tag{3.34}$$

Step 4: *Estimates for* $\left\|g(X(t,\cdot)-g(\bar{X}(t,\cdot))\right\|_{L^2(\mathbb{R})}$.
We consider the constant δ which depends only on M (as we assume $T\leq 1$) given by Lemma 3.3. If ξ is such that $d(X_0(\xi),\bar{X}_0(\xi))<\delta$, then (3.4) holds. If ξ is such that $d(X_0(\xi),\bar{X}_0(\xi))\geq\delta$, we have $\left|g(X(t,\xi))-g(\bar{X}(t,\xi))\right|\leq\frac{C(M)}{\delta}d(X_0(\xi),\bar{X}_0(\xi))$ because $\|g(X(t,\cdot))\|_{L^\infty}$ and $\left\|g(\bar{X}(t,\cdot))\right\|_{L^\infty}$ are bounded by some constant depending only on M. In any case we have

$$\begin{aligned}&\left|g(X(t,\xi))-g(\bar{X}(t,\xi))\right|\\ &\qquad\leq C(M)\big(\left|U(t,\xi)-\bar{U}(t,\xi)\right|+\left|Z(t,\xi)-\bar{Z}(t,\xi)\right|+d(X_0(\xi),\bar{X}_0(\xi))\big).\end{aligned}\tag{3.35}$$

We integrate (3.35) over G and get

$$\begin{aligned}&\left\|g(X)-g(\bar{X})\right\|_{L^\infty_TL^2(G)}\\ &\qquad\leq C(M)(\left\|U-\bar{U}\right\|_{L^\infty_TL^2_\mathbb{R}}+\left\|Z-\bar{Z}\right\|_{L^\infty_TL^2_\mathbb{R}}+\left\|d(X_0,\bar{X}_0)\right\|_{L^2(G)}).\end{aligned}\tag{3.36}$$

Since h_0 and $\bar{h}_0$ are bounded on G by a constant that depends only on M, we have

$$\left\|d(X_0,\bar{X}_0)\right\|_{L^2(G)} \leq C(M)d_\mathbb{R}(X_0,\bar{X}_0).$$

Hence, after using (3.28) and (3.26), (3.36) implies

$$\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^2(G)} \leq C(M)(d_{\mathbb{R}}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_T V}). \tag{3.37}$$

We integrate (3.35) over G^c and get

$$\begin{aligned}\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^1(G^c)} \\ \leq C(M)(\left\|U-\bar{U}\right\|_{L^\infty_T L^1(G^c)}+\left\|Z-\bar{Z}\right\|_{L^\infty_T L^1(G^c)}+\left\|d(X_0,\bar{X}_0)\right\|_{L^1(G^c)}).\end{aligned} \tag{3.38}$$

Since $\operatorname{meas}(G^c)\leq 1$, we get, after using (3.34) and (3.26), that

$$\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^1(G^c)} \leq C(M)(d_{\mathbb{R}}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_T V}). \tag{3.39}$$

We have that $\|g(X)-1\|_{L^\infty_T L^\infty(\mathbb{R})}$ and $\left\|g(\bar{X})-1\right\|_{L^\infty_T L^\infty(\mathbb{R})}$ are bounded by a constant depending only on $C(M)$ and therefore

$$\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^2(G^c)} \leq C(M)\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^1(G^c)}.$$

Hence, (3.39) implies

$$\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^2(G^c)} \leq C(M)(d_{\mathbb{R}}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_T V}),$$

which, combined with (3.37), gives

$$\left\|g(X)-g(\bar{X})\right\|_{L^\infty_T L^2(\mathbb{R})} \leq C(M)(d_{\mathbb{R}}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_T V}). \tag{3.40}$$

Step 4: *Conclusion.*
Gathering (3.26), (3.34) and (3.40), we get

$$\begin{aligned}\sup_{t\in[0,T]} d_{\mathbb{R}}(X(t,\cdot),X(t,\cdot)) &\leq C(M)(d_{\mathbb{R}}(X_0,\bar{X}_0)+T\left\|X-\bar{X}\right\|_{L^\infty_T V}) \\ &\leq C(M)(d_{\mathbb{R}}(X_0,\bar{X}_0)+T\sup_{t\in[0,T]} d_{\mathbb{R}}(X(t,\cdot),X(t,\cdot)))\end{aligned}$$

which, after choosing T small enough, implies (3.3). □

4. Global stability

From the short time stability result, Theorem 3.1, we obtain global stability.

Theorem 4.1. *For any time $T>0$, there exists a constant K only depending on M and T such that*

$$\sup_{t\in[0,T]} d_{\mathbb{R}}(X(t),\bar{X}(t)) \leq K d_{\mathbb{R}}(X_0,\bar{X}_0) \tag{4.1}$$

for any solutions $X(t)$ and $\bar{X}(t)$ with initial data in X_0 and $\bar{X}_0$ in B_M.

We first prove the following result about the continuity in time of the solutions with respect to the distance $d_{\mathbb{R}}$.

Lemma 4.2. *(i) The solutions of* (2.13) *are continuous in time with respect to the distance $d_{\mathbb{R}}$, that is,*

$$\lim_{t\to\bar{t}} d_{\mathbb{R}}(X(t),X(\bar{t}))=0.$$

(ii) Given M, there exists $\bar{T}$ depending only on M such that, for any solution $X(t)$ with initial data X_0 in B_M and $t\leq\bar{T}$, we have

$$d_{\mathbb{R}}(X(t),X_0)\leq C(M)t$$

where $C(M)$ is a constant that only depends on M.

Proof. We proceed as we have done several times now and consider a domain where h_0 is bounded and another where h_0 may be unbounded but which has finite measure. Let $\mathcal{B} = \mathcal{K}_1^c$, applying Lemma 2.10, we have $\|Z\|_{L_T^\infty E(\mathcal{B})} \leq C(M)$ for some constant depending only on M and T. We denote generically by $C(M)$ such constant. We have

$$\|U(t,\cdot\,) - U(\bar{t},\cdot\,)\|_{L^2(\mathbb{R})} \leq \int_t^{\bar{t}} \left\|Q(\tilde{t},\cdot\,)\right\|_{L^2(\mathbb{R})} d\tilde{t} \leq C(M)\left|t-\bar{t}\right|, \tag{4.2}$$

$$\|\zeta(t,\cdot\,) - \zeta(\bar{t},\cdot\,)\|_{L^\infty(\mathbb{R})} \leq \int_t^{\bar{t}} \left\|U(\tilde{t},\cdot\,)\right\|_{L^\infty(\mathbb{R})} d\tilde{t} \leq C(M)\left|t-\bar{t}\right|. \tag{4.3}$$

Since $\|Z\|_{L_T^\infty E(\mathcal{B})} \leq C(M)$, we have

$$\|Z(t,\cdot\,) - Z(\bar{t},\cdot\,)\|_{L^2(\mathcal{B})} \leq \int_t^{\bar{t}} \left\|F(X(\tilde{t},\cdot\,))Z(\tilde{t},\cdot\,)\right\|_{L^2(\mathbb{R})} d\tilde{t} \leq C(M)\left|t-\bar{t}\right|. \tag{4.4}$$

After using (2.23f), we obtain

$$\begin{aligned}
&\left\|h(t,\cdot\,) - \bar{h}(\bar{t},\cdot\,)\right\|_{L^1(\mathcal{B})} \\
&\quad = \left\|(U^2(1+\zeta_\xi) + U_\xi^2 - h\zeta_\xi)(t,\cdot\,) - (U^2(1+\zeta_\xi) + U_\xi^2 - h\zeta_\xi)(\bar{t},\cdot\,)\right\|_{L^1(\mathcal{B})} \\
&\quad \leq C(M)(\left\|U(t,\cdot\,) - \bar{U}(\bar{t},\cdot\,)\right\|_{L^2(\mathcal{B})} + \left\|Z(t,\cdot\,) - \bar{Z}(\bar{t},\cdot\,)\right\|_{L^2(\mathcal{B})}),
\end{aligned}$$

and (4.4) yields

$$\|Z(t,\cdot\,) - Z(\bar{t},\cdot\,)\|_{W(\mathcal{B})} \leq C(M)\left|t-\bar{t}\right|. \tag{4.5}$$

By Chebyshev's inequality, we have $\mathrm{meas}(\mathcal{B}^c) \leq C(M)$ and therefore $\|F(X)\|_{L_T^\infty L^1(\mathcal{B}^c)} \leq \mathrm{meas}(\mathcal{B}^c)\,\|F(X)\|_{L_T^\infty L^\infty(\mathcal{B}^c)} \leq C(M)$ and, after using Cauchy–Schwarz, $\|Z\|_{L_T^\infty L^1(\mathcal{B}^c)} \leq (\mathrm{meas}(\mathcal{B}^c)^{1/2} + 1)\,\|Z\|_{L_T^\infty W} \leq C(M)$. Then

$$\begin{aligned}
\|Z(t,\cdot\,) - Z(\bar{t},\cdot\,)\|_{L^1(\mathcal{B})} &\leq \int_t^{\bar{t}} \left\|F(X(\tilde{t},\cdot\,))Z(\tilde{t},\cdot\,)\right\|_{L^1(\mathbb{R})} d\tilde{t} \\
&\leq C(M)\left|t-\bar{t}\right|,
\end{aligned}$$

which implies, as $\|\zeta_\xi\|_{L^\infty(\mathbb{R})} + \|U_\xi\|_{L^\infty(\mathbb{R})} \leq C(M)$ that

$$\|Z(t,\cdot\,) - Z(\bar{t},\cdot\,)\|_{W(\mathcal{B}^c)} \leq C(M)\left|t-\bar{t}\right|. \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$\|Z(t,\cdot\,) - Z(\bar{t},\cdot\,)\|_{W} \leq C(M)\left|t-\bar{t}\right|,$$

which together with (4.2) and (4.3) implies

$$\left\|X(t,\cdot\,) - \bar{X}(\bar{t},\cdot\,)\right\|_{V} \leq C(M)\left|t-\bar{t}\right|. \tag{4.7}$$

For a given $\xi \in \mathbb{R}$, we have established in the proof of Lemma 3.3 that if $X(\bar{t},\xi) \in \Omega_0$, then $X(t,\xi)$ remains in Ω_- for a short time interval, while, if $X(\bar{t},\xi) \in \Omega_0^c$, $X(t,\xi)$ remains in Ω^c also for a short time interval. The length of this short time interval (given by T in the proof of Lemma 3.3) is controlled by M. Hence, there exists $\bar{T} > 0$ depending only on M such that, for a given $\xi \in \mathbb{R}$, either $X(t,\xi) \in \Omega^-$ for $t \in [\bar{t}-\bar{T}, \bar{t}+\bar{T}]$ or $X(t,\xi) \in \Omega^+$ for $t \in [\bar{t}-\bar{T}, \bar{t}+\bar{T}]$. Then, it follows from Lemma 3.2 that

$$|g(X(t,\xi)) - g(X(\bar{t},\xi))| \leq \kappa(|U(t,\xi) - U(\bar{t},\xi)| + |Z(t,\xi) - Z(\bar{t},\xi)|) \tag{4.8}$$

for a constant κ that depends only on M. From (4.4) and (4.2) we get

$$\begin{aligned}
\|g(X(t,\cdot\,)) - g(X(\bar{t},\cdot\,))\|_{L^2(\mathcal{B})} \leq \kappa(&\|U(t,\cdot\,) - U(\bar{t},\cdot\,)\|_{L^2(\mathcal{B})} \\
&+ \|Z(t,\cdot\,) - Z(\bar{t},\cdot\,)\|_{L^2(\mathcal{B})})
\end{aligned}$$

$$\leq C(M)\,|t-\bar t|\,. \tag{4.9}$$

On $\mathcal{B}^c$, (4.8) gives

$$\begin{aligned}\|g(X(t,\cdot\,))-g(X(\bar t,\cdot\,))\|_{L^1(\mathcal{B})} &\leq \kappa(\|U(t,\cdot\,)-U(\bar t,\cdot\,)\|_{L^1(\mathcal{B})}+\|Z(t,\cdot\,)-Z(\bar t,\cdot\,)\|_{L^1(\mathcal{B})})\\ &\leq C(M)\,\|X(t,\cdot\,)-X(\bar t,\cdot\,)\|_V\,. \end{aligned}\tag{4.10}$$

Since g is bounded (4.10) implies

$$\|g(X(t,\cdot\,))-g(X(\bar t,\cdot\,))\|_{L^2(\mathcal{B})}\leq C(M)\,\|X(t,\cdot\,)-X(\bar t,\cdot\,)\|_V$$

and therefore, by (4.9) and (4.7),

$$\|g(X(t,\cdot\,))-g(X(\bar t,\cdot\,))\|_{L^2(\mathbb{R})}\leq C(M)\,|t-\bar t|\,.$$

Hence,

$$d_{\mathbb{R}}(X(t,\cdot\,),X(\bar t,\cdot\,))\leq C(M)\,|t-\bar t|\,. \tag{4.11}$$

By letting t tends to $\bar t$ we prove *(i)* and by taking $\bar t=0$ we prove *(ii)*. □

Proof of Theorem 4.1. From Lemma 2.10 there exists $\bar M$ depending only on M and T such that $X(t)$ and $\bar X(t)$ belong to $B_{\bar M}$ for all $t\in[0,T]$. From Theorem 3.1, there exist constants $\bar T$, ε and $\bar K$ depending only on $\bar M$ such that

$$d_{\mathbb{R}}(X(t),\bar X(t))\leq K d_{\mathbb{R}}(X(\bar t),\bar X(\bar t)) \tag{4.12}$$

for any t such that $\bar t\leq t\leq \bar t\bar T$ and $\operatorname{meas}(\{\xi\in\mathbb{R}\mid \bar t<\tau(\xi)<t\})+\operatorname{meas}(\{\xi\in\mathbb{R}\mid \bar t<\bar\tau(\xi)<t\})\leq\varepsilon$. To obtain global stability, we have to split the interval $[0,T]$ in small time intervals where (4.12) can be used. We define the increasing sequence of times t_i as follows: Let $t_0=0$ and

$$\begin{aligned}t_{i+1}=\sup\big\{t\in[t_i,t_i+\bar T]\mid \operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\tau(\xi)<t\})&\\ +\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\bar\tau(\xi)<t\})\leq\varepsilon\big\}&.\end{aligned}$$

Introduce subsets I_i of $\mathbb{N}$, with $i=1,2,3$, that characterize how t_{i+1} are chosen:
(i) $i\in I_1$ if $t_{i+1}=t_i+\bar T$;
(ii) $i\in I_2$ if $i\notin I_1$ and $\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\tau(\xi)\leq t_{i+1}\})\geq\frac{\varepsilon}{3}$;
(iii) $i\in I_3$ if $i\notin I_1\cup I_2$ and $\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\bar\tau(\xi)\leq t_{i+1}\})\geq\frac{\varepsilon}{3}$.

Let us prove that we cover all the cases, that is, $I_1\cup I_2\cup I_3=\mathbb{N}$. Assume that there exists $i\notin I_1\cup I_2\cup I_3$. Then, we have $t_i<t_{i+1}<t_i+\bar T$ and

$$\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\tau(\xi)\leq t_{i+1}\})+\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\bar\tau(\xi)\leq t_{i+1}\})\leq\frac{2}{3}\varepsilon. \tag{4.13}$$

Since $\cap_{n\in\mathbb{N}}\big\{\xi\in\mathbb{R}\mid t_{i+1}<\tau(\xi)<t_{i+1}+\frac1n\big\}=\cap_{n\in\mathbb{N}}\big\{\xi\in\mathbb{R}\mid t_{i+1}<\bar\tau(\xi)<t_{i+1}+\frac1n\big\}=\emptyset$, there exists $\tilde t\in(t_{i+1},t_i+\bar T)$ such that

$$\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_{i+1}<\tau(\xi)<\tilde t\})<\frac{\varepsilon}{6}\quad\text{and}\quad\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_{i+1}<\bar\tau(\xi)<\tilde t\})<\frac{\varepsilon}{6}.$$

Hence,

$$\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\tau(\xi)<\tilde t\})+\operatorname{meas}(\{\xi\in\mathbb{R}\mid t_i<\bar\tau(\xi)<\tilde t\})<\varepsilon$$

which contradicts the definition of t_{i+1}, and we have proved that $I_1\cup I_2\cup I_3=\mathbb{N}$.

