CONVERGENCE OF A MIXED METHOD FOR A SEMI-STATIONARY COMPRESSIBLE STOKES SYSTEM

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ABSTRACT. We propose and analyze a finite element method for a semi–stationary Stokes system modeling compressible fluid flow subject to a Navier–slip boundary condition. The velocity (momentum) equation is approximated by a mixed finite element method using the lowest order Nédélec spaces of the first kind. The continuity equation is approximated by a standard piecewise constant upwind discontinuous Galerkin scheme. Our main result states that the numerical method converges to a weak solution. The convergence proof consists of two main steps: (i) To establish strong spatial compactness of the velocity field, which is intricate since the element spaces are only div or curl conforming. (ii) To prove that the discontinuous Galerkin approximations converge strongly, which is required in view of the nonlinear pressure function. Tools involved in the analysis include a higher integrability estimate for the discontinuous Galerkin approximations, a discrete equation for the effective viscous flux, and various renormalized formulations of the discontinuous Galerkin scheme.

CONTENTS

1. Introduction

The purpose of this paper is to prove convergence of a finite element method for the semi–stationary barotropic compressible Stokes system

$$\varrho_t + \operatorname{div}(\varrho \boldsymbol{u}) = 0, \quad \text{in } (0,T) \times \Omega, \tag{1.1}$$

$$-\mu \Delta \boldsymbol{u} - \lambda D \operatorname{div} \boldsymbol{u} + D p(\varrho) = \boldsymbol{f}, \quad \text{in } (0,T) \times \Omega, \tag{1.2}$$

with initial data

$$\varrho|_{t=0} = \varrho_0, \quad \text{on } \Omega. \tag{1.3}$$

Here Ω is a simply connected, bounded, open, polygonal domain in $\mathbb{R}^N$ ($N = 2,3$), with Lipschitz boundary $\partial\Omega$, and $T > 0$ is a fixed final time. The unknowns are the density $\varrho = \varrho(t, \boldsymbol{x}) \geq 0$ and the velocity $\boldsymbol{u} = \boldsymbol{u}(t, \boldsymbol{x}) \in \mathbb{R}^N$, with $\boldsymbol{x} \in \Omega$ and $t \in (0,T)$. We denote by div and D the usual spatial divergence and gradient operators and by Δ the spatial Laplace operator.

The pressure function is assumed to be of the form $p(\varrho) = a\varrho^\gamma$, with $a > 0$ (Boyle's law). Typical values of γ ranges from a maximum of $\frac{5}{3}$ for monoatomic gases, through $\frac{7}{5}$ for diatomic gases *including air*, to lower values close to 1 for polyatomic gases at high temperatures. Throughout this paper we will always assume that $\gamma > 1$. The case $\gamma = 1$ can also be treated; indeed, it is simpler since the pressure function is linear. Furthermore, the viscosity coefficients μ, λ are assumed to be constant and satisfy $\mu > 0, N\lambda + 2\mu \geq 0$.

The study of the system (1.1)–(1.2) can be motivated in several ways. Firstly, the system can be used as a model equation for the barotropic compressible Navier–Stokes equations. This might be a reasonable approximation for strongly viscous fluids, where convection may be neglected. Secondly, Lions [15] use solutions of (1.1)–(1.2) to construct solutions to the barotropic compressible Navier–Stokes equations.

Among many others, the semi–stationary system (1.1)–(1.3) has been studied by Lions in [15, Section 8.2]. He proves the existence of weak solutions and some higher regularity results. In particular, weak solutions was proven to be unique in the case of periodic boundary conditions or when the equations are solved on the hole of $\mathbb{R}^N$. Uniqueness was not obtained in the case of regular Dirichlet boundary conditions and moreover higher regularity results was only shown to hold locally.

In this paper we impose the following boundary conditions, which are relevant in the context of geophysical fluids and shallow water models:

$$\boldsymbol{u} \cdot \nu = 0, \quad \text{on } (0,T) \times \partial\Omega, \tag{1.4}$$

and

$$\begin{aligned} \operatorname{curl} \boldsymbol{u} &= 0, \quad \text{on } (0,T) \times \partial\Omega \text{ if } N = 2, \\ \operatorname{curl} \boldsymbol{u} \times \nu &= 0, \quad \text{on } (0,T) \times \partial\Omega \text{ if } N = 3, \end{aligned} \tag{1.5}$$

where ν denotes the unit outward normal to $\partial\Omega$. The first condition is a natural condition of impermeability type on the normal velocity. The second condition is in the literature often referred to as the Navier–slip condition. It can be interpreted as a viscous dissipation term at the boundary (more precisely "non-dissipation" since this term is equal to zero) [15].

In some geophysical applications, conditions like (1.4)–(1.5) are preferred over the classical Dirichlet condition since the latter necessitates expensive calculations of boundary layers. Of more importance to this paper, the boundary conditions (1.4)–(1.5) will allow us to use the finite element method in a solution space that can be split into two orthogonal parts in terms of a discrete version of the Hodge decomposition, a fact that will play a crucial role in our analysis.

Although many numerical methods have been proposed for the compressible Stokes and Navier–Stokes equations, the convergence properties of these methods are mostly unsettled, especially in several spatial dimensions. Ultimately, it is not clear if these numerical methods converge to a weak solution as the discretization parameters tend to zero. In one dimension, the available results are due to Hoff and his collaborators [22, 23, 24]. All these results apply to the compressible Navier–Stokes equations in Lagrangian coordinates, and moreover require the initial density to be of bounded variation. Interesting results regarding the existence and long time behavior of solutions to the one dimensional compressible Navier–Stokes have also been obtained using semi–discrete finite difference schemes in [12, 13, 4], again in Lagrangian coordinates with the initial density of bounded total variation. In more than one spatial dimension, we refer to a recent paper [10] in which a convergent numerical method for a stationary compressible Stokes system is proposed. The Stokes system considered in [10] is similar to (1.1)–(1.2) with linear pressure and no temporal dependence.

Let us now discuss our choice of numerical method for the semi-stationary Stokes system. For the discretization of (1.1) we utilize a discontinuous Galerkin scheme based on piecewise constant approximations in space and time. The discontinuous Galerkin scheme was introduced more than 30 years ago [14, 19] and has since then undergone a blooming development, cf. [5, 6] for a review. In the context of linear transport equations with rough (i.e., non-Lipschitz) coefficients, a discontinuous Galerkin scheme, with piecewise polynomial approximations of arbitrary degree in the spatial variable and piecewise constant or linear approximations in the temporal variable, has recently been analyzed by Walkington in [21]. The work [21] is further developed in [16] for the variable-density incompressible Navier-Stokes equations.

Let us now turn to the velocity (or momentum) equation (1.2). By introducing the vorticity $\boldsymbol{w} = \operatorname{curl} \boldsymbol{u}$ as an auxiliary unknown, keeping in mind the vector identity $-\Delta = \operatorname{curl}\operatorname{curl} - D\operatorname{div}$, we can recast the momentum equation as

$$\mu \operatorname{curl} \boldsymbol{w} - (\lambda + \mu)D\operatorname{div}\boldsymbol{u} + Dp(\varrho) = \boldsymbol{f}, \qquad (1.6)$$

where we suppress the time variable t (we refer the reader to subsequent sections for more precision). Hence the velocity equation (1.2), together with the boundary conditions (1.4)–(1.5), admits a formulation that lends itself naturally to a mixed finite element method [11, 17, 18].