We have to prove that the partition we have obtained reaches T, that is, that there exists N_0 such that $t_{N_0}\geq T$. Let us consider any N such that $t_N\leq T$. Introduce

$$\mathcal{A}=\big\{\xi\in\mathbb{R}\mid 0<\tau(\xi)\leq T\big\}\text{ and }\bar{\mathcal{A}}=\big\{\xi\in\mathbb{R}\mid 0<\bar\tau(\xi)\leq T\big\}.$$

We have, as $\zeta_\xi(T,\xi)=-1$ for $\xi\in\mathcal{A}$, that

$$\operatorname{meas}(\mathcal{A})=\int_{\mathcal{A}}\zeta_\xi(T,\xi)^2\,d\xi\leq\|\zeta_\xi(T,\cdot\,)\|^2_{L^2(\mathbb{R})}\leq\bar M^2.$$

We have $\bigcup_{i\in I_2\cap[1,N]}\{\xi\in\mathbb{R}\mid t_i<\tau(\xi)\leq t_{i+1}\}\subset\mathcal{A}$ and $\bigcup_{i\in I_3\cap[1,N]}\{\xi\in\mathbb{R}\mid t_i<\bar\tau(\xi)\leq t_{i+1}\}\subset\bar{\mathcal{A}}$, and the sets in each of these unions are disjoint. Hence, from the definition of I_2, we obtain

$$\frac{\varepsilon}{3}\sharp\big(I_2\cap[1,N]\big)\leq\operatorname{meas}\Big(\bigcup_{i\in I_2\cap[1,N]}\{\xi\in\mathbb{R}\mid t_i<\tau(\xi)\leq t_{i+1}\}\Big)\leq\operatorname{meas}(\mathcal{A})\leq\bar M^2$$

and

$$\sharp\big(I_2\cap[1,N]\big)\leq\frac{3\bar M^2}{\varepsilon}. \tag{4.14}$$

Similarly, we have

$$\sharp\big(I_3\cap[1,N]\big)\leq\frac{3\bar M^2}{\varepsilon}. \tag{4.15}$$

The number of elements in I_1 is also bounded,

$$\sharp\big(I_1\cap[1,N]\big)\leq\frac{T}{\bar{\bar T}}. \tag{4.16}$$

Hence, from (4.14), (4.15) and (4.16), as $I_1\cup I_2\cup I_3=\mathbb{N}$, we obtain

$$N\leq\frac{6\bar M^2}{\varepsilon}+\frac{T}{\bar{\bar T}}. \tag{4.17}$$

Let N_0 be the smallest integer which does not satisfy (4.17). We have $t_{N_0}\geq T$, and N_0 is bounded by a constant which depends on M and T but which is independent of the particular solution X and $\bar X$ we have considered. From the definition of t_{i+1} and (4.12) we get

$$d_{\mathbb{R}}(X(t),\bar X(t))\leq\bar K d_{\mathbb{R}}(X(t_i),X(t_i)) \tag{4.18}$$

for all $t\in[t_i,t_{i+1})$, where the constant $\bar K$ only depends on M and $\bar T$. Due to Lemma 4.2, the inequality (4.18) holds for $t\in[t_i,t_{i+1}]$. We can combine those inequalities to obtain

$$d_{\mathbb{R}}(X(t),\bar X(t))\leq\bar K^{N_0}d_{\mathbb{R}}(X_0,X_0)$$

for all $t\in[0,t_{N_0}]$. Since $t_{N_0}\geq T$ and N_0 only depends on M and T, we obtain (4.1) for $K=\bar K^{N_0}$. □

In order to take into account the fact that energy disappears at collisions, we introduce yet another metric. For Ω a subset of $\mathbb{R}$, we define the mapping $\tilde d_\Omega\colon\mathcal{G}\times\mathcal{G}\to\mathbb{R}_+$ as

$$\tilde d_\Omega(X,\bar X)=\big\|\zeta-\bar\zeta\big\|_{L^\infty_\Omega}+\big\|U-\bar U\big\|_{L^2_\Omega}+\big\|\zeta_\xi-\bar\zeta_\xi\big\|_{L^2_\Omega}\\+\big\|U_\xi-\bar U_\xi\big\|_{L^2_\Omega}+\big\|g(X)-g(\bar X)\big\|_{L^2_\Omega}$$

for X, $\bar X$ in $\mathcal{G}$. However, $\tilde d_{\mathbb{R}}$ does not define a distance on $\mathcal{G}$: It satisfies all the axioms of a distance but the Hausdorff condition, as $\tilde d_{\mathbb{R}}(X,\bar X)=0$ does not imply $X=\bar X$. Indeed, consider $X\in\mathcal{G}$ such that $y_\xi=0$ on some interval I of strictly positive and finite measure. Then, let us define $\bar X$ as $\bar y=y$, $\bar U=U$ on $\mathbb{R}$ and $\bar h=h$ on $\mathbb{R}\setminus I$ and $\bar h=h+1$ on I. We have $g(\bar X)=g(X)$ for all $x\in\mathbb{R}$ because $g(\bar X)=g(X)=0$ on I and otherwise X and $\bar X$ coincide. Hence, $\tilde d_{\mathbb{R}}(X,\bar X)=0$ but $X\neq\bar X$. However, it is not a problem and even a necessary condition: Changing the energy density, h, in a region which is not going to interact anymore (because $y_\xi=0$) should not have any consequence on the evolution of the system. Theorem 4.1 gives us that the semigroup $S_t\colon X_0\to X(t)$ is continuous with respect to the distance $d_{\mathbb{R}}$ on bounded sets of $\mathcal{G}$. In the next theorem, we see that S_t is also continuous with respect to the weaker metric $\tilde d_{\mathbb{R}}$.

Theorem 4.3. *The semigroup $S_t : X_0 \to X(t)$ of solutions of* (2.18) *and* (2.19) *is continuous with respect to the distance $\tilde{d}_{\mathbb{R}}$ on bounded sets of $\mathcal{G}$, that is, given $M \geq 0$, for any sequence X_0^n and X_0 in B_M, we have*

$$\lim_{n\to\infty} \tilde{d}_{\mathbb{R}}(X_0^n, X_0) = 0 \text{ implies } \lim_{n\to\infty} \tilde{d}_{\mathbb{R}}(S_t(X_0^n), S_t(X_0)) = 0$$

for all t.

Proof. Let $X(t)$ and $\bar{X}(t)$ be two solutions with initial data X_0 and $\bar{X}_0$ in B_M. We denote by $\mathcal{C}^T$ the set of particles which have collided before a given time T, that is,

$$\mathcal{C}^T = \{\xi \in \mathbb{R} \mid 0 < \tau(\xi) < T \text{ and } 0 < \bar{\tau}(\xi) < T\}.$$

From Lemma 2.9, there exists an ε depending on T and M such that

$$\mathcal{K}_{\frac{1}{\varepsilon}} \cup \bar{\mathcal{K}}_{\frac{1}{\varepsilon}} \cup \mathcal{A}_\varepsilon \cup \bar{\mathcal{A}}_\varepsilon \subset \mathcal{C}^T. \tag{4.19}$$

Let G_ε denote the complement of the set $\mathcal{K}_{\frac{1}{\varepsilon}} \cup \bar{\mathcal{K}}_{\frac{1}{\varepsilon}} \cup \mathcal{A}_\varepsilon \cup \bar{\mathcal{A}}_\varepsilon$. The mapping $\boldsymbol{x} = (x_2, x_3, x_4) \mapsto x_5 = x_2^2 x_3 + \frac{x_4^2}{x_3}$ is Lipschitz on the set $\{\boldsymbol{x} = (x_2, x_3, x_4) \in \mathbb{R}^3 \mid |\boldsymbol{x}| \leq M \text{ and } |x_3| \geq \varepsilon\}$, that is, there exists a constant $C(\varepsilon, M)$ depending only on ε and M such that $|x_5 - \bar{x}_5| \leq C(\varepsilon, M)\,|\boldsymbol{x} - \bar{\boldsymbol{x}}|$. We denote generically by $C(\varepsilon, M)$ such constants. Since $h_0 = U_0^2 y_{0\xi} + \frac{U_{0\xi}^2}{y_{0\xi}}$ and $\bar{h}_0 = \bar{U}_0^2 \bar{y}_{0\xi} + \frac{\bar{U}_{0\xi}^2}{\bar{y}_{0\xi}}$, by (2.23f), it follows that

$$\begin{aligned}\left\|h_0 - \bar{h}_0\right\|_{L^2(G_\varepsilon)} &\leq C(\varepsilon, M)(\left\|U_0 - \bar{U}_0\right\|_{L^2(G_\varepsilon)} \\ &\qquad + \left\|\zeta_{0\xi} - \bar{\zeta}_{0\xi}\right\|_{L^2(G_\varepsilon)} + \left\|U_{0\xi} - \bar{U}_{0\xi}\right\|_{L^2(G_\varepsilon)}) \\ &\leq C(\varepsilon, M)\tilde{d}_{\mathbb{R}}(X_0, \bar{X}_0). \end{aligned} \tag{4.20}$$

After using (2.23f) again, we obtain

$$\begin{aligned}\left\|h_0 - \bar{h}_0\right\|_{L^1(G_\varepsilon)} &= \left\|U_0^2(1+\zeta_{0\xi}) + U_{0\xi}^2 - h_0\zeta_{0\xi} - U_0^2(1+\zeta_{0\xi}) + U_{0\xi}^2 - h_0\zeta_{0\xi}\right\|_{L^1(G_\varepsilon)} \\ &\leq C(\varepsilon, M)(\left\|U_0 - \bar{U}_0\right\|_{L^2(G_\varepsilon)} + \left\|Z_0 - \bar{Z}_0\right\|_{L^2(G_\varepsilon)}) \\ &\leq C(\varepsilon, M)\tilde{d}_{\mathbb{R}}(X_0, \bar{X}_0)\end{aligned}$$

by (4.20). Hence, we have

$$d_{G_\varepsilon}(X_0, \bar{X}_0) \leq C(\varepsilon, M)\tilde{d}_{\mathbb{R}}(X_0, \bar{X}_0). \tag{4.21}$$

From X and $\bar{X}$, we define $X^a(t,\xi)$ and $\bar{X}^a(t,\xi)$ for $t \geq T$ by resetting the energy to 1 for $\xi \in G_\varepsilon^c$, that is,

$$Z^a(t,\xi) = \begin{cases} Z(t,\xi), & \text{if } \xi \in G_\varepsilon, \\ (0,0,1), & \text{if } \xi \in G_\varepsilon^c, \end{cases} \qquad \bar{Z}^a(t,\xi) = \begin{cases} \bar{Z}(t,\xi), & \text{if } \xi \in G_\varepsilon, \\ (0,0,1), & \text{if } \xi \in G_\varepsilon^c, \end{cases}$$

and $\zeta^a = \zeta$, $U^a = U$, $\bar{\zeta}^a = \bar{\zeta}$, $\bar{U}^a = \bar{U}$. Note that for $\xi \in G_\varepsilon^c$, we could set $h(\xi)$ to any other constant $\alpha > 0$. We claim that X^a and $\bar{X}^a$ are solutions to (2.18) and (2.19) for $t \geq T$. For $t \geq T$, if $\tau(\xi) > t \geq T$, then $\xi \in G_\varepsilon$ because $G_\varepsilon^c \subset \mathcal{C}^T$ and then $\tau^a(\xi) = \tau(\xi)$. Hence, we have that, for $t \geq T$,

$$\begin{aligned}\{\xi \in \mathbb{R} \mid \tau(\xi) > t\} &= \{\xi \in G_\varepsilon \mid \tau(\xi) > t\} \\ &= \{\xi \in G_\varepsilon \mid \tau^a(\xi) > t\} = \{\xi \in \mathbb{R} \mid \tau^a(\xi) > t\}\end{aligned}$$

because $\tau^a(\xi) = T$ for $\xi \in G_\varepsilon^c$. It follows that, for $t \geq T$,

$$\begin{aligned}P^a(t,\xi) &= \frac{1}{4}\int_{\tau^a(\eta)>t} \exp\big(-\operatorname{sgn}(\xi-\eta)(y^a(\xi) - y^a(\eta))\big)\big((U^a)^2 y_\xi^a + h^a\big)(\eta)\,d\eta \\ &= \frac{1}{4}\int_{\{\tau^a(\eta)>t\}\cap G_\varepsilon} \exp\big(-\operatorname{sgn}(\xi-\eta)(y^a(\xi) - y^a(\eta))\big)\big((U^a)^2 y_\xi^a + h^a\big)(\eta)\,d\eta\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4}\int_{\{\tau(\eta)>t\}\cap G_\varepsilon} \exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta\\
&= P(t,\xi).
\end{aligned}
$$

Similarly we get that, for $t\geq T$, $Q^a=Q$, so that $F(X^a)=F(X)$. For $\xi\in G_\varepsilon^c$ and $t\geq T$, we have $Z^a(t,\xi)=(0,0,1)$ and therefore $Z_t^a(t,\xi)=0$. For $\xi\in G_\varepsilon$, $Z^a(t,\xi)$ and $Z(t,\xi)$ coincide. Hence, $Z_t^a(t,\xi)=Z_t(t,\xi)=\chi_{\{\tau(\xi)>t\}}F(X)Z=\chi_{\{\tau^a(\xi)>t\}}F(X^a)Z^a$, and we have proved our claim that X^a satisfies (2.18) and (2.19). By a similar argument we obtain the same result for $\bar X^a$. We have introduced X^a and $\bar X^a$ because, first, they coincide on G_ε^c so that

$$
d_{G_\varepsilon}(X^a(t),\bar X^a(t))=d_{\mathbb R}(X^a(t),\bar X^a(t)) \tag{4.22}
$$

and, second, because they are at distance zero from X and $\bar X$, respectively, as we have

$$
\tilde d_{\mathbb R}(X(t),X^a(t))\leq \tilde d_{G_\varepsilon}(X(t),X^a(t))+\tilde d_{G_\varepsilon^c}(\bar X(t),\bar X^a(t))=0 \tag{4.23}
$$

and $\tilde d_{\mathbb R}(\bar X(t),\bar X^a(t))=0$, for $t\geq T$. We can apply Theorem 4.1 and get that there exists a constant K depending on M on $\bar T$ such that, for any $t\leq\bar T$,

$$
d_{\mathbb R}(X^a(t),\bar X^a(t))\leq Kd_{\mathbb R}(X^a(T),\bar X^a(T))\leq Kd_{G_\varepsilon}(X^a(T),\bar X^a(T)),
$$

by (4.22). Hence, after using (4.23),

$$
\begin{aligned}
\tilde d_{\mathbb R}(X(t),\bar X(t)) &= \tilde d_{\mathbb R}(X^a(t),\bar X^a(t))\\
&\leq d_{\mathbb R}(X^a(t),\bar X^a(t))\\
&\leq Kd_{G_\varepsilon}(X^a(T),\bar X^a(T))\\
&= Kd_{G_\varepsilon}(X(T),\bar X(T))\\
&\leq K(d_{G_\varepsilon}(X_0,\bar X_0)+d_{\mathbb R}(X(T),X_0)+d_{\mathbb R}(\bar X(T),\bar X_0))
\end{aligned}
$$

and therefore

$$
\tilde d_{\mathbb R}(X(t),\bar X(t))\leq C(\varepsilon,M)\tilde d_{\mathbb R}(X_0,\bar X_0)+K(d_{\mathbb R}(X(T),X_0)+d_{\mathbb R}(\bar X(T),\bar X_0)) \tag{4.24}
$$

by (4.21). Given any $\bar\varepsilon>0$, by Lemma 4.2, we can choose T small enough so that

$$
K(d_{\mathbb R}(X(T),X_0)+d_{\mathbb R}(\bar X(T),\bar X_0))\leq C(M)T\leq\frac{\bar\varepsilon}{2}.
$$

Then, we can fix $\varepsilon=\varepsilon_0$ so that (4.19) holds and for $\delta=\bar\varepsilon(2C(\varepsilon_0,M))^{-1}$ we have by (4.24) that $\tilde d_{\mathbb R}(X_0,\bar X_0)\leq\delta$ implies

$$
\tilde d_{\mathbb R}(X(t),\bar X(t))\leq\bar\varepsilon.
$$

Hence, we have proved that $\tilde d_{\mathbb R}(X_0,\bar X_0)\to 0$ implies $\tilde d_{\mathbb R}(X(t),\bar X(t))\to 0$, as claimed in the theorem. □

5. Invariance of the system with respect to relabeling

We define the set $\mathcal G_0$ as

$$
\mathcal G_0=\{X\in\mathcal G\mid g(X)=1\}
$$

and the projection Π from $\mathcal G$ to $\mathcal G_0$ as follows. For any $X\in\mathcal G$, we introduce

$$
\varphi(\xi)=\int_{-\infty}^{\xi}(g(X(\eta))-y_\xi(\eta))\,d\eta+y(\xi) \tag{5.1}
$$

so that $\varphi_\xi=g(X)$. We define

$$
\bar y(\xi)=y(\eta)\text{ and }\bar U(\xi)=U(\eta)
$$

for any η such that $\xi = \varphi(\eta)$. We consider the pushforward of $h\,d\xi$ by φ and denote it by ν, that is, $\nu = \varphi_{\#}(h\,d\xi)$. From the Radon–Nikodým theorem, there exists a unique function $\bar h$ in L^1 such that

$$\varphi_{\#}(h\,d\xi) = \nu = \bar h\,d\xi + \nu_{\mathrm{s}} \tag{5.2}$$

where ν_{s} is the singular part of the decomposition of ν and $\bar h\,d\xi$ the absolutely continuous part.