Denote by $\boldsymbol{W}_0^{\mathrm{div},2}$ the vector fields $\boldsymbol{u}$ on Ω for which $\operatorname{div}\boldsymbol{u} \in L^2$ and $\boldsymbol{u} \cdot \nu|_{\partial\Omega} = 0$, and by $\boldsymbol{W}_0^{\mathrm{curl},2}$ the vector fields $\boldsymbol{w}$ on Ω for which $\operatorname{curl}\boldsymbol{w} \in L^2$ and $\boldsymbol{w} \times \nu|_{\partial\Omega} = 0$. We choose corresponding mixed finite element spaces $\boldsymbol{V}_h \subset \boldsymbol{W}_0^{\mathrm{div},2}$ and $\boldsymbol{W}_h \subset \boldsymbol{W}_0^{\mathrm{curl},2}$ based on Nédélec's elements of the first kind [17]. The mixed finite element method seeks functions $\boldsymbol{w}_h \in \boldsymbol{W}_h$ and $\boldsymbol{u}_h \in \boldsymbol{V}_h$ such that

$$\int_{\Omega} \mu \operatorname{curl} \boldsymbol{w}_h \boldsymbol{v}_h + [(\mu + \lambda)\operatorname{div}\boldsymbol{u}_h - p(\varrho_h)]\operatorname{div}\boldsymbol{v}_h \ dx = \int_{\Omega} \boldsymbol{f}_h \boldsymbol{v_h} \ dx,$$

$$\int_{\Omega} \boldsymbol{w}_h \boldsymbol{\eta}_h - \operatorname{curl}\boldsymbol{\eta}_h \boldsymbol{u}_h \ dx = 0,$$

for all $(\boldsymbol{\eta}_h, \boldsymbol{v}_h) \in \boldsymbol{W}_h \times \boldsymbol{V}_h$, where $\varrho_h, \boldsymbol{f}_h$ are given piecewise constant functions.

Let us denote the numerical solution of the semi-stationary Stokes system by $(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h) = (\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)(t, \boldsymbol{x})$. The main goal is to prove that $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution, at least along a subsequence. The challenging issue is to show that the density approximations ϱ_h, which on the outset is only weakly compact in L^2, in fact converges strongly. Strong convergence is mandatory if we want to recover the semi-stationary Stokes system when taking the limit in the discrete equations as $h \to 0$. Related to this issue, the above mixed method enjoys

some advantages over the traditional finite element method based on $\boldsymbol{H}^1$ elements. In particular, the approximation spaces $\boldsymbol{W}_h$ and $\boldsymbol{V}_h$ satisfy

$$\boldsymbol{V}_h = \operatorname{curl} \boldsymbol{W}_h + \boldsymbol{Z}_h,$$

for some $\boldsymbol{Z}_h \subset \boldsymbol{V}_h$ satisfying $\boldsymbol{Z}_h \perp \operatorname{curl} \boldsymbol{W}_h$. An immediate consequence of this discrete Hodge decomposition is that upon writing $\boldsymbol{u}_h = \operatorname{curl} \boldsymbol{\eta}_h + \boldsymbol{z}_h$, we see that *only* $\boldsymbol{z}_h$ is coupled to the density ϱ_h and moreover that $\operatorname{curl} \boldsymbol{w}_h$, and hence $\boldsymbol{w}_h$, only depends on the data $\boldsymbol{f}$. More importantly, equipped with the discrete Hodge decomposition, we can separate the quantity $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h) = P(\varrho_h) - (\lambda + \mu) \operatorname{div} \boldsymbol{u}_h$ from the vorticity. The quanity $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)$ is the so-called effective viscous flux [15] associated with our discrete equations. The fact that we can separate the effective viscous flux from the vorticity makes it possible to prove the following weak continuity property:

$$\lim_{h \to 0} \iint P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)\, \varrho_h \; dxdt = \iint \overline{P_{\text{eff}}}\, \varrho \; dxdt \quad (\overline{P_{\text{eff}}}, \varrho \text{ are weak } L^2 \text{ limits}), \quad (1.7)$$

which is the decisive ingredient in the proof of strong convergence of the density approximations ϱ_h. Related to (1.7), we prove a higher integrability estimate on the pressure ensuring that $p(\varrho_h)$, and thus also $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)$, is weakly compact in L^2. The energy estimate only provides a uniform bound on $p(\varrho_h)$ in $L^\infty(L^1)$, so a priori it is not even clear that $p(\varrho_h)$ converges weakly to an integrable function. Our strong convergence argument is inspired by the work of Lions on the compressible Navier-Stokes equations, cf. [15].

As part of the analysis, we also show that $\varrho_h \boldsymbol{u}_h$ converges weakly to $\varrho \boldsymbol{u}$, where ϱ and $\boldsymbol{u}$ are weak limits of ϱ_h and $\boldsymbol{u}_h$, respectively. This convergence is not immediate since the element spaces utilized for the velocity approximations are merely div or curl conforming. In view of the discrete continuity equation (discontinuous Galerkin scheme), we easily obtain a bound on $(\varrho_h)_t$ in, say, $L^1(W^{-1,1})$. To conclude we need a spatial translation estimate of the form

$$\|\boldsymbol{u}_h - \boldsymbol{u}_h(\cdot, \cdot + \xi)\|_{L^2(\boldsymbol{L}^2)} \to 0 \quad \text{as } |\xi| \to 0, \text{ uniformly in } h. \quad (1.8)$$

In view of the discrete Hodge decomposition, we will actually only need (1.8) for weakly curl free approximations with a $\boldsymbol{L}^2$ bounded divergence.

For velocity fields that are independent of time t, (1.8) implies the $\boldsymbol{L}^2$ compactness of $\{\boldsymbol{u}_h\}_{h>0}$. In the time independent case, it is known that weakly curl free approximations with $\boldsymbol{L}^2$ bounded divergence is compact in $\boldsymbol{L}^2$ provided the approximation spaces satisfy the commuting diagram property [7]. However, despite the fact that the element spaces used here satisfy this property, the inclusion of time in $\boldsymbol{u}_h(t, x)$ makes earlier results inadequate. Specifically, to apply known result we would need L^∞ control in time of the velocity approximations. Unfortunately, this is not available in general for our problem. As a consequence, we shall provide a direct argument for the spatial translation estimate (1.8).

We wish to point out that although the boundary conditions (1.4)–(1.5) are not covered by Lions' results [15], his proofs can be adapted to yield existence, uniqueness, and regularity results for (1.1)–(1.2) with the boundary condtions (1.4)–(1.5). We will not pursue this project here, except for the existence part, which will be an immediate consequence of our convergence result. However, let us remark that the Navier–slip condition (1.5) is technically easier to handle than a Dirichlet condition, both from a mathematical and numerical point of view. The primary reason for this lies in the need for solutions of the auxiliary problem

$$\operatorname{div} \boldsymbol{v} = \boldsymbol{f}, \quad \operatorname{curl} \boldsymbol{v} = 0. \quad (1.9)$$

If $\int_\Omega \boldsymbol{f} \; dx = 0$, the function $\boldsymbol{v}$ will satisfy the boundary conditions (1.4)–(1.5). In other situations, like periodic boundary conditions or when the equations are solved

on $\mathbb{R}^N$, the boundary values of $\boldsymbol{v}$ does not matter. However, it is evident that $\boldsymbol{v}$ cannot be required both to satisfy Dirichlet boundary conditions and (1.9). Thus, (1.9) can only be required to hold locally whenever Dirichlet boundary conditions are imposed. To avoid "localizing" various discrete arguments, which sometimes can require elaborate work, we have chosen to consider the Navier–slip type condition (1.5) instead of the no–slip Dirichlet condition.

This paper is organized as follows: In Section 2, we introduce notation and list some basic results needed for the later analysis. Moreover, we recall the usual notion of weak solution and introduce a mixed weak formulation of the velocity equation. Finally, we introduce the finite element spaces and review some of their basic properties. In Section 3, we present the numerical method and state our main convergence result. The existence of a solution to the discrete equations is confirmed in Section 4. Section 5 is devoted to deriving basic estimates. In Section 6, we prove the main convergence result stated in Section 3. The proof is divided into several steps (subsections), including convergence of the continuity scheme, weak continuity of the discrete viscous flux, strong convergence of the density approximations, and convergence of the velocity scheme.