Proposition 5.1. *We have $\bar X = (\bar\zeta, \bar U, \bar h) \in \mathcal{G}_0$. We define the mapping $\Pi\colon \mathcal{G} \to \mathcal{G}_0$ as $\bar X = \Pi(X)$.*

Note that $\Pi \circ \Pi = \Pi$ so that Π is indeed a projection.

Proof. We have to check that this definition is well-defined. First, let us look at the definition of φ. Let

$$S = \{\xi \in \mathbb{R} \mid X(\xi) \in \Omega\}. \tag{5.3}$$

From the definition of S and Ω, we get that, for any $\xi \in S$,

$$1 \le |\zeta_\xi| + h.$$

We use a kind of Chebyshev inequality to prove that $\mathrm{meas}(S)$ is finite. We have $a \le \frac{1}{2}(a^2+1)$ for all a and therefore

$$\frac{1}{2} \le \frac{1}{2}\,|\zeta_\xi|^2 + h. \tag{5.4}$$

After integrating (5.4) on S, we get that

$$\mathrm{meas}(S) \le \|\zeta_\xi\|_{L^2}^2 + 2\,\|h\|_{L^1} < \infty. \tag{5.5}$$

On S we have that $g(X) - y_\xi = |U_\xi| + (1+2U^2)y_\xi$ is bounded while on S^c, we have $g(X) - y_\xi = h$, which belongs to L^1. Hence, since $\mathrm{meas}(S) < \infty$,

$$g(X) - y_\xi \in L^1 \tag{5.6}$$

so that the integral in (5.1) is well-defined. Let η_1 and η_2 such that $\xi = \varphi(\eta_1) = \varphi(\eta_2)$. Since g is positive, φ is increasing and therefore $\varphi_\xi(\eta) = g(X(\eta)) = 0$ for almost all $\eta \in [\eta_1, \eta_2]$. From the definition of g, we can see that $g(X(\eta))$ vanishes if and only if $y_\xi(\eta) = U_\xi(\eta) = 0$. Hence, $y(\eta_1) = y(\eta_2)$ and $U(\eta_1) = U(\eta_2)$, and the definitions of $\bar y$ and $\bar U$ are wellposed. Let us denote $A = \{\xi \in \mathbb{R} \mid \varphi_\xi(\xi) = 0\}$. We have $\mathrm{meas}(\varphi(A)) = \int_A \varphi_\xi\,d\xi = 0$. By differentating $\bar y \circ \varphi = y$ and $\bar U \circ \varphi = U$ on A^c where the functions φ, y and U are differentiable, we obtain

$$\bar y_\xi \circ \varphi \varphi_\xi = y_\xi \text{ and } \bar U_\xi \circ \varphi \varphi_\xi = U_\xi \tag{5.7}$$

almost everywhere on A^c. The identities (5.7) also hold for $\xi \in A$ because, as we alreasy saw, $\varphi_\xi = 0$ implies $y_\xi = U_\xi = 0$. Let us prove that $\bar X \in \mathcal{G}$. From the definition of $\bar\zeta$, we obtain directly that $\zeta \in L^\infty$ and $\|\bar\zeta\|_{L^\infty_{\mathbb{R}}} \le \|\zeta\|_{L^\infty_{\mathbb{R}}}$. From the defintion of g, we get that $0 \le y_\xi \le g(X) = \varphi_\xi$. Hence,

$$|\bar y(\xi_2) - \bar y(\xi_1)| = \left|\int_{\eta_1}^{\eta_2} y_\xi(\eta)\,d\eta\right| \le \left|\int_{\eta_1}^{\eta_2} \varphi_\xi(\eta)\,d\eta\right| = |\xi_2 - \xi_1|\,,$$

and $\bar y$ is Lipschitz and therefore differentiable almost everywhere. Let $B = \left\{\xi \in \mathbb{R} \mid \varphi_\xi \le \frac{1}{2}\right\}$. We have $\mathrm{meas}(S \cap B) < \infty$ and on $B \cap S^c$, we have $\varphi_\xi = y_\xi + h \le \frac{1}{2}$, that is $h + \zeta_\xi \le -\frac{1}{2}$. By a Chebyshev type of inequality, since $\zeta_\xi \in L^2$ and $h \in L^1$, we infer that $\mathrm{meas}(B \cap S^c) < \infty$, and therefore $\mathrm{meas}(B) < \infty$. Then,

$$\int_{\mathbb{R}} \bar\zeta_\xi^2(\xi)\,d\xi = \int_{\varphi(B)} \bar\zeta_\xi^2(\xi)\,d\xi + \int_{\varphi(B^c)} \bar\zeta_\xi^2(\xi)\,d\xi$$

$$
\begin{aligned}
&\leq \|\zeta_\xi\|_{L^\infty}^2 \int_{\varphi(B)} d\xi + \int_{B^c} \bar{\zeta}_\xi^2 \circ \varphi(\xi)\varphi_\xi(\xi)\, d\xi \\
&\leq \|\zeta_\xi\|_{L^\infty}^2 \int_{B} \varphi_\xi(\xi) d\xi + 2\int_{B^c} \bar{\zeta}_\xi^2 \circ \varphi(\xi)\varphi_\xi(\xi)^2\, d\xi \\
&\leq \|\zeta_\xi\|_{L^\infty}^2 \|\varphi_\xi\|_{L^\infty} \operatorname{meas}(B) + 2\|\zeta_\xi\|_{L^2}^2,
\end{aligned}
$$

and $\zeta_\xi \in L^2$. Let us prove that $U \in L^2$. We have

$$
\int_{\mathbb{R}} \bar{U}^2\, d\xi = \int_{\mathbb{R}} \bar{U}^2 \circ \varphi\varphi_\xi\, d\xi \leq \|\varphi_\xi\|_{L^\infty_{\mathbb{R}}} \|U\|_{L^2_{\mathbb{R}}}^2.
$$

Next we prove that $U_\xi \in L^2$. We have

$$
\int_{\mathbb{R}} \bar{U}\psi_\xi\, d\xi = \int_{\mathbb{R}} \bar{U}\circ\varphi\psi_\xi\circ\varphi\varphi_\xi\, d\xi = \int_{\mathbb{R}} U(\psi\circ\varphi)_\xi\, d\xi = -\int_{\mathbb{R}} U_\xi(\psi\circ\varphi)\, d\xi.
$$

Since $U_\xi(\xi) = 0$ whenever ξ is such that $y_\xi(\xi) = 0$ and therefore $\varphi_\xi(\xi) = 0$, the domain of integration in the last integral can be restricted to $\{\xi \in \mathbb{R} \mid \varphi_\xi(\xi) > 0\}$ and we get, by the Cauchy–Schwarz inequality,

$$
\left|\int_{\mathbb{R}} \bar{U}\psi\, d\xi\right| = \int_{\varphi_\xi(\xi)>0} \frac{U_\xi}{\sqrt{\varphi_\xi}}(\psi\circ\varphi)\sqrt{\varphi_\xi}\, d\xi \leq \Big(\int_{\varphi_\xi(\xi)>0} \frac{U_\xi^2}{\varphi_\xi}\, d\xi\Big)^{1/2} \|\psi\|_{L^2}.
$$

On the other hand, we have

$$
\int_{\mathbb{R}} \frac{U_\xi^2}{\varphi_\xi}\, d\xi = \int_S \frac{U_\xi^2}{\varphi_\xi}\, d\xi + \int_{S^c} \frac{U_\xi^2}{\varphi_\xi}\, d\xi \leq \operatorname{meas}(S)\|U_\xi\|_{L^\infty} + \left\|\frac{1}{y_\xi + h}\right\|_{L^\infty} \|U_\xi\|_{L^2}^2 < \infty
$$

because $|U_\xi| \leq \varphi_\xi$ on S and $\frac{1}{\varphi_\xi} = \frac{1}{y_\xi + h}$ on S^c. Hence, for some constant C, $\left|\int_{\mathbb{R}} \bar{U}\psi_\xi\, d\xi\right| \leq C\|\psi\|_{L^2}$ for all $\psi \in L^2$, which proves that $U_\xi \in L^2$. We have

$$
\int_{\mathbb{R}} \bar{h}\, d\xi = \nu_{\text{ac}}(\mathbb{R}) \leq \nu(\mathbb{R}) = \int_{\mathbb{R}} h\, d\xi. \tag{5.8}
$$

Here we clearly see that the L^1 norm of h is not preserved by the projection: We may lose some energy. We claim that

$$
\bar{h}\circ\varphi\varphi_\xi = h \tag{5.9}
$$

for almost every $\xi \in A^c$. First we prove that, for almost every $\xi \in A^c$, φ is injective. Indeed, assume that it is not the case, and that there exists $\xi \in A^c$ and $\xi' \neq \xi$ such that $\varphi(\xi) = \varphi(\xi')$. Since φ is monotone increasing, it implies that $\varphi(\eta) = \varphi(\xi)$ for all $\eta \in [\xi, \xi']$. Hence, $\varphi_\xi(\xi) = 0$, which contradicts the fact that $\xi \in A^c$. Let us prove that $\nu_{\text{ac}} = \nu|_{\varphi(A^c)}$ and $\nu_{\text{s}} = \nu|_{\varphi(A)}$. For any given Borel set B of Lebesgue measure zero, we have

$$
\nu(B\cap\varphi(A^c)) = \int_{\varphi^{-1}(B\cap\varphi(A^c))} h\, d\xi = \int_{\varphi^{-1}(B)\cap A^c} \frac{h}{\varphi_\xi}\varphi_\xi\, d\xi
$$

because $\varphi^{-1}(\varphi(A^c)) = A^c$ as φ is injective on A^c. The generalized area formula yields

$$
\int_{\varphi^{-1}(B)\cap A^c} \frac{h}{\varphi_\xi}\varphi_\xi\, d\xi = \int_{\mathbb{R}} \left(\sum_{\eta\in\varphi^{-1}(B)\cap A^c\cap\varphi^{-1}(\{\xi\})} \frac{h}{\varphi_\xi}(\eta)\right) d\xi \tag{5.10}
$$

where $\mathcal{H}^0$ is the multiplicity function, see [1] for more details on this formula and the precise definition of $\mathcal{H}^0$. Since φ is injective on A^c, the set $\varphi^{-1}(B)\cap A^c\cap\varphi^{-1}(\{\xi\})$ reduces to one point, $\varphi^{-1}(\xi)$, and we have

$$
\nu(B\cap\varphi(A^c)) = \int_{B\cap\varphi(A^c)} \frac{h}{\varphi_\xi}(\varphi^{-1}(\xi))\, d\xi = 0
$$

because $\operatorname{meas}(B) = 0$. Hence, $\nu|_{\varphi(A^c)}$ is absolutely continuous. On the other hand, since $\operatorname{meas}(\varphi(A)) = 0$, $\nu|_{\varphi(A)}$ is nonzero only on a set of measure zero and, as $\nu = \nu|_{\varphi(A^c)} + \nu|_{\varphi(A)}$, it follows that $\nu_{\mathrm{ac}} = \nu|_{\varphi(A^c)}$ and $\nu_{\mathrm{s}} = \nu|_{\varphi(A)}$. For any subset B of A^c, we have

$$\int_{\varphi(B)} \bar{h}\, d\xi = \nu_{\mathrm{ac}}(\varphi(B)) = \nu(\varphi(B)) = \int_B h\, d\xi$$

which implies, after a change of variables, that

$$\int_B h\, d\xi = \int_B \bar{h} \circ \varphi \varphi_\xi\, d\xi.$$

Since B is an arbitrary subset of A^c, this concludes the proof of the claim (5.9). Since $\varphi(A^c)$ has full measure, from (5.7) and (5.9), we infer that $\bar{X}$ fulfills (2.23c). Since $\operatorname{meas}(\varphi(A)) = 0$, for almost every ξ there exists $\eta \in A^c$ such that $\xi = \varphi(\eta)$. We have, after using (5.7) and (5.9) that

$$\bar{y}_\xi(\xi)\bar{h}(\xi) = \bar{y}_\xi \circ \varphi(\eta)\bar{h} \circ \varphi(\eta) = \frac{1}{\varphi_\xi^2(\eta)} y_\xi(\eta)h(\eta) = \frac{1}{\varphi_\xi^2(\eta)}(U^2 y_\xi^2 + U_\xi^2)(\eta)$$

as $X \in \mathcal{G}$ and satisfies (2.23c). Hence, again using (5.7), we get $(\bar{y}_\xi \bar{h})(\xi) = (\bar{U}^2 \bar{y}_\xi^2 + \bar{U}_\xi^2)(\xi)$ and $\bar{X}$ satisfies (2.23c). In the same way, one obtains

$$\frac{1}{\bar{y}_\xi(\xi) + \bar{h}(\xi)} = \frac{1}{\bar{y}_\xi \circ \varphi(\eta) + \bar{h} \circ \varphi(\eta)} = \frac{\varphi_\xi(\eta)}{y_\xi(\eta) + h(\eta)} \leq \|\varphi_\xi\|_{L^\infty} \left\| \frac{1}{y_\xi + h} \right\|_{L^\infty},$$

and $\bar{X}$ fulfills (2.23e). The function g is positive homogenous in the three last variables, that is, $g(x_1, x_2, \lambda x_3, \lambda x_4, \lambda x_5) = \lambda g(x_1, x_2, x_3, x_4, x_5)$ for all $\lambda \geq 0$ and $\boldsymbol{x} \in \mathbb{R}^5$. Hence, we have, for any $\eta \in A^c$,

$$\begin{aligned} g(\bar{X} \circ \varphi)(\eta) &= \frac{1}{\varphi_\xi(\eta)} g(\bar{y} \circ \varphi, \bar{U} \circ \varphi, \bar{y}_\xi \circ \varphi \varphi_\xi, \bar{U}_\xi \circ \varphi \varphi_\xi, \bar{h} \circ \varphi \varphi_\xi)(\eta) \\ &= \frac{1}{\varphi_\xi(\eta)} g(X(\eta)) = 1. \end{aligned}$$

Hence, $g(\bar{X}) = 1$ almost everywhere and $\bar{X}$ belongs to $\mathcal{G}_0$. □

Proposition 5.2. *The projection* Π *is a continuous mapping with respect to the distance* $\tilde{d}_{\mathbb{R}}$ *on bounded set of* $\mathcal{G}$*, that is, for any sequence* X^n *and* X *in* B_M*, we have*

$$\lim_{n\to\infty} \tilde{d}_{\mathbb{R}}(X^n, X) = 0 \textit{ implies } \lim_{n\to\infty} \tilde{d}_{\mathbb{R}}(\Pi(X^n), \Pi(X)) = 0.$$

Proof. First we prove that

$$\varphi^n \to \varphi \text{ in } L^\infty. \tag{5.11}$$

To simplify the notation, we denote $g(X^n)$ and $g(X)$ by g^n and g, respectively. We have

$$\|\varphi^n - \varphi\|_{L^\infty} \leq \left\| g^n - y_\xi^n - g + y_\xi \right\|_{L^1} + \|y^n - y\|_{L^\infty}. \tag{5.12}$$

We define the set A_n and A as

(5.13)
$$A_n = \{\xi \in \mathbb{R} \mid X^n(\xi) \in \Omega \text{ or } y_\xi^n(\xi) \leq \frac{1}{2}\} \text{ and } A = \{\xi \in \mathbb{R} \mid X(\xi) \in \Omega \text{ or } y_\xi(\xi) \leq \frac{1}{2}\}.$$

We have $\operatorname{meas}(\{\xi \in \mathbb{R} \mid y_\xi^n(\xi) \leq \frac{1}{2}\}) \leq \frac{1}{4}\int_{\mathbb{R}} {\zeta_\xi^n}^2\, d\xi \leq \frac{1}{4}M^2$ and $\operatorname{meas}(\{\xi \in \mathbb{R} \mid X(\xi) \in \Omega\}) \leq C(M)$, for some constant $C(M)$ depending only on M, see (5.5). Hence, $\operatorname{meas}(A_n) \leq C(M)$ and $\operatorname{meas}(A) \leq C(M)$ for another constant $C(M)$. We denote generically by $C(M)$ constants which depend only on M. Note that on A_n^c, we have $g^n - y^n_\xi = h^n$. We split the integration domain in two, $A_n \cup A$ and $A_n^c \cap A^c$. On $A_n \cup A$, we have

$$\begin{aligned}\int_{A_n\cup A} \left|g^n - y^n_\xi - g_\xi + y_\xi\right| d\xi &\leq (\operatorname{meas}(A_n\cup A))^{1/2}(\|g^n-g\|_{L^2} + \left\|y^n_\xi - y_\xi\right\|_{L^2})\\ &\leq C(M)(\|g^n-g\|_{L^2} + \left\|y^n_\xi - y_\xi\right\|_{L^2})\end{aligned} \tag{5.14}$$

while, on $A_n^c \cap A^c$, we have

$$\begin{aligned}|h^n - h| &= \left|(U^n)^2 y^n_\xi - \frac{(U^n_\xi)^2}{y^n_\xi} + U^2 y_\xi - \frac{U_\xi^2}{y_\xi}\right|\\ &\leq 4\left|(U^n)^2 2(y^n_\xi)^2 2y_\xi - (U^n_\xi)^2 y_\xi - U^2 y_\xi^2 y^n_\xi - U^2_\xi y^n_\xi\right|\end{aligned}$$

so that

$$\begin{aligned}\int_{A_n^c\cap A^c} \left|g^n - y^n_\xi - g_\xi + y_\xi\right| d\xi &= \int_{A_n^c\cap A^c} |h^n - h|\, d\xi\\ &\leq C(M)(\|U^n - U\|^2_{L^2} + \left\|U^n_\xi - U_\xi\right\|^2_{L^2}\\ &\qquad + \|y_\xi - y_\xi\|^2_{L^2})\end{aligned} \tag{5.15}$$

for some constant $C(M)$ depending only on M. Combining (5.12), (5.14) and (5.15), we obtain (5.11). We denote $\Pi(X^n)$ and $\Pi(X)$ by $\bar{X}^n$ and $\bar{X}$, respectively. We have $\bar{y}^n(\xi) - \bar{y}(\xi) = y^n(\eta_n) - y(\eta)$ where η and η_n are chosen so that $\xi = \varphi^n(\eta_n) = \varphi(\eta)$. We have

$$\begin{aligned}|y(\eta_n) - y(\eta)| &= \left|\int_\eta^{\eta_n} y_\xi(\bar\eta)\, d\bar\eta\right| \leq \left|\int_\eta^{\eta_n} \varphi_\xi(\bar\eta)\, d\bar\eta\right| \text{ (because } y_\xi \leq \varphi_\xi)\\ &\leq |\varphi(\eta_n) - \varphi(\eta)| = |\varphi(\eta_n) - \varphi^n(\eta_n)|\\ &\leq \|\varphi - \varphi^n\|_{L^\infty}.\end{aligned}$$

Hence, as $\bar{y}^n(\xi) - \bar{y}(\xi) = y^n(\eta_n) - y(\eta_n) + y(\eta_n) - y(\eta)$,

$$\|\bar{y}^n - \bar{y}\|_{L^\infty} \leq \|y^n - y\|_{L^\infty} + \|\varphi^n - \varphi\|_{L^\infty}$$

which implies, by (5.11), that $\bar{y}^n \to \bar{y}$ in L^∞. Let us prove that $\bar\zeta^n_\xi \to \bar\zeta_\xi$, $\bar{U}^n \to \bar{U}$ and $\bar{U}^n_\xi \to \bar{U}_\xi$ in L^2. We will only treat the first limit, i.e., that $\bar\zeta^n_\xi \to \bar\zeta_\xi$, as the others can be treated in exactly the same way. Note that $g(\bar{X}_n) = g(\bar{X}) = 1$ so that it is clear that $g(\bar{X}_n) \to g(\bar{X})$. Let us introduce

$$A^n_\delta = \{\xi \in \mathbb{R} \mid g(X^n(\xi)) = \varphi^n_\xi(\xi) \leq \delta\} \text{ and } A_\delta = \{\xi\in\mathbb{R} \mid g(X(\xi)) = \varphi_\xi(\xi) \leq \delta\}.$$

Without loss of generality, we may assume that $\delta \leq (2M)^{-1}$. If $\xi \in \mathcal{A}_\delta$, then $g(X(\xi)) \leq \delta \leq (2M)^{-1} < (y_\xi + h)(\xi)$ and therefore $\xi \in S$. Hence, $A_\delta \subset S$ and $\operatorname{meas}(A_\delta) \leq C(M)$ from (5.5). Similarly, one gets $\operatorname{meas}(A^n_\delta) \leq C(M)$. We consider the decomposition $\mathbb{R} = \varphi^n(A^c_\delta \cap (A^n_\delta)^c) \cup \varphi^n(A_\delta) \cup \varphi^n(A^n_\delta)$. Since $\bar{y}^n_\xi \leq \bar{g}^n = 1$, we have $\left\|\bar\zeta_\xi\right\|_{L^\infty} \leq 1$ and

$$\int_{\varphi^n(A^n_\delta)} \left|\bar\zeta^n_\xi - \bar\zeta_\xi\right|^2 d\xi \leq 4\int_{\varphi^n(A^n_\delta)} d\xi \leq 4\int_{A^n_\delta} \varphi^n_\xi\, d\xi \leq 4\delta \operatorname{meas}(A^n_\delta) \leq C(M)\delta. \tag{5.16}$$

We have

$$\begin{aligned}\int_{\varphi^n(A_\delta)} \left|\bar\zeta^n_\xi - \bar\zeta_\xi\right|^2 d\xi &\leq 4\int_{\varphi^n(A_\delta)} d\xi = 4\int_{A_\delta} \varphi^n_\xi\, d\xi\\ &\leq 8\delta \operatorname{meas}(A_\delta \cap A^n_{2\delta}) + C(M)\operatorname{meas}(A_\delta \cap (A^n_{2\delta})^c).\end{aligned}$$

Since $\operatorname{meas}(A_\delta \cap (A_{2\delta}^n)^c) \leq \frac{1}{\delta}\int_{A_\delta\cap(A_{2\delta}^n)^c} \left|\varphi_\xi - \varphi_\xi^n\right| d\xi \leq \operatorname{meas}(A_\delta)^{1/2}\delta^{-1}\left\|\varphi_\xi - \varphi_\xi^n\right\|_{L^2}$, the last inequality gives us

$$\int_{\varphi^n(A_\delta)} \left|\bar\zeta_\xi^n - \bar\zeta_\xi\right|^2 d\xi \leq C(M)(\delta + \frac{1}{\delta}\left\|\varphi_\xi - \varphi_\xi^n\right\|_{L^2}). \tag{5.17}$$

It remains to evaluate $\int_{\varphi^n(A_\delta^c\cap(A_\delta^n)^c)} \left|\bar\zeta_\xi^n - \bar\zeta_\xi\right|^2 d\xi$. We have

$$\begin{aligned}\int_{\varphi^n(A_\delta^c\cap(A_\delta^n)^c)} \left|\bar\zeta_\xi^n - \bar\zeta_\xi\right|^2 d\xi &= \int_{A_\delta^c\cap(A_\delta^n)^c} \left|\bar\zeta_\xi^n\circ\varphi^n - \bar\zeta_\xi\circ\varphi^n\right|^2 \varphi_\xi^n\, d\xi\\ &\leq 2\int_{A_\delta^c\cap(A_\delta^n)^c} \left|\bar\zeta_\xi^n\circ\varphi^n - \bar\zeta_\xi\circ\varphi\right|^2 \varphi_\xi^n\, d\xi\\ &\quad + 2\int_{A_\delta^c\cap(A_\delta^n)^c} \left|\bar\zeta_\xi\circ\varphi - \bar\zeta_\xi\circ\varphi^n\right|^2 \varphi_\xi^n\, d\xi.\end{aligned} \tag{5.18}$$

We denote by I_1^n and I_2^n, the integrals on the right-hand side of (5.18). We have

$$\begin{aligned}I_1^n &= \int_{A_\delta^c\cap(A_\delta^n)^c} \left|\frac{\zeta_\xi^n}{\varphi_\xi^n} - \frac{\zeta_\xi}{\varphi_\xi}\right|^2 \varphi_\xi^n\, d\xi \leq \frac{C(M)}{\delta^3}\int_{A_\delta^c\cap(A_\delta^n)^c} \left|\zeta_\xi^n\varphi_\xi - \zeta_\xi\varphi_\xi^n\right|^2 d\xi\\ &\leq \frac{C(M)}{\delta^3}(\left\|\zeta_\xi^n - \zeta_\xi\right\|_{L^2}^2 + \left\|g^n - g\right\|_{L^2}^2).\end{aligned} \tag{5.19}$$

Given a continuous function with compact support f, we have

$$\begin{aligned}I_2^n &\leq 3\int_{A_\delta^c\cap(A_\delta^n)^c} \left|\bar\zeta_\xi\circ\varphi - f\circ\varphi\right|^2 \varphi_\xi^n\, d\xi + 3\int_{A_\delta^c\cap(A_\delta^n)^c} \left|f\circ\varphi - f\circ\varphi^n\right|^2 \varphi_\xi^n\, d\xi\\ &\quad + 3\int_{A_\delta^c\cap(A_\delta^n)^c} \left|\bar\zeta_\xi\circ\varphi^n - f\circ\varphi^n\right|^2 \varphi_\xi^n\, d\xi.\end{aligned} \tag{5.20}$$

We denote by I_{21}^n I_{22}^n and I_{23}^n the three integrals in the inequality above. We have, after a change of variables,

$$I_{23}^n = \int_{\varphi^n(A_\delta^c\cap(A_\delta^n)^c)} \left|\bar\zeta_\xi - f\right|^2 d\xi \leq \left\|\bar\zeta_\xi - f\right\|_{L^2}^2 \tag{5.21}$$

and

$$I_{21}^n \leq \frac{C(M)}{\delta}\int_{A_\delta^c\cap(A_\delta^n)^c} \left|\bar\zeta_\xi\circ\varphi - f\circ\varphi\right|^2 \varphi_\xi\, d\xi \leq \frac{C(M)}{\delta}\left\|\bar\zeta_\xi - f\right\|_{L^2}^2. \tag{5.22}$$

Since f is continuous with compact support and $\varphi^n \to \varphi$ in L^∞, it follows from the Lebesgue dominated convergence theorem that

$$\lim_{n\to\infty}\int_{\mathbb{R}} \left|f\circ\varphi - f\circ\varphi^n\right|^2 \varphi_\xi^n\, d\xi = 0$$

so that $\lim_{n\to\infty} I_{22}^n = 0$, independently of δ. Gathering (5.16), (5.17), (5.19), (5.20), (5.21) and (5.22), we finally obtain

$$\left\|\bar\zeta_\xi^n - \bar\zeta_\xi\right\|_{L^2}^2 \leq C(M)\Big(\delta + \frac{1}{\delta^3}(\left\|g - g^n\right\|_{L^2}^2 + \left\|\zeta_\xi - \zeta_\xi^n\right\|_{L^2}^2 + \left\|\bar\zeta_\xi - f\right\|_{L^2}^2) + I_{22}^n\Big) \tag{5.23}$$

where we have assumed without loss of generality that $\delta \leq 1$. For any $\varepsilon > 0$, we take $\delta \leq \varepsilon^2(3C(M))^{-1}$. Since the space of continuous functions with compact support is dense in L^2, there exists f, continuous with compact support, such that $C(M)\delta^{-3}\left\|\bar\zeta_\xi - f\right\|_{L^2}^2 \leq \varepsilon^2/3$. Since $\zeta_\xi^n \to \zeta_\xi$, $g^n \to g$ in L^2 and $I_{22}^n \to 0$, there exists N such that for all $n \geq N$, $C(M)\delta^{-3}(\left\|g - g^n\right\|_{L^2}^2 + \left\|\zeta_\xi - \zeta_\xi^n\right\|_{L^2}^2 + I_{22}^n) \leq \varepsilon^2/3$ and then it follows that $\|\bar\zeta_\xi^n - \bar\zeta_\xi\| \leq \varepsilon$. Hence, we have proved that $\bar\zeta_\xi^n \to \bar\zeta_\xi$ in L^2. $\square$

The system is invariant with respect to relabeling. Let us explain what we mean by relabeling.

Definition 5.3. *We say that $\bar X\in\mathcal{G}$ is a* relabeling *of $X\in\mathcal{G}$ if there exists a ψ which satisfies*

$$\psi(\xi)-\xi\in L^\infty(\mathbb{R}),\ \psi_\xi-1\in L^\infty(\mathbb{R})\cap L^2(\mathbb{R}),\ \psi_\xi\geq 0,\ \lim_{\xi\to-\infty}(\psi(\xi)-\xi)=0 \tag{5.24a}$$

and such that

$$\bar y=y\circ\psi,\quad \bar U=U\circ\psi. \tag{5.24b}$$

Note that in the definition above, we do not require any relationship between $\bar h$ and h. However, we have $y_\xi(\psi(\xi))h(\psi(\xi))=U^2(\psi(\xi))y_\xi^2(\psi(\xi))+U_\xi^2(\psi(\xi))$ and $\bar y(\xi)\bar h(\xi)=\bar U^2(\xi)\bar y_\xi^2(\xi)+\bar U_\xi^2(\xi)$. Since $\bar y(\xi)=y_\xi(\psi(\xi))\circ\psi_\xi(\xi)$ and $\bar U(\xi)=U_\xi(\psi(\xi))\circ\psi_\xi(\xi)$, it implies that for almost all $\xi\in\mathbb{R}$ such that $\bar y_\xi(\xi)\neq 0$,

$$\bar h(\xi)=h(\psi(\xi))\psi_\xi(\xi). \tag{5.25}$$

Proposition 5.4. *If $\bar X_0$ is a relabeling of X_0, then, for any $t\geq 0$, $S_t(\bar X_0)$ is a relabeling of $S_t(X_0)$. More precisely, if $\bar y_0=y_0\circ\psi$, $\bar U_0=U_0\circ\psi$ for a ψ satisfying* (5.24a)*, then*

$$\bar y(t,\xi)=y(t,\psi(\xi))\ \text{and}\ \bar U(t,\xi)=U(t,\psi(\xi))$$

where $X(t)=S_t(X_0)$ and $\bar X(t)=S_t(\bar X_0)$.

Proof. Let ψ be the function such that $\bar y_0=y_0\circ\psi$ and $\bar U_0=U_0\circ\psi$ which is defined in Definition 5.3. Let us introduce $\tilde X(t,\xi)$ as follows: $\tilde y(t,\xi)=y(t,\psi(\xi))$, $\tilde U(t,\xi)=U(t,\psi(\xi))$,

$$\tilde h(t,\xi)=\begin{cases}h(t,\psi(\xi))\psi_\xi(\xi) & \text{if } \psi_\xi(\xi)\neq 0,\\ \bar h_0(\xi) & \text{otherwise.}\end{cases}$$

We have $\tilde X(0,\xi)=\bar X_0$, see (5.25), and we claim that $\tilde X$ satisfies (2.18) and (2.19) and therefore, by uniqueness of the solution, we have $\tilde X(t)=\bar X(t)$ for all t and, since $\tilde X$ is a relabeling of X, the lemma is proved. For any ξ such that $\psi_\xi(\xi)\neq 0$, we have that $\tilde y_\xi(t,\xi)=0$ if and only of $y_\xi(t,\psi(\xi))=0$ and therefore, for such ξ, $\tilde\tau(\xi)=\tau(\psi(\xi))$. We have, after a change of variables,

$$\begin{aligned}P(t,\psi(\xi))&=\frac14\int_{\tau(\eta)>t}\exp\big(-\operatorname{sgn}(\psi(\xi)-\eta)(y(\psi(\xi))-y(\eta))\big)\big(U^2y_\xi+h\big)(\eta)\,d\eta\\ &=\frac14\int_{\tau(\psi(\eta))>t}\exp\big(-\operatorname{sgn}(\psi(\xi)-\psi(\eta))(y(\psi(\xi))-y(\psi(\eta)))\big)\\ &\qquad\qquad\times\big(U^2y_\xi+h\big)(\psi(\eta))\psi_\xi(\eta)\,d\eta.\end{aligned}$$

Let us denote $B=\{\xi\in\mathbb{R}\mid\psi_\xi(\xi)=0\}$. The integrand vanishes for $\eta\in B$. For $\eta\in B^c$, we have $\tau(\psi(\eta))=\tilde\tau(\eta)$ and $h(\psi(\eta))\psi_\xi(\eta)=\tilde h(\eta)$. Let us consider ξ and η such that $\xi>\eta$ (the case $\xi<\eta$ would be handled in the same way) and $\eta\in B$. Since ψ is increasing, we have $\psi(\xi)\geq\psi(\eta)$. We cannot have $\psi(\xi)=\psi(\eta)$ as it would imply that $\psi_\xi(\eta)=0$. Hence, $\psi(\xi)>\psi(\eta)$ and we have proved that $\operatorname{sgn}(\psi(\xi)-\psi(\eta))=\operatorname{sgn}(\xi-\eta)$ on B^c. Hence,

$$P(t,\psi(\xi))=\frac14\int_{\tilde\tau(\eta)>t}\exp\big(-\operatorname{sgn}(\xi-\eta)(\tilde y(\xi)-\tilde y(\eta))\big)\big(\tilde U^2\tilde y_\xi+\tilde h\big)(\eta)\,d\eta=\tilde P(t,\xi). \tag{5.26}$$