2. Preliminary material

2.1. **Some functional spaces and analysis results.** We make frequent use of the divergence and curl operators and denote these by div and curl, respectively. In the 2D case, we will denote both the rotation operator taking vectors into scalars and the curl operator taking scalars into vectors by curl. This confusing but rather standard notation greatly simplifies all subsequent arguments allowing identical treatment of the 2D and 3D cases.

We will also make use of the spaces

$$\boldsymbol{W}^{\mathrm{div},2}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega) \right\},$$
$$\boldsymbol{W}^{\mathrm{curl},2}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \right\},$$

where ν denotes the unit outward pointing normal vector on $\partial\Omega$. If $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{div},2}(\Omega)$ satisfies $\boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0$, we write $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{div},2}(\Omega)$. Similarly, $\boldsymbol{v} \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ means $\boldsymbol{v} \in \boldsymbol{W}^{\mathrm{div},2}(\Omega)$ and $\boldsymbol{v} \times \nu|_{\partial\Omega} = 0$. In two dimensions, $\boldsymbol{w}$ is a scalar function and the space $\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$ is to be understood as $W_0^{1,2}(\Omega)$. To define weak solutions, we shall use the space

$$\mathcal{W}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(\Omega) : \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega), \mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{L}^2(\Omega), \boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0 \right\},$$

which coincides with $\boldsymbol{W}_0^{\mathrm{div},2}(\Omega) \cap \boldsymbol{W}^{\mathrm{curl},2}(\Omega)$. The space $\mathcal{W}(\Omega)$ is equipped with the norm $\|\boldsymbol{v}\|_{\mathcal{W}}^2 = \|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\mathrm{div}\,\boldsymbol{v}\|_{L^2(\Omega)}^2 + \|\mathrm{curl}\,\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}^2$. It is known that $\|\cdot\|_{\mathcal{W}}$ is equivalent to the $\boldsymbol{H}^1$ norm on the space $\left\{ v \in \boldsymbol{H}^1(\Omega) : \boldsymbol{v} \cdot \nu|_{\partial\Omega} = 0 \right\}$, see, e.g., [15].

The space $\mathcal{W}(\Omega)$ admits a unique orthogonal Hodge decomposition

$$\mathcal{W}(\Omega) = \mathrm{curl}\,S(\Omega) + D\Delta^{-1}L_0^2(\Omega), \tag{2.1}$$

where $S(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega) : \mathrm{curl}\,\boldsymbol{v} \in \boldsymbol{W}^{1,2}(\Omega)\}$, Δ^{-1} is the inverse Neumann Laplace operator, and L_0^2 denotes the L^2 functions on Ω that have zero mean.

For the convenience of the reader we list some basic functional analysis results to be used in the subsequent arguments (for proofs, see, e.g.,[9]). Throughout the paper we use overbars to denote weak limits, with the underlying spaces being (silently) given by the context.

Lemma 2.1. *Let O be a bounded open subset of $\mathbb{R}^M$, $M \geq 1$. Suppose $g\colon \mathbb{R} \to (-\infty, \infty]$ is a lower semicontinuous convex function and $\{v_n\}_{n \geq 1}$ is a sequence of functions on O for which $v_n \rightharpoonup v$ in $L^1(O)$, $g(v_n) \in L^1(O)$ for each n, $g(v_n) \rightharpoonup$*

$\overline{g(v)}$ *in* $L^1(O)$. *Then* $g(v) \leq \overline{g(v)}$ *a.e. on* O, $g(v) \in L^1(O)$, *and* $\int_O g(v) \, dy \leq$ $\liminf_{n\to\infty} \int_O g(v_n) \, dy$. *If, in addition,* g *is strictly convex on an open interval* $(a,b) \subset \mathbb{R}$ *and* $g(v) = \overline{g(v)}$ *a.e. on* O, *then, passing to a subsequence if necessary,* $v_n(y) \to v(y)$ *for a.e.* $y \in \{y \in O \mid v(y) \in (a,b)\}$.

Let X be a Banach space and denote by $X^\star$ its dual. The space $X^\star$ equipped with the weak-$\star$ topology is denoted by $X^\star_{\text{weak}}$, while X equipped with the weak topology is denoted by X_{weak}. By the Banach-Alaoglu theorem, a bounded ball in $X^\star$ is $\sigma(X^\star, X)$-compact. If X separable, then the weak-$\star$ topology is metrizable on bounded sets in $X^\star$, and thus one can consider the metric space $C([0,T]; X^\star_{\text{weak}})$ of functions $v : [0,T] \to X^\star$ that are continuous with respect to the weak topology. We have $v_n \to v$ in $C([0,T]; X^\star_{\text{weak}})$ if $\langle v_n(t), \phi \rangle_{X^\star, X} \to \langle v(t), \phi \rangle_{X^\star, X}$ uniformly with respect to t, for any $\phi \in X$. The following lemma is a consequence of the Arzelà-Ascoli theorem:

Lemma 2.2. *Let* X *be a separable Banach space, and suppose* $v_n \colon [0,T] \to X^\star$, $n = 1, 2, \ldots$, *is a sequence for which* $\|v_n\|_{L^\infty([0,T]; X^\star)} \leq C$, *for some constant* C *independent of* n. *Suppose the sequence* $[0,T] \ni t \mapsto \langle v_n(t), \Phi \rangle_{X^\star, X}$, $n = 1, 2, \ldots$, *is equi-continuous for every* Φ *that belongs to a dense subset of* X. *Then* v_n *belongs to* $C([0,T]; X^\star_{\text{weak}})$ *for every* n, *and there exists a function* $v \in C([0,T]; X^\star_{\text{weak}})$ *such that along a subsequence as* $n \to \infty$ *there holds* $v_n \to v$ *in* $C([0,T]; X^\star_{\text{weak}})$.

In what follows, we will often obtain a priori estimates for a sequence $\{v_n\}_{n \geq 1}$ that we write as "$v_n \in_{\text{b}} X$" for some functional space X. What this really means is that we have a bound on $\|v_n\|_X$ that is independent of n.

2.2. **Topological degree in finite dimensions.** Our numerical method constitutes a nonlinear–implicit discrete problem. We will prove the existence of a solution to this problem by a topological degree argument [8].

Denote by $d(F, \Omega, y)$ the $\mathbb{Z}$–valued (Brouwer) degree of a continuous function $F : \bar{\Omega} \to \mathbb{R}^M$ at a point $y \in \mathbb{R}^N \backslash F(\partial S)$ relative to an open and bounded set $\Omega \subset \mathbb{R}^M$. For notational convenience, let us reformulate the definition of degree so that it applies directly in our finite element setting. Indeed, below we define $d_{S_h}(F, \tilde{S}_h, q_h)$ with $F : \tilde{S}_h \to S_h$ being a continuous finite element mapping, $\tilde{S}_h$ being a bounded subset of a finite element space S_h, and q_h being a function in S_h.

Definition 2.3. Let S_h be a finite element space, $\| \cdot \|$ be a norm on this space, and introduce the bounded set

$$\tilde{S}_h = \{q_h \in S_h; \|q_h\| \leq C\},$$

where $C > 0$ is a constant. Let $\{\sigma_i\}_{i=1}^M$ be a basis such that $\text{span}\{\sigma_i\}_{i=1}^M = S_h$ and define the operator $\Pi_{\mathcal{B}} : S_h \to \mathbb{R}^M$ by

$$\Pi_{\mathcal{B}} q_h = (q_1, q_2, \ldots, q_M), \qquad q_h = \sum_{i=1}^M q_i \sigma_i.$$

The degree $d_{S_h}(F, \tilde{S}_h, q_h)$ of a continuous mapping $F : \tilde{S}_h \to S_h$ at $q_h \in S_h \backslash F(\partial \tilde{S}_h)$ relative to $\tilde{S}_h$ is defined as

$$d_{S_h}(F, \tilde{S}_h, q_h) = d\left(\Pi_{\mathcal{B}} F(\Pi_{\mathcal{B}}^{-1}), \Pi_{\mathcal{B}} \tilde{S}_h, \Pi_{\mathcal{B}} q_h\right).$$

The next lemma is a consequence of the properties of the degree $d(F, \Omega, y)$, cf. [8].