In the same way, one proves $Q(t,\psi(\xi))=\tilde Q(t,\xi)$. Since X is a solution to (2.18) and (2.19), we have

$$y_t(t,\psi(\xi))=U(t,\psi(\xi)),\quad U_t(t,\psi(\xi))=-Q(t,\psi(\xi))$$

and

$$(5.27)\qquad \begin{cases} y_{\xi t}(t,\psi(\xi)) = \chi_{\{\tau(\psi(\xi))>t\}} U_\xi(t,\psi(\xi)),\\ U_{\xi t}(t,\psi(\xi)) = \chi_{\{\tau(\psi(\xi))>t\}} \big(\frac{1}{2}h + \left(\frac{1}{2}U^2 - P\right) y_\xi\big)(t,\psi(\xi)),\\ h_t(t,\psi(\xi)) = \chi_{\{\tau(\psi(\xi))>t\}} \big(-2Q\,U y_\xi + \big(3U^2 - 2P\big)\,U_\xi\big)(t,\psi(\xi)). \end{cases}$$

Hence, $\tilde{X}$ satisfies (2.18). For $\xi\in B^c$, we have $\tau(\psi(\xi)) = \tilde{\tau}(\xi)$, $h(\psi(\xi))\psi_\xi(\xi) = \tilde{h}(\xi)$, $P(t,\psi(\xi)) = \tilde{P}(t,\xi)$, $Q(t,\psi(\xi)) = \tilde{Q}(t,\xi)$ and therefore, after multiplying (5.27) by $\varphi_\xi(\xi)$ we get that $\tilde{X}$ fulfills (2.19) on B^c. For $\xi\in B$, we have $\tilde{y}_\xi(t,\xi) = \tilde{U}_\xi(t,\xi) = 0$, $\tilde{\tau}(\xi) = 0$ and $\tilde{h}(t,\xi) = h_0(\xi)$ so that $\tilde{X}$ satisfies (2.19) also on B. Hence, we have proved our claim that $\tilde{X}$ satisfies (2.18) and (2.19) and therefore it coincides with $\bar{X}$. ☐

In [22] for the conservative case, we define an equivalence relation between elements that are equal up to a relabeling. In the dissipative case, we cannot formulate this in the same way. We can check from Definition 5.3 that if $\bar{X}$ is a relabeling of X, then X is not necessary a relabeling of $\bar{X}$, basically because ψ^{-1} is either not well-defined or not sufficiently regular. However, we have the following result.

Lemma 5.5. *If X_2 is a relabeling of X_1, then $\Pi(X_2) = \Pi(X_1)$.*

Proof. There exists ψ which satisfies (5.24a) and such that $y_2 = y_1\circ\psi$, $U_2 = U_1\circ\psi$ and, for almost all $\xi\in B^c$, where $B = \{\xi\in\mathbb{R} \mid \psi_\xi(\xi) = 0\}$, $h_2(\xi) = h_1(\psi(\xi))\psi_\xi(\xi)$. We claim that

$$(5.28)\qquad g(X_1\circ\psi)\psi_\xi = g(X_2)$$

almost everywhere. Let us prove this claim. We have

$$(5.29)\qquad g(X_1(\psi(\xi)))\psi_\xi(\xi) = |U_{1\xi}(\psi(\xi))|\,\psi_\xi(\xi) + 2(1+U_1^2(\psi(\xi)))y_{1\xi}(\psi(\xi))\psi_\xi(\xi)$$

if

$$(5.30)\qquad U_{1\xi}(\psi(\xi)) + 2(1+U_1^2(\psi(\xi)))y_{1\xi}(\psi(\xi)) \le y_{1\xi}(\psi(\xi)) + h_1(\psi(\xi))$$

and

$$(5.31)\qquad g(X_1(\psi(\xi)))\psi_\xi(\xi) = y_\xi(\psi(\xi))\psi_\xi(\xi) + h(\psi(\xi))\psi_\xi(\xi)$$

otherwise. For $\xi\in B^c$, after multiplying both sides of the inequality by $\varphi_\xi(\xi)$, we obtain that (5.30) is equivalent to

$$(5.32)\qquad U_{2\xi}(\xi) + 2(2+U_2^2(\xi))y_{2\xi}(\xi) \le y_{2\xi}(\xi) + h_2(\xi)$$

and if ξ satisfies (5.32), we have, from (5.29), that

$$(5.33)\qquad g(X_1(\psi(\xi)))\psi_\xi(\xi) = |U_{2\xi}(\xi)| + 2(1+U_2^2(\xi))y_{2\xi}(\xi) = g(X_2(\xi)).$$

Similarly, if ξ does not satisfy (5.30), then (5.31) yields

$$g(X_1(\psi(\xi)))\psi_\xi(\xi) = y_{2\xi}(\xi) + h_2(\xi) = g(X_2(\xi)).$$

Hence, (5.28) holds on B^c. For $\xi\in B$, $y_{2\xi}(\xi) = 0$ which implies $g(X_2(\xi)) = g(X_1\circ\psi)\psi_\xi = 0$ and therefore (5.28) also holds on B^c. Hence, we have proved our claim that (5.28) holds almost everywhere. Let us denote $\bar{X}_1 = \Pi(X_1)$, $\bar{X}_2 = \Pi(X_2)$. We have

$$(5.34)\qquad \bar{y}_1\circ\varphi_1 = y_1,\ \bar{U}_1\circ\varphi_1 = U_1 \text{ and } \bar{y}_2\circ\varphi_2 = y_2,\ \bar{U}_2\circ\varphi_2 = U_2,$$

where φ_1 and φ_2 are defined as in (5.1). Since $\varphi_{1\xi} = g(X_1)$ and $\varphi_{2\xi} = g(X_2)$, equation (5.28) yields $\varphi_{1\xi}\circ\psi\psi_\xi = \varphi_{2\xi}$ and, after integration, we obtain $\varphi_1\circ\psi(\xi) = \varphi_2(\xi) + \varphi_1\circ\psi(\eta) - \varphi_2(\eta)$ for any ξ and η. We have

$$\varphi_1\circ\psi(\eta) - \varphi_2(\eta) = \varphi_1\circ\psi(\eta) - \psi(\eta) + \psi(\eta) - \eta + \varphi_2(\eta) - \eta$$

and therefore $\lim_{\eta\to\infty}(\varphi_1\circ\psi(\eta)-\varphi_2(\eta))=0$ because

$$\lim_{\eta\to\infty}(\varphi_1(\eta)-\eta)=\lim_{\eta\to\infty}(\varphi_2(\eta)-\eta)=\lim_{\eta\to\infty}(\psi(\eta)-\eta)=0.$$

Hence, $\varphi_1\circ\psi=\varphi_2$ and, from (5.34), it follows that $\bar y_1\circ\varphi_2=\bar y_2\circ\varphi_2$ and $\bar U_1\circ\varphi_2=\bar U_2\circ\varphi_2$ which implies $\bar y_1=\bar y_2$ and $\bar y_1=\bar U_2$ because φ_2 is surjective. Since $\bar X_1$ and $\bar X_2$ belong to $\mathcal{G}_0$, we have $g(X_1)=g(X_2)=1$ and therefore, for almost all ξ, we have $y_{1\xi}(\xi)\neq 0$ and $y_{2\xi}(\xi)\neq 0$ as $y_{1\xi}(\xi)=0$ implies $g(X_1(\xi))=0$. Hence, for almost all ξ, we have

$$h_1(\xi)=U_1^2(\xi)y_{1\xi}(\xi)+\frac{U_{1\xi}^2(\xi)}{y_{1\xi}(\xi)}=U_2^2(\xi)y_{2\xi}(\xi)+\frac{U_{2\xi}^2(\xi)}{y_{2\xi}(\xi)}=h_2(\xi),$$

and we have proved that $\bar X_1=\bar X_2$. □

We can define an equivalence relation in $\mathcal{G}$ as follows: X_1 and X_2 are equivalent if $\Pi(X_1)=\Pi(X_2)$. The preceding lemma tells us that this equivalence relation is related to relabeling in the sense that if X_2 is a relabeling of X_1, then X_1 and X_2 are equivalent. By considering equivalence classes, we suppress somehow the arbitrariness of the choice of relabeling we introduced by setting the equivalent system of the first section and which is inherent to any lagrangian formalism. This is a condition to obtain a bijection with the Eulerian coordinates. The set of equivalent classes is in bijection with $\mathcal{G}_0$ and that is why we now define a semigroup on this set.

Theorem 5.6. *The mapping $\tilde S_t$ defined as $\tilde S_t=\Pi\circ S_t$ is a continuous semigroup on bounded sets of $\mathcal{G}_0$.*

Proof. From Theorem 4.3 and Proposition 5.2, we get that $\tilde S_t$ is continuous. It remains to check the semigroup property. We claim that

$$\Pi\circ S_t\circ\Pi=\Pi\circ S_t. \tag{5.35}$$

Consider a given $X\in\mathcal{G}$. We have that X is a relabeling of $\Pi(X)$. Hence, from Lemma 5.4, it follows that $S_t(X)$ is a relabeling of $S_t(\Pi(X))$. Then Lemma 5.5 implies $\Pi(S_t(X))=\Pi(S_t(\Pi(X)))$ and we have proved the claim (5.35). By using (5.35), we prove the semigroup property

$$\tilde S_{t+t'}=\Pi\circ S_{t+t'}=\Pi\circ S_t\circ S_{t'}=\Pi\circ S_t\circ\Pi\circ S_{t'}=\tilde S_t\circ\tilde S_{t'}.$$

□

6. From Lagrangian to Eulerian coordinates

In this section we define the mappings between the Eulerian variable $u\in H^1$ and the Lagrangian variable $X\in\mathcal{G}_0$ and vice versa. We use the fact the Eulerian variable u formally is just a particular relabeling of X, a relabeling $\bar X$ for which we have $\bar y(\xi)=\xi$.

Definition 6.1. *Given $u\in H^1$, let us denote $\bar X(x)=(x,u,1,u_x,u^2+u_x^2)$. We define y as*

$$\int_{-\infty}^{y(\xi)}(g(\bar X(x))-1)\,dx+y(\xi)=\xi \tag{6.1}$$

and set

$$U=u\circ y\ \text{ and } h=(u^2+u_x^2)\circ y\,y_\xi.$$

Then, $X=(y,U,h)$ belongs to $\mathcal{G}_0$ and we denote by $\boldsymbol{L}$ the mapping $u\mapsto X$ from H^1 to $\mathcal{G}_0$.

Note the similarity between (6.1) and (5.1).

Proposition 6.2. *The mapping $\boldsymbol{L}$ sends bounded set of H^1 into bounded sets of $\mathcal{G}_0$, that is, for any $M > 0$ and $u \in H^1$, $\|u\|_{H^1} \leq M$ implies $\boldsymbol{L}(u) \in B_{\bar{M}}$ for some constant $\bar{M}$ depending only on M.*

Proof. We prove the well-posedness of the definition of $\boldsymbol{L}$ and Proposition 6.2 at the same time. We consider $u \in H^1$ such that $\|u\|_{H^1} \leq M$. Let

$$A = \{x \in \mathbb{R} \mid |u_x(x)| + 2(1+u^2)(x) \leq 1 + u^2(x) + u_x^2(x) \text{ and } u_x(x) \leq 0\}, \tag{6.2}$$

that is, $A = \{x \in \mathbb{R} \mid \bar{X}(x) \in \Omega\}$. Since $A \subset \{x \in \mathbb{R} \mid 1 \leq (u^2 + u_x^2)(x)\}$ and $u \in H^1$, the set A has finite measure and $\operatorname{meas}(A) \leq C(M)$ for some constant depending only on M. We will denote generically by $C(M)$ such constant. The function

$$z \mapsto \int_{-\infty}^{z} l(x)\,dx + z, \tag{6.3}$$

where $l(x) = g(\bar{X}(x)) - 1 = (|u_x| + (2u^2+1))(x)\chi_A(x) + (u^2 + u_x^2)(x)\chi_{A^c}(x)$, is continuous and strictly increasing. Therefore it is bijective and its inverse, $y(\xi)$, is well-defined. Let $\xi \leq \xi'$, since y is increasing and l positive, we get

$$y(\xi') - y(\xi) \leq \xi' - \xi. \tag{6.4}$$

Hence, y is Lipschitz with a Lipschitz constant smaller than one. Assuming again without loss of generality that $\xi \leq \xi'$, we have

$$\begin{aligned}
|U(\xi') - U(\xi)| &= |u \circ y(\xi') - u \circ y(\xi)| \\
&\leq \int_{y(\xi)}^{y(\xi')} |u_x(x)|\,dx \\
&\leq \int_{y(\xi)}^{y(\xi')} |u_x(x)|\,\chi_A(x)\,dx + \int_{y(\xi)}^{y(\xi')} |u_x(x)|\,\chi_{A^c}(x)\,dx.
\end{aligned}$$

From (6.1), we obtain $\int_{y(\xi)}^{y(\xi')} |u_x(x)|\,\chi_A(x)\,dx \leq |\xi' - \xi|$ and $\int_{y(\xi)}^{y(\xi')} u_x^2(x)\chi_{A^c}(x)\,dx \leq |\xi' - \xi|$. Therefore, after using Cauchy–Schwarz, we get

$$\begin{aligned}
|U(\xi') - U(\xi)| &\leq |\xi' - \xi| + |y(\xi') - y(\xi)|^{1/2} \Big(\int_{y(\xi)}^{y(\xi')} u_x^2(x)\chi_{A^c}(x)\,dx \Big)^{1/2} \\
&\leq 2(\xi' - \xi)
\end{aligned}$$

and U is Lipschitz. Let B_1 be the set where y and U are differentiable. Since y and U are Lipschitz, we have $\operatorname{meas}(B_1^c) = 0$. Let B_2 be the set where u and $z \mapsto \int_{-\infty}^{z} l(x)\,dx$ are differentiable. We have $\operatorname{meas}(B_2^c) = 0$. We denote

$$B = y^{-1}(B_2) \cap B_1. \tag{6.5}$$

By (6.3), for any interval I, we have

$$\int_{y(I)} (l(x) + 1)\,dx = \operatorname{meas}(I). \tag{6.6}$$

Using the Lebesgue monotone convergence theorem, as y is one-to-one, one can prove that $B \mapsto \int_{y(B)} (l(x)+1)\,dx$ is a measure. Since it coincides with the Lebesgue measure on any interval, it must coincide for any Borel set. Hence, (6.6) holds for any Borel set and we get $\operatorname{meas}(y^{-1}(B_2^c)) = \int_{B_2^c} (l(x)+1)\,dx = 0$ because $\operatorname{meas}(B_2^c) = 0$. Hence, $\operatorname{meas}(B^c) = 0$. For any $\xi \in B$, by differentiating (6.1), we obtain

$$y_\xi(\xi) = \frac{1}{1 + |u_x| + (2u^2+1)\chi_A + (u^2 + u_x^2)\chi_{A^c}} \circ y(\xi). \tag{6.7}$$

For any $\xi \in B$, we have

$$U_\xi(\xi) = u_x \circ y(\xi) y_\xi(\xi) \tag{6.8}$$

and, since $y_\xi > 0$ on B, we get that $h = U^2 y_\xi + \frac{U_\xi^2}{y_\xi}$ and X satisfies (2.23f) almost everywhere. From (2.23f), we get

$$\int_{\mathbb{R}} U_\xi^2(\xi)\, d\xi \leq \int_{\mathbb{R}} h y_\xi \, d\xi \leq \int_{\mathbb{R}} h \, d\xi = \int_{\mathbb{R}} (u^2 + u_x^2) \circ y y_\xi \, d\xi = \int_{\mathbb{R}} (u^2 + u_x^2)\, dx \leq M^2.$$

Hence, $U_\xi \in L^2$ and $\|U_\xi\|_{L^2} \leq C(M)$. We have $y(\xi) \leq \xi$ and therefore $\lim_{\xi \to -\infty} y(\xi) = -\infty$. Since $\zeta(\xi) = \int_{-\infty}^{y(\xi)} l(x)\, dx$ and $l \in L^1$, we have $\lim_{\zeta \to -\infty} \zeta(\xi) = 0$ and $\zeta \in L^\infty$. We define, as in (5.3), the set $S = \{\xi \in \mathbb{R} \mid X(\xi) \in \Omega\}$. For any $\xi \in y^{-1}(A) \cap B$,

$$|u_x| \circ y(\xi) + 2(1 + U^2(\xi)) \leq 1 + (u^2 + u_x^2) \circ y(\xi) \text{ and } u_x \circ y(\xi) \leq 0 \tag{6.9}$$

which after multiplying each side of the inequalities by y_ξ gives

$$|U_\xi(\xi)| + 2(1 + U^2(\xi)) y_\xi(\xi) \leq y_\xi(\xi) + h(\xi) \text{ and } U_\xi(\xi) \leq 0, \tag{6.10}$$

that is, $\xi \in S$ and we have proved that $y^{-1}(A) \cap B \subset S$. For any $\xi \in S \cap B$, (6.10) holds and implies (6.9) and therefore $y(\xi) \in A$. Hence, $y^{-1}(A) \cap B \subset S \subset y^{-1}(A) \cup B^c$ and

$$\chi_{y^{-1}(A)}(\xi) = \chi_S(\xi) \tag{6.11}$$

for almost every ξ because $\operatorname{meas}(B^c) = 0$. After differentiating (6.1) and using (6.8), we obtain

$$(|U_\xi| + 2(U^2 + 1) y_\xi)(\xi) \chi_A(y(\xi)) + (y_\xi + h)(\xi) \chi_{A^c}(y(\xi)) = 1 \tag{6.12}$$

for any $\xi \in B$. Since $\chi_A(y(\xi)) = \chi_S(\xi)$ almost everywhere from (6.11), (6.12) implies

$$g(X(\xi)) = (|U_\xi| + 2(U^2 + 1) y_\xi)(\xi) \chi_S(\xi) + (y_\xi + h)(\xi) \chi_{S^c}(\xi) = 1$$

almost everywhere. We have

$$\begin{aligned}
\operatorname{meas}(S) &= \int_S g(X(\xi))\, d\xi = \int_S (|U_\xi| + 2(U^2 + 1) y_\xi)\, d\xi \\
&= \int_{y(S)} (|u_x| + 2(u^2 + 1))\, dx = \int_A (|u_x| + 2(u^2 + 1))\, dx \\
&\leq \operatorname{meas}(A)^{1/2} \|u_x\|_{L^2} + \operatorname{meas}(A)(2 \|u\|_{L^\infty}^2 + 1) \leq C(M).
\end{aligned} \tag{6.13}$$

Using the fact that $h + \zeta_\xi = 0$ on S^c and $|\zeta_\xi| \leq 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}} |\zeta_\xi|^2 \, d\xi &= \int_S |\zeta_\xi|^2 \, d\xi + \int_{S^c} |\zeta_\xi|^2 \, d\xi \\
&\leq \|\zeta_\xi\|_{L^\infty} \operatorname{meas}(S) + \int_{S^c} h \, d\xi \\
&\leq \|\zeta_\xi\|_{L^\infty} \operatorname{meas}(S) + \|h\|_{L^1} \leq C(M).
\end{aligned}$$

Hence, $\zeta_\xi \in L^2$ and $\|\zeta_\xi\|_{L^2} \leq C(M)$. Since $U = u \circ y$, $U \in L^\infty$. Let $B_3 = \{\xi \in \mathbb{R} \mid y_\xi \leq \frac{1}{2}\}$. Since $\zeta_\xi \in L^2$, using the Chebyshev inequality, one can prove that $\operatorname{meas}(B_3) < C(M)$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} U^2 \, d\xi &= \int_{B_3} U^2 \, d\xi + \int_{B_3^c} U^2 \, d\xi \\
&\leq \|U\|_{L^\infty}^2 \operatorname{meas}(B_3) + 2 \int_{B_3^c} (u^2 \circ y y_\xi)\, d\xi \\
&\leq \|u\|_{L^\infty}^2 \operatorname{meas}(B_3) + 2 \|u\|_{L^2}^2 \leq C(M).
\end{aligned}$$

Hence, $U \in L^2$ and $\|U\|_{L^2} \leq C(M)$. We have $g(X) \leq y_\xi + h$. Thus, $\left\|\frac{1}{y_\xi+h}\right\|_{L^\infty} \leq 1$ and (2.23e) is fulfilled. This concludes the proof that X belongs to $\mathcal{G}_0$. $\square$

We define the mapping $\boldsymbol{M}$ from Lagrangian coordinates to Eulerian coordinates as follows.

Definition 6.3. *Given $X \in \mathcal{G}$, the function*

$$u(x) = U(\xi) \text{ if } x = y(\xi)$$

is well-defined and belongs to H^1. We denote $\boldsymbol{M}$ the mapping $X \mapsto u$ from $\mathcal{G}_0$ to H^1.

Proposition 6.4. *We have*

$$\boldsymbol{M} \circ \Pi = \boldsymbol{M}.$$

From this proposition we recover the fact that if two elements are equivalent in Lagrangian coordinates up to a relabeling, then they have the same Eulerian representantive, that is, with our terminology, $\Pi(X) = \Pi(\bar{X})$ implies $\boldsymbol{M}(X) = \boldsymbol{M}(\bar{X})$.

Proof. We prove the well-posedness of the definition of $\boldsymbol{M}$ and Proposition 6.4. Given $x \in \mathbb{R}$, assume there are $\xi < \xi'$ such that $y(\xi) = y(\xi') = x$. Then, $y_\xi = 0$ on $[\xi, \xi']$. By (2.23f), it implies that $U_\xi = 0$ on $[\xi, \xi']$ and therefore $U(\xi) = U(\xi')$. Let us prove that $u \in L^2$. For any smooth function ψ, we get, after a change of variables and using Cauchy–Schwarz,

$$\begin{aligned}\int_{\mathbb{R}} u\psi\,dx &= \int U\psi \circ y y_\xi\,d\xi \\ &\leq \Big(\int_{\mathbb{R}} U^2 y_\xi\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} \psi^2 \circ y y_\xi\Big)^{1/2} \\ &\leq \|h\|_{L^1}^{1/2}\,\|\psi\|_{L^2}\end{aligned}$$

as $U^2 y_\xi \leq h$ by (2.23f). Hence, $u \in L^2$. We have

$$\int_{\mathbb{R}} u\psi_x\,dx = \int_{\mathbb{R}} U(\xi)\psi_x \circ y(\xi) y_\xi(\xi)\,d\xi = -\int_{\mathbb{R}} U_\xi \psi \circ y\,d\xi = -\int_{\{\xi\in\mathbb{R} \mid y_\xi(\xi)>0\}} U_\xi \psi \circ y\,d\xi.$$

We can reduce the domain of integration because $U_\xi = 0$ on $\{\xi \in \mathbb{R} \mid y_\xi(\xi) = 0\}$, by (2.23f). Then,

$$\int_{\mathbb{R}} u\psi_x\,dx \leq \Big(\int_{\{\xi\in\mathbb{R} \mid y_\xi(\xi)>0\}} \frac{U_\xi^2}{y_\xi}\,d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} \psi^2 \circ y y_\xi\,d\xi\Big)^{1/2} \leq \|h\|_{L^1}^{1/2}\,\|\psi\|_{L^2}$$

as $\frac{U_\xi^2}{y_\xi} \leq h$ by (2.23f). Hence, $u_x \in L^2$ and we have proved that $u \in H^1$. Let us prove Proposition 6.4. Given $X \in \mathcal{G}$, we denote $u = \boldsymbol{M}(X)$, $\bar{X} = \Pi(X)$ and $\bar{u} = \boldsymbol{M}(\bar{X})$. We have $u \circ y = U$, $\bar{u} \circ \bar{y} = \bar{U}$ and $\bar{y} \circ \varphi = y$, $\bar{U} \circ \varphi = U$ for φ as defined in (5.1). Hence,

$$u \circ y = U = \bar{U} \circ \varphi = \bar{u} \circ \bar{y} \circ \varphi = \bar{u} \circ y$$

and $u = \bar{u}$. $\square$

We prove that H^1 is in bijection with $\mathcal{G}_0$. More precisely we have the following theorem.

Theorem 6.5.

$$\boldsymbol{M} \circ \boldsymbol{L} = \mathrm{Id} \ \textit{ and } \boldsymbol{L} \circ \boldsymbol{M} = \Pi.$$

Proof. Given $u \in H^1$, we denote $X = \boldsymbol{L}(u)$ and $\bar{u} = \boldsymbol{M}(X)$. We have $U = u \circ y$. Since $X \in \mathcal{G}_0$, y is invertible and therefore $\bar{u} = U \circ y^{-1}$. Hence, $\bar{u} = u$ and $\boldsymbol{M} \circ \boldsymbol{L} = \mathrm{Id}$.

Given $X \in \mathcal{G}_0$, we denote $u = \boldsymbol{M}(X)$ and $\bar{X} = \boldsymbol{L}(u)$. Let S be defined as earlier, $S = \{\xi \in \mathbb{R} \mid X(\xi) \in \Omega\}$. We know that $\mathrm{meas}(S) < \infty$, see (5.5). We have $g(X) - y_\xi \in L^1$, see (5.6). Since $g(X) = 1$, we have $g(X) - y_\xi = 1 - y_\xi = \zeta_\xi$ and

$$\int_{\xi'}^{\xi} (g(X) - y_\xi)\, d\xi = -\zeta(\xi) + \zeta(\xi').$$

Letting ξ' tend to $-\infty$, as $\lim_{\xi \to -\infty} \zeta(\xi) = 0$, we get

$$\int_{-\infty}^{\xi} (g(X) - y_\xi)\, d\eta + y(\xi) = \xi$$

which can be rewritten as

$$\int_{-\infty}^{\xi} \big((|U_\xi| + (2U^2 + 1)y_\xi)(\xi)\chi_S(\xi) + h(\xi)\chi_{S^c}(\xi)\big)\, d\xi + y(\xi) = \xi$$

or

$$\int_{-\infty}^{\xi} \big((|U_\xi| + (2U^2 + 1)y_\xi)(\xi)\chi_{y^{-1}(A)}(\xi) + h(\xi)\chi_{y^{-1}(A^c)}(\xi)\big)\, d\xi + y(\xi) = \xi \tag{6.14}$$

by (6.11). Since, $U_\xi = u \circ y y_\xi$ and $h = (u^2 \circ y + u_x^2 \circ y)y_\xi$ almost everywhere, after a change of variables in (6.14), we get

$$\int_{-\infty}^{y(\xi)} \big((|u_x| + (2u^2 + 1))\chi_A + (u^2 + u_x^2)\chi_{y(A^c)}\big)\, dx + y(\xi) = \xi. \tag{6.15}$$

Hence, y and, by definition, $\bar{y}$ satisfy (6.1) and therefore they coincide, i.e., $\bar{y} = y$. We have $\bar{U} = u \circ \bar{y} = u \circ y = U$ and $\bar{h} = \bar{U}^2\bar{y}_\xi + \frac{\bar{U}_\xi^2}{y_\xi} = U^2 y_\xi + \frac{U_\xi^2}{y_\xi} = h$ almost everywhere. □

Earlier we noted that $\tilde{d}_{\mathbb{R}}$ does not satisfy the Hausdorff property on $\mathcal{G}$. However, for $X, \bar{X} \in \mathcal{G}_0$, we have that $\tilde{d}_{\mathbb{R}}(X, \bar{X}) = 0$ implies $X = \bar{X}$, and therefore $\tilde{d}_{\mathbb{R}}$ is a metric on $\mathcal{G}_0$. Let us prove that. We have

$$\left\|\zeta - \bar{\zeta}\right\|_{L^\infty} + \left\|U - \bar{U}\right\|_{L^2} \leq \tilde{d}_{\mathbb{R}}(X, \bar{X}) = 0.$$

Hence, $\zeta = \bar{\zeta}$ and $U = \bar{U}$. Since for $X \in \mathcal{G}_0$, we have $y_\xi > 0$ almost everywhere (and similarly for $\bar{X}$), we get $\bar{h} = \bar{U}^2\bar{y}_\xi + \frac{\bar{U}_\xi^2}{y_\xi} = U^2 y_\xi + \frac{U_\xi^2}{y_\xi} = h$. Hence, $X = \bar{X}$.

The bijective mapping $\boldsymbol{L}$ allows us to transport the metric $\tilde{d}_{\mathbb{R}}$ and the semigroup $\tilde{S}_t$ from $\mathcal{G}_0$ to H^1. We define the metric d_{H^1} on H^1 as follows

$$d_{H^1}(u, \bar{u}) = \tilde{d}_{\mathbb{R}}(\boldsymbol{L}(u), \boldsymbol{L}(\bar{u}))$$

and the semigroup T_t on H^1 as follows

$$T_t = \boldsymbol{M} \circ \tilde{S} \circ \boldsymbol{L}.$$

The main result of this paper is the following theorem.

Theorem 6.6. *The semigroup T_t constitutes a semigroup of weak solutions of the Camassa–Holm equation, that is, for any initial data u_0 in H^1, $u(t, x) = T_t(u_0)$ is a weak solution of* (2.1).

The semigroup T_t is continuous with respect to the metric d_{H^1} on bounded sets of H^1, that is, for any $M > 0$ and any sequence $u_n \in H^1$ such that $\|u_n\|_{H^1} \leq M$, we have $\lim_{n\to\infty} d_{H^1}(u_n, u) = 0$ implies $\lim_{n\to\infty} d_{H^1}(T_t(u_n), T_t(u)) = 0$.

Proof. Let us start by proving that $u(t,x)$ is a weak solution of (2.1). By definition, we have $u(t,x) = \boldsymbol{M}\circ\Pi\circ S_t\circ\boldsymbol{L}(u_0)$. From Proposition 6.4, we get that $u(t,x) = \boldsymbol{M}\circ S_t\circ\boldsymbol{L}(u_0)$. Let us denote $X_0 = \boldsymbol{L}(u_0)$ and $X(t) = S_t(X_0)$. We want to prove that, for all $\varphi\in C^\infty(\mathbb{R}_+\times\mathbb{R})$ with compact support,

$$\int_{\mathbb{R}_+\times\mathbb{R}} \left[-u(t,x)\varphi_t(t,x) + u(t,x)u_x(t,x)\varphi(t,x)\right]\,dxdt = \int_{\mathbb{R}_+\times\mathbb{R}} -P_x(t,x)\varphi(t,x)\,dxdt \tag{6.16}$$

where P is given by (2.1b) or equivalently (2.2). We use the change of variables $x = y(t,\xi)$ and, since $U_\xi = u_x\circ y\, y_\xi$, we get

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} &\left[-u(t,x)\varphi_t(t,x) + u(t,x)u_x(t,x)\varphi(t,x)\right]\,dxdt\\ &= \int_{\mathbb{R}_+\times\mathbb{R}} \left[-U(t,\xi)y_\xi(t,\xi)\varphi_t(t,y(t,\xi)) + U(t,\xi)U_\xi(t,\xi)\varphi(t,y(t,\xi))\right]\,d\xi dt.\end{aligned} \tag{6.17}$$

We have $y_{\xi t}(t,\xi) = \chi_{\{\tau(\xi)>t\}}(\xi)U_\xi(t,\xi)$, from (2.19). Since $U_\xi(t,\xi) = 0$ for $t\geq\tau(\xi)$, we get $y_{\xi t}(t,\xi) = U_\xi(t,\xi)$. Then, using the fact that $y_t = U$, one easily check that

$$(Uy_\xi\varphi\circ y)_t - (U^2\varphi)_\xi = Uy_\xi\varphi_t\circ y - UU_\xi\varphi\circ y + U_t y_\xi\varphi\circ y. \tag{6.18}$$

After integrating (6.18) over $\mathbb{R}_+\times\mathbb{R}$, the left-hand side of (6.18) vanishes and we obtain

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} &\left[-Uy_\xi\,\varphi_t\circ y + UU_\xi\,\varphi\circ y\right]\,d\xi dt\\ &= \frac{1}{4}\int_{\mathbb{R}_+\times\mathbb{R}\times\{\tau(\eta)>t\}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\\ &\qquad\qquad\times\big(U^2y_\xi + h\big)(\eta)y_\xi(\xi)\varphi\circ y(\xi)\,d\eta d\xi dt\end{aligned} \tag{6.19}$$

by (2.18). Since $y_\xi(t,\xi) = 0$ for $t\geq\tau(\xi)$, we can change the integration domain to $\mathbb{R}_+\times\{\tau(\xi)>t\}\times\{\tau(\eta)>t\}$, and we have

$$\begin{aligned}\int_{\mathbb{R}_+\times\mathbb{R}} &\left[-Uy_\xi\,\varphi_t\circ y + UU_\xi\,\varphi\circ y\right]\,d\xi dt\\ &= \frac{1}{4}\int_{\mathbb{R}_+\times\{\tau(\xi)>t\}\times\{\tau(\eta)>t\}} \operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\\ &\qquad\qquad\times\big(U^2y_\xi + h\big)(\eta)y_\xi(\xi)\varphi\circ y(\xi)\,d\eta d\xi dt.\end{aligned}$$