Lemma 2.4. *Fix a finite element space* S_h, *and let* $d_{S_h}(F, \tilde{S}_h, q_h)$ *be the associated degree of Definition 2.3. The following properties hold:*
  (1) $d_{S_h}(F, \tilde{S}_h, q_h)$ *does not depend on the choice of basis for* S_h.
  (2) $d_{S_h}(\text{Id}, \tilde{S}_h, q_h) = 1$.

(3) $d_{S_h}(H(\cdot, \alpha), \tilde{S}_h, q_h(\alpha))$ *is independent of* $\alpha \in J := [0, 1]$ *for* $H : \tilde{S}_h \times J \to S_h$
   continuous, $q_h : J \to S_h$ *continuous, and* $q_h(\alpha) \notin H(\partial \tilde{S}_h, \alpha)$ $\forall \alpha \in [0, 1]$.
(4) $d_{S_h}(F, \tilde{S}_h, q_h) \neq 0 \iff F^{-1}(q_h) \neq \emptyset$.

2.3. **Weak and renormalized solutions.**

Definition 2.5 (Weak solutions). We say that a pair $(\varrho, \boldsymbol{u})$ of functions constitutes a weak solution of the semi-stationary compressible Stokes system (1.1)–(1.2) with initial data (1.3) and Navier-slip type boundary conditions (1.4)–(1.5) provided the following conditions hold:

(1) $(\varrho, \boldsymbol{u}) \in L^\infty(0, T; L^\gamma(\Omega)) \times L^2(0, T; \mathcal{W}(\Omega))$;
(2) $\varrho_t + \operatorname{div}(\varrho \boldsymbol{u}) = 0$ in the weak sense, i.e, $\forall \phi \in C^\infty([0, T] \times \overline{\Omega})$,

$$\int_0^T \int_\Omega \varrho \left(\phi_t + \boldsymbol{u} D\phi \right) \, dx dt + \int_\Omega \varrho_0 \phi|_{t=0} \, dx = 0; \qquad (2.2)$$

(3) $-\mu \Delta \boldsymbol{u} - \lambda D \operatorname{div} \boldsymbol{u} + Dp(\varrho) = \boldsymbol{f}$ in the weak sense, i.e, $\forall \boldsymbol{\phi} \in \boldsymbol{C}^\infty((0, T) \times \overline{\Omega})$
   for which $\boldsymbol{\phi} \cdot \nu = 0$ on $(0, T) \times \partial \Omega$,

$$\int_0^T \int_\Omega \mu \operatorname{curl} \boldsymbol{u} \operatorname{curl} \boldsymbol{\phi} + [(\mu + \lambda) \operatorname{div} \boldsymbol{u} - p(\varrho)] \operatorname{div} \boldsymbol{\phi} \, dx dt = \int_0^T \int_\Omega \boldsymbol{f} \boldsymbol{\phi} \, dx dt.$$

For the convergence analysis we shall also need the DiPerna-Lions concept of renormalized solutions of the continuity equation.

Definition 2.6 (Renormalized solutions). Given $\boldsymbol{u} \in L^2(0, T; \mathcal{W}(\Omega))$, we say that $\varrho \in L^\infty(0, T; L^\gamma(\Omega))$ is a renormalized solution of (1.1) provided

$$B(\varrho)_t + \operatorname{div}(B(\varrho)\boldsymbol{u}) + b(\varrho) \operatorname{div} \boldsymbol{u} = 0 \quad \text{in the weak sense on } [0, T) \times \overline{\Omega},$$

for any $B \in C[0, \infty) \cap C^1(0, \infty)$ with $B(0) = 0$ and $b(\varrho) := \varrho B'(\varrho) - B(\varrho)$.

We shall need the following lemma.

Lemma 2.7. *Suppose* $(\varrho, \boldsymbol{u})$ *is a weak solution according to Definition* 2.5. *If* $\varrho \in L^2((0, T) \times \Omega))$, *then* ϱ *is a renormalized solution according to Definition* 2.6.

Proof. Let $(\varrho, \boldsymbol{u})$ be a weak solution. Then $\boldsymbol{u} \in L^2(0, T; \boldsymbol{H}^1(\Omega))$. As the boundary of Ω is Lipschitz, the velocity field $\boldsymbol{u}(t)$ can be extended to the full space $\mathbb{R}^3$ such that $\tilde{\boldsymbol{u}}(t)|_\Omega = \boldsymbol{u}(t)$ and

$$\|\tilde{\boldsymbol{u}}\|_{L^2(0, T; \boldsymbol{H}^1(\mathbb{R}^N))} \leq C(\Omega) \|\boldsymbol{u}\|_{L^2(0, T; \boldsymbol{H}^1(\Omega))},$$

where $\tilde{\boldsymbol{u}}(t)$ denotes the extension of $\boldsymbol{u}(t)$. If we extend $\varrho(t)$ to $\mathbb{R}^N$ by setting $\tilde{\varrho}(t) = \varrho(t)\boldsymbol{1}_\Omega$, we get

$$\tilde{\varrho}_t + \operatorname{div}(\tilde{\varrho} \, \tilde{\boldsymbol{u}}) = 0 \quad \text{in the weak sense on } [0, T) \times \mathbb{R}^3.$$

Now, to conclude the proof, we appeal to a well-known lemma from [15] stating that the square-integrable weak solution $\tilde{\varrho}$ is also a renormalized solution. $\qquad \square$

2.4. **A mixed formulation.**

In view of the Navier-slip boundary condition (1.5), it is natural to introduce the vorticity $\boldsymbol{w} = \operatorname{curl} \boldsymbol{u}$ as an independent variable, thereby turning the velocity equation into (1.6). This immediately leads to the following mixed formulation, which acts as a motivation for our choice of numerical method: Determine functions

$$(\boldsymbol{w}, \boldsymbol{u}) \in L^2(0, T; \boldsymbol{W}_0^{\operatorname{curl}, 2}(\Omega)) \times L^2(0, T; \boldsymbol{W}_0^{\operatorname{div}, 2}(\Omega))$$

such that

$$\int_0^T \int_\Omega \mu \operatorname{curl} \boldsymbol{w} \boldsymbol{v} + [(\mu + \lambda) \operatorname{div} \boldsymbol{u} - p(\varrho)] \operatorname{div} \boldsymbol{v} \ dxdt = \int_0^T \int_\Omega \boldsymbol{f} \boldsymbol{v} \ dxdt,$$

$$\int_0^T \int_\Omega \boldsymbol{w} \boldsymbol{\eta} - \operatorname{curl} \boldsymbol{\eta} \boldsymbol{u} \ dxdt = 0,$$
(2.3)

for all $(\eta, \boldsymbol{v}) \in L^2(0, T; \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)) \times L^2(0, T; \boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$.

In order to arrive at the weak formulation (2.3), we have utilized the integration by parts formula

$$\int_\Omega \boldsymbol{\eta} \operatorname{curl} \boldsymbol{u} \ dx = \int_\Omega \boldsymbol{u} \operatorname{curl} \boldsymbol{\eta} \ dx + \int_{\partial\Omega} \boldsymbol{u}(\boldsymbol{\eta} \times \nu) \ dS(x).$$
(2.4)

It follows as an immediate consequence of the Stokes Theorem and will be applied multiple times throughout the paper.

The upcoming goal is to prove that a sequence of approximate solutions, denoted by $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$, converge to a limit $(\varrho, \boldsymbol{w}, \boldsymbol{u})$ satisfying (2.2) and (2.3); the term "converge" is made precise in a forthcoming section. Having constructed such a limit, it follows immediately that the pair $(\varrho, \boldsymbol{u})$ is a weak solution according to Definition 2.5, thereby completing the analysis.

2.5. **Finite element spaces and some basic results.** Upon inspection of the spatial spaces entering the weak formulations stated above, we see that they can be related through a De Rham sequence. In two dimensions this reads

$$0 \xrightarrow{\ \subset\ } W_0^{1,2} \xrightarrow{\ \mathrm{curl}\ } \boldsymbol{W}_0^{\mathrm{div},2} \xrightarrow{\ \mathrm{div}\ } L_0^2 \longrightarrow 0,$$

while in three dimensions the corresponding sequence is

$$0 \xrightarrow{\ \subset\ } W_0^{1,2} \xrightarrow{\ \mathrm{grad}\ } \boldsymbol{W}_0^{\mathrm{curl},2} \xrightarrow{\ \mathrm{curl}\ } \boldsymbol{W}_0^{\mathrm{div},2} \xrightarrow{\ \mathrm{div}\ } L_0^2 \longrightarrow 0.$$

These sequences are exact in the sense that the null space of one operator exactly matches the image of the next. This perspective on the spaces is actually useful as we see that this is precisely how the quantities $\boldsymbol{w}, \boldsymbol{u}$, and ϱ relate to each other.

It follows from (2.3) that the vorticity $\boldsymbol{w}$ is decoupled from the density ϱ, which is an important consequence of our choice boundary condition and this fact is of relevance to the convergence analysis. Moreover, the subsequent analysis relies heavily on the solvability of the problem (or more precisely a discrete version of it)

$$\operatorname{div} \boldsymbol{v} = q, \quad \boldsymbol{v} \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

for some given right-hand side q in L^2. In particular, it is important for us to extract from this problem some control on $\operatorname{curl} \boldsymbol{v}$. From the above De Rham sequence, we see immediately that there exists solution $\boldsymbol{v}$ which is weakly curl free. In the continuous setting this is enough to conclude that $\operatorname{curl} \boldsymbol{v} = 0$; indeed, the Hodge decomposition (2.1) combined with the fact that $\boldsymbol{v}$ is weakly curl free implies $v = Ds$ for some scalar s.

Motivated by these remarks, we shall in the next section present a numerical method that utilizes finite element spaces satisfying a discrete version of the above De Rham sequence. More precisely, we will replace $\boldsymbol{W}_0^{\mathrm{curl},2}$ and $\boldsymbol{W}_0^{\mathrm{div},2}$ by the lowest order Nédélec finite element spaces of the first kind (but other spaces are possible) with vanishing degrees of freedom at the boundary $\partial\Omega$. Let us denote these spaces by $\boldsymbol{W}_h$ and $\boldsymbol{V}_h$ respectively. It is well known that the spaces $\boldsymbol{W}_h, \boldsymbol{V}_h$ together with the space Q_h of piecewise constants (cf. the ensuing section for missing details) satisfies in three dimensions the following exact discrete De Rham sequence:

$$0 \xrightarrow{\ \subset\ } S_h \xrightarrow{\ \mathrm{grad}\ } \boldsymbol{W}_h \xrightarrow{\ \mathrm{curl}\ } \boldsymbol{V}_h \xrightarrow{\ \mathrm{div}\ } Q_h \cap L_0^2(\Omega) \longrightarrow 0,$$

where S_h is the usual scalar linear Lagrange element. In the two dimensional case this sequence still holds, but now the spaces S_h and $\boldsymbol{W}_h$ are equal and thus the sequence does not contain the gradient operator. All finite element spaces are defined with respect to a given tetrahedral mesh E_h of Ω.

We introduce the canonical interpolation operators:

$$\Pi_h^S : W_0^{1,2} \cap W^{2,2} \to S_h, \quad \Pi_h^W : \boldsymbol{W}_0^{\mathrm{curl},2} \cap \boldsymbol{W}^{2,2} \to \boldsymbol{W}_h,$$

$$\Pi_h^V : \boldsymbol{W}_0^{\mathrm{div},p} \cap \boldsymbol{W}^{1,2} \to \boldsymbol{V}_h, \quad \Pi_h^Q : L_0^2 \to Q_h,$$

using the available degrees of freedom of the involved spaces. That is, the operators are defined

$$\left(\Pi_h^S s\right)(x_i) = s(x_i), \quad \forall x_i \in \mathcal{N}_h;$$

$$\int_e \left(\Pi_h^W \boldsymbol{w}\right) \times \nu \, dS(x) = \int_e \boldsymbol{w} \times \nu \, dS(x), \quad \forall e \in \mathcal{E}_h;$$

$$\int_\Gamma \left(\Pi_h^V \boldsymbol{v}\right) \cdot \nu \, dS(x) = \int_\Gamma \boldsymbol{v} \cdot \nu \, dS(x), \quad \forall \Gamma \in \Gamma_h;$$

$$\int_E \Pi_h^Q q \, dx = \int_E q \, dx, \quad \forall E \in E_h,$$

where Γ_h, $\mathcal{E}_h$, and $\mathcal{N}_h$, denote the set of faces, edges, and vertices, respectively, of E_h. Then it is well known that the following diagram commutes:

$$
\begin{array}{ccccccc}
W_0^{1,2} \cap W^{2,2} & \xrightarrow{\mathrm{grad}} & \boldsymbol{W}_0^{\mathrm{curl},2} \cap \boldsymbol{W}^{2,2} & \xrightarrow{\mathrm{curl}} & \boldsymbol{W}_0^{\mathrm{div},p} \cap \boldsymbol{W}^{1,2} & \xrightarrow{\mathrm{div}} & L_0^2 \\
\Pi_h^S \downarrow & & \Pi_h^W \downarrow & & \Pi_h^V \downarrow & & \Pi_h^Q \downarrow \\
S_h & \xrightarrow{\mathrm{grad}} & \boldsymbol{W}_h & \xrightarrow{\mathrm{curl}} & \boldsymbol{V}_h & \xrightarrow{\mathrm{div}} & Q_h.
\end{array}
$$

Remark 2.8. The interpolation operators Π_h^S, Π_h^W, and Π_h^V, are defined on function spaces with enough regularity to ensure that the corresponding degrees of freedom are functionals on these spaces. This is reflected in writing $\boldsymbol{W}_0^{\mathrm{curl},2} \cap \boldsymbol{W}^{2,2}$ instead of merely $\boldsymbol{W}^{\mathrm{curl},2}$ and so on.

In view of the above commuting diagram, we can define the spaces orthogonal to the range of the previous operator, i.e.,

$$\boldsymbol{W}_h^{0,\perp} := \{\boldsymbol{w}_h \in \boldsymbol{W}_h; \mathrm{curl}\, \boldsymbol{w}_h = 0\}^\perp \cap \boldsymbol{W}_h,$$

$$\boldsymbol{V}_h^{0,\perp} := \{\boldsymbol{v}_h \in \boldsymbol{V}_h; \mathrm{div}\, \boldsymbol{v}_h = 0\}^\perp \cap \boldsymbol{V}_h,$$

to obtain decompositions

$$\boldsymbol{W}_h = DS_h + \boldsymbol{W}_h^{0,\perp},$$

$$\boldsymbol{V}_h = \mathrm{curl}\, \boldsymbol{W}_h + \boldsymbol{V}_h^{0,\perp}, \tag{2.5}$$

and the discrete Poincaré inequalities

$$\|\boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)} \le C \|\mathrm{div}\, \boldsymbol{v}_h\|_{L^2(\Omega)}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h^{0,\perp}, \tag{2.6}$$

$$\|\boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} \le C \|\mathrm{curl}\, \boldsymbol{w}_h\|_{L^2(\Omega)}, \quad \forall \boldsymbol{w} \in \boldsymbol{W}_h^{0,\perp}. \tag{2.7}$$

Thus, with this configuration of elements we are able to perform unique Hodge type decompositions of the discrete vector fields. As an example, we immediately have the existence of a function $\boldsymbol{v}_h \in \boldsymbol{V}_h^{0,\perp}$ satisfying

$$\mathrm{div}\, \boldsymbol{v}_h|_E = q_h|_E, \ \forall E \in E_h,$$

for any given $q_h \in Q_h \cap \left\{\int_\Omega q_h \, dx = 0\right\}$.