To simplify the notation, we deliberately omitted the t variable. On the other hand, by using the change of variables $x = y(t,\xi)$ and $z = y(t,\eta)$, we have

$$\begin{aligned}-\int_{\mathbb{R}_+\times\mathbb{R}} &P_x(t,x)\varphi(t,x)\,dxdt\\ &= \frac{1}{2}\int_{\mathbb{R}_+\times\mathbb{R}^2} \operatorname{sgn}(y(\xi)-y(\eta))e^{-|y(\xi)-y(\eta)|}\\ &\qquad\times\Big(u^2(t,y(\eta)) + \frac{1}{2}u_x^2(t,y(\eta))\Big)\varphi(t,y(\xi))y_\xi(\eta)y_\xi(\xi)\,d\eta d\xi dt.\end{aligned} \tag{6.20}$$

Again, we can restrict the integration domain to $\mathbb{R}_+\times\{\tau(\xi)>t\}\times\{\tau(\eta)>t\}$ because $y_\xi(t,\xi) = 0$ for $t\geq\tau(\xi)$ and $y_\xi(t,\eta) = 0$ for $t\geq\tau(\eta)$. Moreover when $y_\xi(t,\eta) > 0$, we have $u_x(t,y(t,\eta)) = \frac{U_\xi}{y_\xi}(t,\eta)$. Hence, as y is an increasing function and after using (2.23f), we get

$$-\int_{\mathbb{R}_+\times\mathbb{R}} P_x(t,x)\varphi(t,x)\,dxdt \tag{6.21}$$

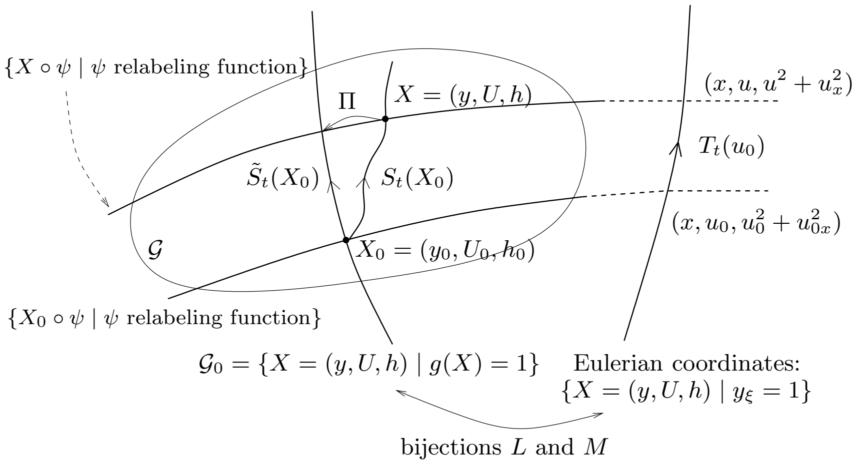

FIGURE 4. Summary of the method in a picture. In section 2-4, we establish the existence of the continuous semigroup of solution S_t in $\mathcal{G}$. Then, in section 5, by using the projection Π we construct the continuous semigroup $\tilde{S}_t$ in $\mathcal{G}_0$. Finally, in section 6, by using the bijection L between Eulerian coordinates and $\mathcal{G}_0$, we construct the continuous semigroup T_t of dissipative solutions of the Camassa-Holm equation in H^1 for the metric d_{H^1}.

$$
\begin{aligned}
= \frac{1}{4}\int_{\mathbb{R}_+\times\{\tau(\xi)>t\}\times\{\tau(\eta)>t\}} &\operatorname{sgn}(\xi-\eta)\exp\big(-\operatorname{sgn}(\xi-\eta)(y(\xi)-y(\eta))\big)\\
&\times\big(U^2y_\xi+h\big)(\eta)y_\xi(\xi)\varphi(t,y(\xi))\,d\eta d\xi dt.
\end{aligned}
$$

Thus, comparing (6.20) and (6.21), we get

$$
\int_{\mathbb{R}_+\times\mathbb{R}}[-Uy_\xi\,\varphi_t(t,y)+UU_\xi\,\varphi]\,d\xi dt = -\int_{\mathbb{R}_+\times\mathbb{R}}P_x(t,x)\varphi(t,x)\,dxdt
$$

and (6.16) follows from (6.17). The continuity of the semigroup follows from Proposition 6.2 and Theorem 5.6. Given a converging sequence u_n in d_{H^1} such that $\|u_n\|_{H^1}\le M$, we have $X_n = \boldsymbol{L}(u_n)\to X = \boldsymbol{L}(u)$ in $\tilde{d}_{\mathbb{R}}$ and $\boldsymbol{L}(u_n)\in B_{\bar{M}}$ for some $\bar{M}$, by Proposition 6.2. From Theorem 5.6, we get $\tilde{S}_t(X_n)\to\tilde{S}_t(X)$ in $\tilde{d}_{\mathbb{R}}$ and therefore $T_t(u_n)\to T_t(u)$ in d_{H^1}. □

Proposition 6.7. *Given any initial data $u_0\in H^1$ and the corresponding dissipative solution u, we have the following one-sided estimate on the derivative: For almost every x and all $t\ge 0$,*

$$
u_x(t,x)\le\frac{2}{t}+\sqrt{2}\,\|u_0\|_{H^1}\,. \tag{6.22}
$$

Proof. Given $\bar{t}\ge 0$, let us denote

$$
B = \{\xi\in\mathbb{R}\mid y(\bar{t},\xi), U(\bar{t},\xi)\text{ are differentiable and }y_\xi(\bar{t},\xi)>0\}\,.
$$

The set $y(B)$ has full measure (we drop $\bar{t}$ in the notation). Indeed, we have, after a change of variables

$$
\operatorname{meas}(y(B)^c)=\int_{y(B)^c}dx=\int_{B^c}y_\xi\,d\xi=0.
$$

Hence, for all $\bar{t}$ and for almost every x, there exists ξ such that $y(\bar{t},\xi)>0$. From the definition of $\tau(\xi)$, it follows that $\bar{t}<\tau(\xi)$ and $y_\xi(t,\xi)>0$ for all $t\in[0,\bar{t}]$. The variable $\alpha = \frac{U_\xi}{y_\xi}$ is thus well-defined on $[0,\bar{t}]$ and, since $U_\xi = u_x \circ y y_\xi$, we have

$$u_x(\bar{t},x) = u_x(\bar{t},y(\bar{t},\xi)) = \frac{U_\xi(\bar{t},\xi)}{y_\xi(\bar{t},\xi)} = \alpha(\bar{t},\xi). \tag{6.23}$$

From (2.19), we get

$$\alpha_t = \frac{U_{\xi,t}y_\xi - y_{\xi,t}U_\xi}{y_\xi^2} = \frac{\frac{1}{2}hy_\xi + (\frac{1}{2}U^2 - P)y_\xi - U_\xi^2}{y_\xi^2}$$

which yields

$$\alpha_t + \frac{1}{2}\alpha = (U^2 - P) \tag{6.24}$$

after using (2.23f).[1] Using (2.106), we get that $\|U\|_{L^\infty_T L^\infty_\mathbb{R}} \leq \frac{1}{\sqrt{2}}\|h_0\|_{L^1}^{1/2} = \frac{1}{\sqrt{2}}\|u_0\|_{H^1}$ and, from (2.108), we have

$$Q(t,\xi) \leq \frac{1}{2}\|U\|^2_{L^\infty_T L^\infty_\mathbb{R}} + \frac{1}{4}\|h_0\|_{L^1} \leq \frac{1}{2}\|u_0\|^2_{H^1}.$$

The same estimate holds for P, and therefore we have

$$U^2 - P \leq \|u_0\|^2_{H^1} \tag{6.25}$$

for all t and ξ. We claim that

$$\alpha(t,\xi) \leq \frac{2}{t} + \sqrt{2}\|u_0\|_{H^1} \tag{6.26}$$

Let us assume the opposite. We denote $\beta(t) = \frac{2}{t} + \sqrt{2}\|u_0\|_{H^1}$ and t_0 the first time when $\alpha(t_0) = \beta(t_0)$. For $t \leq t_0$ we have $\alpha(t) < \beta(t)$ which implies that $\beta_t(t_0) \leq \alpha_t(t_0)$. On the other hand,

$$\begin{aligned}\alpha_t(t_0) &= -\frac{1}{2}\alpha^2(t_0) + (U^2 - P) = -\frac{1}{2}\beta^2(t_0) + (U^2 - P)\\ &= \beta_t(t_0) - \frac{2\sqrt{2}}{t}\|u_0\|_{H^1} + U^2 - P - \|u_0\|^2_{H^1},\end{aligned}$$

which implies, after using (6.25), that $\alpha_t(t_0) < \beta_t(t_0)$, and we get a contradiction. We conclude the proof of the proposition by comparing (6.26) and (6.23). □

7. The metric d_{H^1}

The metric d_{H^1} is defined implicitly through the mapping from Eulerian to Lagrangian coordinates. In this section we give more explicit characterizations. We prove that convergence with respect to the H^1-norm implies convergence with respect to d_{H^1} which itself implies convergence with respect to the L^∞-norm.

Proposition 7.1. *Given a sequence $u_n \in H^1$ and $u \in H^1$, we have*

$$\lim_{n\to\infty}\|u_n - u\|_{H^1} = 0 \text{ implies } \lim_{n\to\infty} d_{H^1}(u_n,u) = 0.$$

Proof. Let $\bar{X}_n = (x, u_n, 1, u_{n,x}, u^2 + u_{n,x}^2)$ and $\bar{X} = (x,u,1,u_x,u^2+u_x^2)$. We write $l_n = g(X_n) - 1$ and $l = g(X) - 1$. We denote $X_n = \boldsymbol{L}(u_n)$ and $X = \boldsymbol{L}(u)$. We have

$$\int_{-\infty}^{y(\xi)} l(x)\,dx + y(\xi) = \xi\ , \qquad \int_{-\infty}^{y_n(\xi)} l_n(x)\,dx + y_n(\xi) = \xi. \tag{7.1}$$

***Step 1**: $l_n \to l$ in $L^1(\mathbb{R})$.*
We define the set A as in (6.2)

$$A = \{x \in \mathbb{R} \mid |u_x|(x) + 2(1+u^2)(x) \leq 1 + u^2(x) + u_x^2(x) \text{ and } u_x(x) \leq 0\}$$

[1] Equation (6.24) also appears in [25] and [8] in the proof of the same estimate.

and A_n as

$$A_n = \{x \in \mathbb{R} \mid |u_{n,x}|(x) + 2(1+u_n^2)(x) \leq 1 + u_n^2(x) + u_{n,x}^2(x) \text{ and } u_{n,x}(x) \leq 0\}.$$

We have $A \subset \{x \in \mathbb{R} \mid 1 \leq (u^2 + u_x^2)(x)\}$ so that $\operatorname{meas}(A) \leq \|u\|_{H^1}$ and, similarly, $\operatorname{meas}(A_n) \leq \|u_n\|_{H^1}$. On $A \cap A_n$, we have $(l_n - l)(x) = |u_x| + 2u^2(x) - (|u_x| + 2u^2(x))$ and therefore

$$\|l_n - l\|_{L^1(A_n \cap A)} \leq C \operatorname{meas}(A_n \cap A)^{1/2} \|u - u_n\|_{H^1} \tag{7.2}$$

for a constant C that depends only on $\|u\|_{H^1}$. We denote generically by C such constants. On $A^c \cap A_n^c$, we have $(l_n - l)(x) = (u^2 + u_x^2)(x) - (u_n^2 + u_{n,x}^2)(x)$ and therefore

$$\|l_n - l\|_{L^1(A_n^c \cap A^c)} \leq C \|u - u_n\|_{H^1}. \tag{7.3}$$

We want to estimate $\|l_n - l\|_{L^1(A^c \cap A_n)}$. Let introduce the sets

$$B_1 = \{x \in \mathbb{R} \mid |u_x|(x) + 2(1+u^2)(x) > 1 + u^2(x) + u_x^2(x)\}$$

and

$$B_2 = \{x \in \mathbb{R} \mid |u_x|(x) + 2(1+u^2)(x) \leq 1 + u^2(x) + u_x^2(x) \text{ and } u_x(x) > 0\}.$$

We have $A^c \subset B_1 \cup B_2$. For $x \in B_1 \cap A_n$, we have

$$\begin{aligned} l(x) - l_n(x) &= (1 + u^2 + u_x^2)(x) - |u_{n,x}|(x) - 2(1+u_n^2)(x) \\ &\leq |u_x|(x) + 2(1+u^2)(x) - |u_{n,x}|(x) - 2(1+u_n^2)(x) \\ &\leq |u_x - u_{n,x}|(x) + 2\left|u^2 - u_n^2\right|(x) \end{aligned}$$

and

$$\begin{aligned} l_n(x) - l(x) &= |u_{n,x}|(x) - 2(1+u_n^2)(x) - (1 + u^2 + u_x^2)(x) \\ &\leq (1 + u_n^2 + u_{n,x}^2)(x) - (1 + u^2 + u_x^2)(x). \end{aligned}$$

Hence, $|l - l_n|(x) \leq |u - u_x|(x) + 2\left|u^2 - u_n^2\right|(x) + \left|u_x^2 - u_{n,x}^2\right|(x)$ and, since $\operatorname{meas}(B_1 \cap A_n) \leq C$ for n large enough,

$$\|l - l_n\|_{L^1(B_1 \cap A_n)} \leq C \|u - u_n\|_{H^1}. \tag{7.4}$$

For $x \in B_2 \cap A_n$, we have that $|u_x|(x) + (1+u^2)(x) \leq u_x^2(x)$ implies $|u_x|(x) \leq u_x^2(x)$ so that $|u_x|(x) \leq 1$. Since $u_x(x) > 0$, it yields $u_x(x) \geq 1$. Similarly we get $|u_{n,x}| \geq 1$ but, as $u_{n,x} \leq 0$, it gives $u_{n,x} \leq -1$. Then, $(u_x - u_{n,x})(x) \geq 2$ for all $x \in B_2 \cap A_n$ and therefore

$$\operatorname{meas}(B_2 \cap A_n) \leq \frac{1}{4} \|u_x - u_{n,x}\|_{L^2(\mathbb{R})}^2$$

and $\lim_{n\to\infty} \operatorname{meas}(B_2 \cap A_n) = 0$ as n tends to ∞. Hence,

$$\begin{aligned} \|l - l_n\|_{L^1(B_2 \cap A_n)} &= \left\|1 + u^2 + u_x^2 - (|u_{n,x}| + 2(1+u_n^2))\right\|_{L^1(B_2 \cap A_n)} \\ &\leq \left\|1 + u^2 + u_x^2 - (|u_x| + 2(1+u^2))\right\|_{L^1(B_2 \cap A_n)} \\ &\quad + \left\|(|u_x| + 2(1+u^2)) - (|u_{n,x}| + 2(1+u_n^2))\right\|_{L^1(B_2 \cap A_n)} \\ &\leq \left\|-u^2 + u_x^2\right\|_{L^1(B_2 \cap A_n)} + \operatorname{meas}(B_2 \cap A_n) \\ &\quad + \operatorname{meas}(B_2 \cap A_n)^{1/2} \|u_x\|_{L^2(\mathbb{R})} + C \|u - u_n\|_{H^1} \end{aligned}$$

and $\lim_{n\to\infty} \|l - l_n\|_{L^1(B_2 \cap A_n)} = 0$. From (7.4), it follows that $\lim_{n\to\infty} \|l_n - l\|_{L^1(A^c \cap A_n)} = 0$. Similarly, one proves that $\lim_{n\to\infty} \|l_n - l\|_{L^1(A \cap A_n^c)} = 0$. By (7.2) and (7.3), we conclude that $\lim_{n\to\infty} \|l_n - l\|_{L^1_{\mathbb{R}}} = 0$.