The following lemma summarizes well–known error estimates satisfied by the interpolation operators. The estimates are derived from the Bramble–Hilbert lemma

using scaling arguments. We however note that care must be taken when mapping functions in $\boldsymbol{W}_h$ and $\boldsymbol{V}_h$ to a reference element (cf. [3, 17]).

Lemma 2.9. *There exists a constant $C > 0$, depending only on the shape regularity of E_h and the size of Ω, such that for any $1 \le p < \infty$,*

$$\left\| \phi - \Pi_h^Q \phi \right\|_{L^p(\Omega)} \le Ch \left\| D\phi \right\|_{\boldsymbol{L}^p(\Omega)},$$

$$\left\| \boldsymbol{v} - \Pi_h^V \boldsymbol{v} \right\|_{\boldsymbol{L}^p(\Omega)} + h \left\| \mathrm{div}(\boldsymbol{v} - \Pi_h^V \boldsymbol{v}) \right\|_{L^p(\Omega)} \le Ch^s \left\| D^s \boldsymbol{v} \right\|_{\boldsymbol{L}^p(\Omega)}, \quad r = 1, 2,$$

$$\left\| \boldsymbol{w} - \Pi_h^W \boldsymbol{w} \right\|_{\boldsymbol{L}^p(\Omega)} + h \left\| \mathrm{curl}(\boldsymbol{w} - \Pi_h^W \boldsymbol{w}) \right\|_{\boldsymbol{L}^p(\Omega)} \le Ch^s \| D^s \boldsymbol{w} \|_{\boldsymbol{L}^p(\Omega)}, \quad s = 1, 2,$$

for all $\phi \in W^{1,p}(\Omega)$, $\boldsymbol{v} \in W^{s,p}(\Omega)$, and $\boldsymbol{w} \in W^{2,p}(\Omega)$.

In what follows, we will need the following lemma. It follows from scaling arguments and the equivalence of finite dimensional norms.

Lemma 2.10. *There exists a constant $C > 0$, depending only on the shape regularity of E_h, such that for $1 \le q, p \le \infty$, and $r = 0, 1$,*

$$\|\phi_h\|_{W^{r,p}(E)} \le Ch^{-r + \min\{0, \frac{N}{p} - \frac{N}{q}\}} \|\phi_h\|_{L^q(E)},$$

for any $E \in E_h$ and all polynomial functions $\phi_h \in \mathbb{P}_k(E)$, $k = 0, 1, \ldots$.

The next result follows from scaling arguments and the trace theorem.

Lemma 2.11. *Fix any $E \in E_h$ and let $\phi \in W^{1,2}(E)$ be arbitrary. There exists a constant $C > 0$, depending only on the shape regularity of E_h such that,*

$$\|\phi\|_{L^2(\Gamma)} \le Ch^{-\frac{1}{2}} \left(\|\phi\|_{L^2(E)} + h\|D\phi\|_{\boldsymbol{L}^2(E)} \right), \quad \forall \Gamma \in \Gamma_h \cap \partial E.$$

3. NUMERICAL METHOD AND MAIN RESULT

In this section we define the numerical method and the state the convergence theorem. The proof of this theorem is deferred to subsequent sections.

Given a time step $\Delta t > 0$, we discretize the time interval $[0, T]$ in terms of the points $t^m = m\Delta t$, $m = 0, \ldots, M$, where we assume that $M\Delta t = T$. Regarding the spatial discretization, we let $\{E_h\}_h$ be a shape regular family of tetrahedral meshes of Ω, where h is the maximal diameter. It will be a standing assumption that h and Δt are related such that $\Delta t = ch$, for some constant c. By shape regular we mean that there exists a constant $\kappa > 0$ such that every $E \in E_h$ contains a ball of radius $\lambda_E \ge \frac{h_E}{\kappa}$, where h_E is the diameter of E. Furthermore, we let Γ_h denote the set of faces in E_h. Throughout the paper, we will use the three dimensional terminology (tetrahedron, face, etc.) to denote both the three dimensional and the two dimensional case (triangle, edge, etc).

On each element $E \in E_h$, we denote by $Q(E)$ the constants on E. The functions that are piecewise constant with respect to the elements of a mesh E_h are denoted by $Q_h = Q_h(\Omega)$. Next, on each $E \in E_h$, we denote by $\boldsymbol{W}(E)$ the lowest order space of *curl–conforming Nédélec* polynomials of first kind [17]. In two dimensions, $\boldsymbol{W}(E)$ is the space of linear scalar polynomials on E and is totally determined by it's value at the vertices of E. In three dimensions, each member of $\boldsymbol{W}(E)$ is of the form

$$\boldsymbol{a} + \boldsymbol{b} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3,$$

and is totally determined by the following degrees of freedom: $\int_e \boldsymbol{w} \cdot \tau_e \, dS(x)$ for all edges (not faces) e of the element E, where τ_e is the unit tangential vector on e.

On each element $E \in E_h$, we denote by $\boldsymbol{V}(E)$ the lowest order space of *div–conforming Nédélec* polynomials of first kind [17]. In two dimensions, it is the Raviart–Thomas polynomial space on E. Each member of $\boldsymbol{V}(E)$ is of the form

$$\boldsymbol{a} + b \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{a} \in \mathbb{R}^3, b \in \mathbb{R},$$

and is totally determined by the following degrees of freedom: $\int_\Gamma \boldsymbol{v} \cdot \nu \; dS(x)$, for all faces Γ of the element E, where ν is a unit normal vector on Γ.

The element spaces $\boldsymbol{W}_h = \boldsymbol{W}_h(\Omega)$ and $\boldsymbol{V}_h = \boldsymbol{V}_h(\Omega)$ are formed on the entire mesh E_h by matching the degrees of freedom of the polynomial space $\boldsymbol{W}(E)$ and $\boldsymbol{V}(E)$, respectively, on each face $\Gamma \in \Gamma_h$. In addition, we incorporate the boundary conditions by letting the degrees of freedom of the spaces $\boldsymbol{W}_h$ and $\boldsymbol{V}_h$ vanish at the faces on the boundary.

Before defining our numerical method, we shall need to introduce some additional notation related to the discontinuous Galerkin scheme. Concerning the boundary ∂E of an element E, we write f_+ for the trace of the function f achieved from within the element E and f_- for the trace of f achieved from outside E. Concerning a face Γ that is shared between two elements E_- and E_+, we will write f_+ for the trace of f achieved from within E_+ and f_- for the trace of f achieved from within E_-. Here E_- and E_+ are defined such that ν points from E_- to E_+, where ν is fixed (throughout) as one of the two possible normal components on each face Γ. We also write $[f]_\Gamma = f_+ - f_-$ for the jump of f across the face Γ, while forward time-differencing of f is denoted by $[f^m] = f^{m+1} - f^m$. To denote the set of inner faces of Γ_h we will use the notation $\Gamma_h^I = \{\Gamma \in \Gamma_h; \Gamma \not\subset \partial\Omega\}$.

Let us now define our numerical method for the semi-stationary Stokes system (1.1)–(1.2) augmented with the boundary conditions (1.4) and (1.5) (note, however, that in the definition below the boundary conditions are built into the finite element spaces and not listed explicitly).

Definition 3.1 (Numerical method). Let $\left\{ \varrho_h^0(x) \right\}_{h>0}$ be a sequence in $Q_h(\Omega)$ that satisfies $\varrho_h^0 > 0$ for each fixed $h > 0$ and $\varrho_h^0 \to \varrho_0$ a.e. in Ω and in $L^1(\Omega)$ as $h \to 0$. Set $\boldsymbol{f}_h(t, \cdot) = \boldsymbol{f}_h^m(\cdot) := \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \Pi_h^Q \boldsymbol{f}(s, \cdot) \; ds$, for $t \in (t_{m-1}, t_m)$, $m = 1, \ldots, M$.

Now, determine functions

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad m = 1, \ldots, M,$$

such that for all $\phi_h \in Q_h(\Omega)$,

$$\int_\Omega \varrho_h^m \phi_h \; dx - \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h]_\Gamma \; dS(x)$$

$$= \int_\Omega \varrho_h^{m-1} \phi_h \; dx, \tag{3.1}$$

and for all $(\boldsymbol{\eta}_h, \boldsymbol{v}_h) \in \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega)$,

$$\int_\Omega \mu \operatorname{curl} \boldsymbol{w}_h^m \boldsymbol{v}_h + [(\mu + \lambda) \operatorname{div} \boldsymbol{u}_h^m - p(\varrho_h^m)] \operatorname{div} \boldsymbol{v}_h \; dx = \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h \; dx,$$

$$\int_\Omega \boldsymbol{w}_h^m \boldsymbol{\eta}_h - \boldsymbol{u}_h^m \operatorname{curl} \boldsymbol{\eta}_h \; dx = 0, \tag{3.2}$$

for $m = 1, \ldots, M$.

In (3.1), $(\boldsymbol{u}_h \cdot \nu)^+ = \max\{\boldsymbol{u}_h \cdot \nu, 0\}$ and $(\boldsymbol{u}_h \cdot \nu)^+ = \min\{\boldsymbol{u}_h \cdot \nu, 0\}$, so that $\boldsymbol{u}_h \cdot \nu = (\boldsymbol{u}_h \cdot \nu)^+ + (\boldsymbol{u}_h \cdot \nu)^-$, i.e., in the evaluation of $\varrho(\boldsymbol{u} \cdot \nu)$ at the face Γ the trace of ϱ is taken in the upwind direction.

Remark 3.2. Using the identity

$$\Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(\varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) \phi_h \, dS(x)$$

$$= -\Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h]_\Gamma \, dS(x).$$

we can state (3.1) on the following form:

$$\int_\Omega \varrho_h^m \phi_h \, dx + \Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(\varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) \phi_h \, dS(x)$$

$$= \int_\Omega \varrho_h^{m-1} \phi_h \, dx. \tag{3.3}$$

For each fixed $h > 0$, the numerical solution $\{(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m)\}_{m=0}^M$ is extended to the whole of $(0, T] \times \Omega$ by setting

$$(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)(t) = (\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m), \qquad t \in (t_{m-1}, t_m], \quad m = 1, \dots, M. \tag{3.4}$$

In addition, we set $\varrho_h(0) = \varrho_h^0$.

Our main result is that, passing if necessary to a subsequence, $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution. More precisely, there holds

Theorem 3.3 (Convergence). *Suppose $\boldsymbol{f} \in \boldsymbol{L}^2((0,T) \times \Omega)$ and $\varrho_0 \in L^\gamma(\Omega)$, $\gamma > 1$. Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.4) and Definition 3.1. Then, passing if necessary to a subsequence as $h \to 0$, $\boldsymbol{w}_h \rightharpoonup \boldsymbol{w}$ in $L^2(0,T; \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega))$, $\boldsymbol{u}_h \rightharpoonup \boldsymbol{u}$ in $L^2(0,T; \boldsymbol{W}_0^{\mathrm{div},2}(\Omega))$, $\varrho_h \boldsymbol{u}_h \rightharpoonup \varrho \boldsymbol{u}$ in the sense of distributions on $(0,T) \times \Omega$, and $\varrho_h \to \varrho$ a.e. in $(0,T) \times \Omega$, where the limit $(\varrho, \boldsymbol{w}, \boldsymbol{u})$ satisfies the mixed formulation (2.3), and consequently $(\varrho, \boldsymbol{u})$ is also a weak solution according to Definition 2.5.*

This theorem will be an immediate consequence of the results stated and proved in Sections 4–6.

4. NUMERICAL METHOD IS WELL DEFINED

In this section we show that there exists a solution to the discrete problem given in Definition 3.1. However, we commence by obtaining a positive lower bound for the density, recalling that the approximate initial density $\varrho_h^0(\cdot)$ is strictly positive.

Lemma 4.1. *Fix any $m = 1, \dots, M$, and suppose $\varrho_h^{m-1} \in Q_h(\Omega)$, $\boldsymbol{u}_h^m \in \boldsymbol{V}_h(\Omega)$ are given bounded functions. Then the solution $\varrho_h^m \in Q_h(\Omega)$ of the discontinuous Galerkin scheme (3.1) satisfies*

$$\min_{x \in \Omega} \varrho_h^m(x) \geq \min_{x \in \Omega} \varrho_h^{m-1}(x) \left(\frac{1}{1 + \Delta t \| \operatorname{div} \boldsymbol{u}_h^m \|_{L^\infty(\Omega)}} \right).$$

Consequently, if $\varrho_h^{m-1}(\cdot) > 0$, then $\varrho_h^m(\cdot) > 0$.

Proof. Let $\tilde{E} \in E_h$ be such that $\varrho_h^m|_{\tilde{E}} \leq \varrho_h^m|_E \; \forall E \in E_h$, and insert into (3.3) the test function $\phi_h \in Q_h(\Omega)$, defined by

$$\phi_h(x) = \begin{cases} \frac{1}{|\tilde{E}|}, & x \in \tilde{E}, \\ 0, & \text{otherwise.} \end{cases}$$

Integrating by parts then yields

$$\varrho_h^m|_{\tilde{E}} = -\frac{\Delta t}{|\tilde{E}|} \int_{\partial \tilde{E} \setminus \partial \Omega} \left(\varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m (\boldsymbol{u}_h \cdot \nu)^- \right) dS(x) + \varrho_h^{m-1}|_{\tilde{E}}$$

$$= -\Delta t \left(\varrho_h^m \operatorname{div} \boldsymbol{u}_h^m \right)\big|_{\tilde{E}} - \frac{\Delta t}{|\tilde{E}|} \int_{\partial \tilde{E} \setminus \partial \Omega} (\varrho_-^m - \varrho_+^m)(\boldsymbol{u}_h \cdot \nu)^- dS(x) + \varrho_h^{m-1}\big|_{\tilde{E}},$$

$$\geq -\Delta t \left(\varrho_h^m \operatorname{div} \boldsymbol{u}_h^m \right)\big|_{\tilde{E}} + \varrho_h^{m-1}\big|_{\tilde{E}},$$

where we have also used the relation $\boldsymbol{u}_h \cdot \nu = (\boldsymbol{u}_h \cdot \nu)^+ + (\boldsymbol{u}_h \cdot \nu)^-$, ϱ_h^m is constant on $\tilde{E}$, and that ϱ_h^m attains its minimal value on $\tilde{E}$. Consequently,

$$\varrho_h^m\big|_{\tilde{E}} \geq \varrho_h^{m-1}\big|_{\tilde{E}} \left(\frac{1}{1 + \Delta t \| \operatorname{div} \boldsymbol{u}_h^m \|_{L^\infty(\Omega)}} \right).$$

$\square$

We now turn to the existence of solutions to our nonlinear–implicit discrete problem. We will apply a topological degree argument, thereby reducing the proof to exhibiting a solution to a linear problem.