Step 2: *$\zeta_n \to \zeta$ in $L^\infty(\mathbb{R})$ and $\zeta_{n,\xi} \to \zeta_\xi$ in $L^2(\mathbb{R})$.*
After taking the difference between the two equations in (7.1), we obtain

$$\int_{-\infty}^{y(\xi)} (l - l_n)(x)\,dx + \int_{y_n(\xi)}^{y(\xi)} l_n(x)\,dx + y(\xi) - y_n(\xi) = 0. \tag{7.5}$$

Since l_n is positive, $\left|y - y_n + \int_{y_n}^{y} l_n(x)\,d\xi)\right| = |y - y_n| + \left|\int_{y_n}^{y} l_n(x)\,d\xi)\right|$ and (7.5) implies

$$|y(\xi) - y_n(\xi)| \leq \int_{-\infty}^{y(\xi)} |l - l_n|\,dx \leq \|l - l_n\|_{L^1(\mathbb{R})},$$

and it follows that $\zeta_n \to \zeta$ in $L^\infty(\mathbb{R})$. We have

$$y_\xi = \frac{1}{l \circ y + 1} \quad \text{and} \quad y_{n,\xi} = \frac{1}{l_n \circ y_n + 1} \tag{7.6}$$

almost everywhere, see (6.7). Hence,

$$\begin{aligned} \zeta_{n,\xi} - \zeta_\xi &= (l \circ y - l_n \circ y_n) y_{n,\xi} y_\xi \\ &= (l \circ y - l \circ y_n) y_{n,\xi} y_\xi + (l \circ y_n - l_n \circ y_n) y_{n,\xi} y_\xi. \end{aligned} \tag{7.7}$$

Since $0 \leq y_\xi \leq 1$, see (6.4), we have

$$\int_{\mathbb{R}} |l \circ y_n - l_n \circ y_n|\, y_{n,\xi} y_\xi\,d\xi \leq \int_{\mathbb{R}} |l \circ y_n - l_n \circ y_n|\, y_{n,\xi}\,d\xi = \|l - l_n\|_{L^1(\mathbb{R})}. \tag{7.8}$$

For any $\varepsilon > 0$, there exists a continuous function $\tilde{l}$ with compact support such that $\|l - \tilde{l}\|_{L^1(\mathbb{R})} \leq \varepsilon/3$. We can decompose the first term in the right-hand side of (7.7) into

$$\begin{aligned} (l \circ y - l \circ y_n) y_{n,\xi} y_\xi &= (l \circ y - \tilde{l} \circ y) y_{n,\xi} y_\xi \\ &\quad + (\tilde{l} \circ y - \tilde{l} \circ y_n) y_{n,\xi} y_\xi + (\tilde{l} \circ y_n - l \circ y_n) y_{n,\xi} y_\xi. \end{aligned} \tag{7.9}$$

Then, we have

$$\int_{\mathbb{R}} \left| l \circ y - \tilde{l} \circ y \right| y_{n,\xi} y_\xi\,d\xi \leq \int \left| l \circ y - \tilde{l} \circ y \right| y_\xi\,d\xi = \|l - \tilde{l}\|_{L^1(\mathbb{R})} \leq \varepsilon/3$$

and, similarly, we obtain $\int_{\mathbb{R}} |l \circ y_n - \tilde{l} \circ y_n| y_{n,\xi} y_\xi\,d\xi \leq \varepsilon/3$. Since $y_n \to y$ in $L^\infty(\mathbb{R})$ and $\tilde{l}$ is continuous with compact support, by applying the Lebesgue dominated convergence theorem, we obtain $\tilde{l} \circ y_n \to \tilde{l} \circ y$ in $L^1(\mathbb{R})$, and we can choose n big enough so that

$$\int_{\mathbb{R}} \left| \tilde{l} \circ y - \tilde{l} \circ y_n \right| y_{n,\xi} y_\xi\,d\xi \leq \|\tilde{l} \circ y - \tilde{l} \circ y_n\|_{L^1(\mathbb{R})} \leq \varepsilon/3.$$

Hence, from (7.9), we get that $\int_{\mathbb{R}} |l \circ y - l \circ y_n|\, y_{n,\xi} y_\xi\,d\xi \leq \varepsilon$ so that we have proved that

$$\lim_{n \to \infty} \int_{\mathbb{R}} |l \circ y - l \circ y_n|\, y_{n,\xi} y_\xi\,d\xi = 0,$$

and, from (7.7) and (7.8), it follows that $\zeta_{n,\xi} \to \zeta_\xi$ in $L^1(\mathbb{R})$. Since $\zeta_{n,\xi}$ is bounded in $L^\infty(\mathbb{R})$, we finally get that $\zeta_{n,\xi} \to \zeta_\xi$ in $L^2(\mathbb{R})$.

Step 3: *$U_n \to U$ in $H^1(\mathbb{R})$.*
Given two functions f_1 and f_2 in $H^1(\mathbb{R})$, we claim that

$$\|f_1 \circ y_n - f_2 \circ y_n\|_{H^1(\mathbb{R})} \leq C\, \|f_1 - f_2\|_{H^1(\mathbb{R})} \tag{7.10}$$

for some constant C which does not depend on n. Let us prove this claim. Let $C_n = \{x \in \mathbb{R} \mid l_n(x) > 1\}$. Chebychev's inequality yields $\operatorname{meas}(C_n) \leq \|l_n\|_{L^1(\mathbb{R})}$.

Let $B_n = \{\xi \in \mathbb{R} \mid y_{n,\xi}(\xi) < \frac{1}{2}\}$. Since $y_{n,\xi}(l_n \circ y_n + 1) = 1$ almost everywhere, $l_n \circ y_n > 1$ on B_n and therefore $y_n(B_n) \subset C_n$. From (6.6), we get

$$\operatorname{meas}(B_n) = \int_{y(B_n)} (l_n(x)+1)\,dx \leq \operatorname{meas}(y_n(B_n)) + \|l_n\|_{L^1(\mathbb{R})} \leq \operatorname{meas}(C_n) + \|l_n\|_{L^1(\mathbb{R})},$$

and therefore $\operatorname{meas}(B_n) \leq 2\,\|l_n\|_{L^1(\mathbb{R})} \leq C$. We have

$$\|f_1 \circ y_n - f_2 \circ y_n\|^2_{L^2(\mathbb{R})} = \int_{B_n} (f_1\circ y_n - f_2\circ y_n)^2\,d\xi + \int_{B_n^c} (f_1\circ y_n - f_2\circ y_n)^2\,d\xi, \tag{7.11}$$

and, as $y_{n,\xi} \geq \frac{1}{2}$ on B_n^c,

$$\int_{B_n^c} (f_1 \circ y_n - f_2 \circ y_n)^2\,d\xi \leq 2\int_{B_n^c} (f_1 \circ y_n - f_2 \circ y_n)^2 y_{n,\xi}\,d\xi \leq 2\,\|f_1 - f_2\|^2_{L^2(\mathbb{R})}.$$

Hence,

$$\|f_1 \circ y_n - f_2 \circ y_n\|^2_{L^2(\mathbb{R})} \leq \operatorname{meas}(B_n)\,\|f_1 - f_2\|^2_{L^\infty(\mathbb{R})} + 2\,\|f_1 - f_2\|^2_{L^2(\mathbb{R})}$$

and, since $\operatorname{meas}(B_n) \leq 2\,\|l_n\|_{L^1(\mathbb{R})}$,

$$\begin{aligned}\|f_1 \circ y_n - f_2 \circ y_n\|^2_{L^2(\mathbb{R})} &\leq 2\,\|l_n\|_{L^1(\mathbb{R})}\,\|f_1 - f_2\|^2_{L^\infty(\mathbb{R})} + 2\,\|f_1 - f_2\|^2_{L^2(\mathbb{R})} \\ &\leq C\,\|f_1 - f_2\|^2_{H^1(\mathbb{R})}. \end{aligned}\tag{7.12}$$

We have

$$\begin{aligned}\|(f_1 \circ y_n)_\xi - (f_2 \circ y_n)_\xi\|^2_{L^2(\mathbb{R})} &= \|f_{1,x} \circ y_n y_{n,\xi} - f_{2,x} \circ y_n y_{n,\xi}\|^2_{L^2(\mathbb{R})} \\ &\leq \int_{\mathbb{R}} |f_{1,x} \circ y_n - f_{2,x} \circ y_n|^2\, y_{n,\xi}\,d\xi \qquad (\text{because } y_{n,\xi} \leq 1) \\ &\leq \|f_{1,x} - f_{2,x}\|^2_{L^2(\mathbb{R})}. \end{aligned}\tag{7.13}$$

Combining (7.12) and (7.13), we prove the claim (7.10). We have

$$\begin{aligned}\|U_n - U\|_{H^1(\mathbb{R})} &= \|u_n \circ y_n - u \circ y\|_{H^1(\mathbb{R})} \\ &\leq \|u_n \circ y_n - u \circ y_n\|_{H^1(\mathbb{R})} + \|u \circ y - u \circ y_n\|_{H^1(\mathbb{R})} \\ &\leq C\,\|u_n - u\|_{H^1(\mathbb{R})} + \|u \circ y - u \circ y_n\|_{H^1(\mathbb{R})}, \end{aligned}\tag{7.14}$$

from (7.10). The class of smooth functions with compact support is dense in H^1 and therefore, by (7.10), for any $\varepsilon \geq 0$, there exists a smooth function $\tilde{u}$ with compact support such that $\|u \circ y - \tilde{u} \circ y\|_{H^1(\mathbb{R})} \leq \frac{\varepsilon}{4}$ and $\|u \circ y_n - \tilde{u} \circ y_n\|_{H^1(\mathbb{R})} \leq \frac{\varepsilon}{4}$ for all n. Hence,

$$\|u \circ y - u \circ y_n\|_{H^1(\mathbb{R})} \leq \frac{2\varepsilon}{4} + \|\tilde{u} \circ y - \tilde{u} \circ y_n\|_{H^1(\mathbb{R})}. \tag{7.15}$$

Since the support of $\tilde{u}$ is finite and $y_n \to y$ in $L^\infty(\mathbb{R})$, we have

$$\lim_{n\to\infty} \|\tilde{u} \circ y - \tilde{u} \circ y_n\|_{L^2(\mathbb{R})} = 0. \tag{7.16}$$

We have

$$\begin{aligned}\|(\tilde{u} \circ y)_\xi - (\tilde{u} \circ y_n)_\xi\|_{L^2(\mathbb{R})} &= \|\tilde{u}_x \circ y y_\xi - \tilde{u}_x \circ y_n y_{n,\xi}\|_{L^2(\mathbb{R})} \\ &\leq \|y_\xi(\tilde{u}_x \circ y - \tilde{u}_x \circ y_n)\|_{L^2(\mathbb{R})} \\ &\qquad + \|(y_\xi - y_{n,\xi})\,\tilde{u}_x \circ y_n\|_{L^2(\mathbb{R})} \\ &\leq \|y_\xi\|_{L^\infty}\,\|\tilde{u}_x \circ y - \tilde{u}_x \circ y_n\|_{L^2(\mathbb{R})} \\ &\qquad + \|\tilde{u}_x\|_{L^\infty(\mathbb{R})}\,\|y_\xi - y_{n,\xi}\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence, $\lim_{n\to\infty} \|(\tilde{u}\circ y)_\xi - (\tilde{u}\circ y_n)_\xi\|_{L^2(\mathbb{R})} = 0$ because we have proved that $\zeta_{n,\xi} \to \zeta$ in $L^2(\mathbb{R})$ and $\lim_{n\to\infty} \|\tilde{u}_x \circ y - \tilde{u}_x \circ y_n\|_{L^2(\mathbb{R})} = 0$ as $y_n \to y$ in $L^\infty(\mathbb{R})$ and $\tilde{u}_x$ is smooth with compact support. By (7.16), it implies

$$\lim_{n\to\infty} \|\tilde{u}\circ y - \tilde{u}\circ y_n\|_{H^1(\mathbb{R})} = 0.$$

Combining (7.14) and (7.15), we obtain that $\|U_n - U\|_{H^1(\mathbb{R})} \leq \varepsilon$ for n large enough and we have proved that $U_n \to U$ in $H^1(\mathbb{R})$. □

Proposition 7.2. *Given u_n and u in $H^1(\mathbb{R})$, we have that*

$$\lim_{n\to\infty} d_{H^1}(u_n, u) = 0 \text{ implies } \lim_{n\to\infty} \|u_n - u\|_{L^\infty(\mathbb{R})} = 0.$$

Proof. Let $X_n = \boldsymbol{L}(u_n)$ and $X = \boldsymbol{L}(u)$. By definition, $\lim_{n\to\infty} d_{H^1(\mathbb{R})}(u_n, u) = 0$ implies that $X_n \to X$ in V. In particular, $y_n - y$ and $U_n - U$ tend to zero in $L^\infty(\mathbb{R})$. Given $x \in \mathbb{R}$, let $\xi_n = {y_n}^{-1}(x)$, $\xi = y^{-1}(x)$ and $x_n = y(\xi_n)$. We have

$$u(x) - u_n(x) = u(x) - u(x_n) + U(\xi_n) - U_n(\xi_n) \tag{7.17}$$

and

$$\begin{aligned} |u(x) - u(x_n)| &= \left| \int_{x_n}^{x} u_x(\bar{x})\, d\bar{x} \right| \\ &\leq |x - x_n|^{1/2} \|u_x\|_{L^2(\mathbb{R})} = |y_n(\xi_n) - y(\xi_n)|^{1/2} \|u_x\|_{L^2(\mathbb{R})} \\ &\leq \|y - y_n\|_{L^\infty(\mathbb{R})}^{1/2} \|u_x\|_{L^2(\mathbb{R})}. \end{aligned} \tag{7.18}$$

Hence,

$$|u_n(x) - u(x)| \leq \|y - y_n\|_{L^\infty(\mathbb{R})}^{1/2} \|u_x\|_{L^2(\mathbb{R})} + \|U - U_n\|_{L^\infty}$$

and $u_n \to u$ in $L^\infty(\mathbb{R})$. □

References

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of Bounded Variation and Free Discontinuity Problems*. Clarendon Press, New York, 2000.

[2] R. Beals, D. Sattinger, and J. Szmigielski. Peakon-antipeakon interaction. *J. Nonlinear Math. Phys.* 8:23–27, 2001.

[3] A. Bressan and A. Constantin. Global conservative solutions of the Camassa–Holm equation. *Arch. Ration. Mech. Anal.*, 183:215–239, 2007.

[4] A. Bressan and A. Constantin. Global dissipative solutions of the Camassa–Holm equation. *Analysis and Applications*, 5:1–27, 2007.

[5] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys. Rev. Lett.*, 71(11):1661–1664, 1993.

[6] R. Camassa, D. D. Holm, and J. Hyman. A new integrable shallow water equation. *Adv. Appl. Mech.*, 31:1–33, 1994.

[7] G. M. Coclite, H. Holden, and K. H, Karlsen. Well-posedness for a parabolic-elliptic system. *Discrete Cont. Dynam. Systems* 13:659–682, 2005.

[8] G. M. Coclite, H. Holden, and K. H, Karlsen. Global weak solutions to a generalized hyperelastic-rod wave equation. *SIAM J. Math. Anal.*, 37:1044–1069, 2005.

[9] G. M. Coclite, K. H, Karlsen, and N. H. Risebro. A convergent finite difference scheme for the Camassa–Holm equation with general H^1 data. *Preprint*, submitted.

[10] G. M. Coclite, K. H, Karlsen, and N. H. Risebro. An explicit finite difference scheme for the Camassa–Holm equation. *Preprint*, submitted.

[11] A. Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann. Inst. Fourier (Grenoble)*, 50(2):321–362, 2000.

[12] A. Constantin and J. Escher. Global existence and blow-up for a shallow water equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 26(2):303–328, 1998.

[13] A. Constantin and J. Escher. Global weak solutions for a shallow water equation. *Indiana Univ. Math. J.*, 47:1527–1545, 1998.

[14] A. Constantin and J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.*, 181:229–243, 1998.

[15] A. Constantin and L. Molinet. Global weak solutions for a shallow water equation. *Comm. Math. Phys.*, 211(1):45–61, 2000.

[16] A. Fokas and B. Fuchsteiner. Symplectic structures, their Bäcklund transformations and hereditary symmetries. *Phys.* D 4:47–66, 1981/82.

[17] M. Fonte. Conservative solution of the Camassa Holm equation on the real line. *arXiv:math.AP/0511549*.

[18] H. Holden and X. Raynaud. Convergence of a ?nite di?erence scheme for the Camassa–Holm equation. *SIAM J. Num. Anal.*, 44(4):1655?1680, 2006.

[19] H. Holden and X. Raynaud. Global multipeakon solutions of the Camassa–Holm equation. *J. Hyperbolic Differ. Equ.*, 4:39–64, 2007.

[20] H. Holden and X. Raynaud. Global conservative solutions of the generalized hyperelastic-rod wave equation. *J. Differential Equations*, 233:448–484, 2007.

[21] H. Holden and X. Raynaud. Periodic conservative solutions of the Camassa–Holm equation. *Annales de l'Institut Fourier,* to appear.

[22] H. Holden and X. Raynaud. Global conservative solutions of the Camassa–Holm equation — a Lagrangian point of view. *Comm. Partial Differential Equations*, 32:1511–1549, 2007.

[23] A. W. Knapp. *Basic Real Analysis*. Birkhäuser, Boston, MA, 2005.

[24] E. Wahlén. On the peakon-antipeakon interaction. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, to appear.

[25] Z. Xin and P. Zhang. On the weak solutions to a shallow water equation. *Comm. Pure Appl. Math.*, 53:1411–1433, 2000.

[26] Z. Xin and P. Zhang. On the uniqueness and large time behavior of the weak solutions to a shallow water equation. *Comm. Partial Differential Equations*, 27:1815–1844, 2002.

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