Lemma 4.2. *For each fixed $h > 0$, there exists a solution*

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in Q_h(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega), \quad \varrho_h^m(\cdot) > 0, \quad m = 1, \ldots, M,$$

to the nonlinear–implicit discrete problem posed in Definition 3.1.

Proof. We argue by induction. Assume for $m = 1, \ldots, k-1$ that there exists a solution

$$(\varrho_h^m, \boldsymbol{w}_h^m, \boldsymbol{u}_h^m) \in S_h := Q_h^+(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega)$$

to the discrete problem of Definition 3.1. Here and below we denote by $Q_h^+(\Omega)$ the strictly positive functions in $Q_h(\Omega)$. Moreover, the norm $\|\cdot\|$ on S_h is defined by $\|(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\|^2 = \|\varrho_h\|_{L^2(\Omega)}^2 + \|\boldsymbol{w}_h\|_{L^2(\Omega)}^2 + \|\boldsymbol{u}_h\|_{L^2(\Omega)}^2$.

The claim is that we can find a solution for $m = k$:

$$(\varrho_h^k, \boldsymbol{w}_h^k, \boldsymbol{u}_h^k) \in S_h. \tag{4.1}$$

To this end, we introduce the mapping

$$H : Q_h^+(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega) \times [0,1] \to Q_h(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega),$$

$$H(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h, \alpha) = (z_h(\alpha), \boldsymbol{y}_h(\alpha), \boldsymbol{x}_h(\alpha)),$$

where the triplet $(z_h(\alpha), \boldsymbol{y}_h(\alpha), \boldsymbol{x}_h(\alpha))$ is defined by

$$\int_\Omega z_h(\alpha)\phi_h \, dx = \alpha \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(\varrho_+(\boldsymbol{u}_h \cdot \nu)^+ + \varrho_-(\boldsymbol{u}_h \cdot \nu)^- \right) \phi_h \, dS(x)$$

$$+ \int_\Omega \frac{\varrho_h - \varrho_h^{k-1}}{\Delta t} \phi_h \, dx, \quad \forall \phi_h \in Q_h(\Omega),$$

$$\int_\Omega \boldsymbol{x}_h(\alpha)\boldsymbol{v}_h \, dx = \int_\Omega \mu \operatorname{curl} \boldsymbol{w}_h \boldsymbol{v}_h + (\lambda + \mu) \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h \, dx$$

$$- \alpha \int_\Omega P(\varrho_h) \operatorname{div} \boldsymbol{v}_h \, dx - \int_\Omega \boldsymbol{f}_h \boldsymbol{v}_h \, dx, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega),$$

$$\int_\Omega \boldsymbol{y}_h \boldsymbol{\eta}_h \, dx = \int_\Omega \boldsymbol{w}_h \boldsymbol{\eta}_h - \boldsymbol{u}_h \operatorname{curl} \boldsymbol{\eta}_h \, dx, \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{W}_h(\Omega).$$

Solving $H(\varrho_h, \boldsymbol{u}_h, \boldsymbol{w}_h, 1) = 0$ is equivalent to finding a solution (4.1) to the nonlinear–implicit discrete problem posed in Definition 3.1.

Let us fix an arbitrary $\alpha \in [0,1]$, and consider a solution $(\varrho_h(\alpha), \boldsymbol{u}_h(\alpha), \boldsymbol{w}_h(\alpha))$ belonging to $Q_h^+(\Omega) \times \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega)$ of the corresponding problem

$$H(\varrho_h(\alpha), \boldsymbol{u}_h(\alpha), \boldsymbol{w}_h(\alpha), \alpha) = 0.$$

We claim that there is a constant $C_\dagger > 0$, independent of α, such that

$$\|\boldsymbol{w}_h(\alpha)\|_{\boldsymbol{W}^{\operatorname{curl},2}(\Omega)}^2 + \|\boldsymbol{u}_h(\alpha)\|_{\boldsymbol{W}^{\operatorname{div},2}(\Omega)}^2 + \|\varrho_h(\alpha)\|_{L^\gamma(\Omega)}^\gamma \leq C_\dagger. \tag{4.2}$$

Indeed, repeating the arguments leading to estimate (5.15) in Section 5 we conclude that (4.2) holds with $C_\dagger = C \|\boldsymbol{f}\|^2_{L^2(0,T;\boldsymbol{L}^2(\Omega))} + \|\varrho_h^{k-1}\|^\gamma_{L^\gamma(\Omega)}$, where the constant C is the constant appearing in (5.15). Here, we have also used that $\Delta t = O(h)$. Let $\tilde{S}_h \subset S_h = Q_h^+(\Omega) \times \boldsymbol{V}_h(\Omega) \times \boldsymbol{W}_h(\Omega)$ be a ball of sufficiently large radius, cf (4.2). Then, since every solution of $H(\varrho_h, \boldsymbol{u}_h, \boldsymbol{w}_h, \alpha) = 0$ lies strictly inside $\tilde{S}_h$,

$$0 \notin H(\partial \tilde{S}_h, \alpha), \qquad \forall \alpha \in [0,1]. \tag{4.3}$$

We claim that $H(\cdot, \cdot)$ is continuous on $\tilde{S}_h \times [0,1]$. Let $(\varrho_h, \boldsymbol{u}_h, \boldsymbol{w}_h) \in \tilde{S}_h$. By equivalence of norms on finite dimensional spaces, the functions $\varrho_h, \boldsymbol{u}_h, \boldsymbol{w}_h$ are bounded on Ω. In view of this and Lemma 2.10, the claim follows.

By the virtue of (4.3) and the continuity of $H(\cdot, \cdot)$, we have by Lemma 2.4 that

$$d_{S_h}(H(\cdot, \alpha), \tilde{S}_h, 0) \quad \text{is independent of } \alpha \in [0,1].$$

The proof will be completed by proving that $d_{S_h}(H(\cdot, \alpha = 0), \tilde{S}_h, 0) \neq 0$. To see this, observe that the problem $H(\varrho_h, \boldsymbol{u}_h, \boldsymbol{w}_h, 0) = 0$ is equivalent to finding a triplet $(\varrho_h, \boldsymbol{u}_h, \boldsymbol{w}_h) \in \tilde{S}_h$ satisfying

$$\int_\Omega \varrho_h \phi_h \, dx = \int_\Omega \varrho_h^{k-1} \phi_h \, dx, \quad \forall \phi_h \in Q_h(\Omega), \tag{4.4}$$

and

$$\int_\Omega \mu \operatorname{curl} \boldsymbol{w}_h \boldsymbol{v}_h + (\lambda + \mu) \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h \, dx = \int_\Omega \boldsymbol{f}_h \boldsymbol{v}_h \, dx, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h(\Omega),$$

$$\int_\Omega \operatorname{curl} \boldsymbol{\eta}_h \boldsymbol{u}_h - \boldsymbol{w}_h \boldsymbol{\eta}_h \, dx = 0, \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{W}_h(\Omega). \tag{4.5}$$

Clearly, (4.4) has the solution $\varrho_h = \varrho_h^{k-1}$. Moreover, (4.5) is a system on mixed form admitting a unique solution provided that the finite element spaces satisfy the Babsuka–Brezzi condition. However, the commuting diagram property satisfied by our finite element spaces immediately renders the Babuska–Brezzi condition satisfied (cf. Theorem A.4). $\square$

5. Basic estimates

In this section we establish a few estimates to be used later on, including square-integrability of the pressure and weak time-continuity of the density. However, we begin with the following lemma providing us with a renormalized formulation of the continuity scheme (3.1).

Lemma 5.1 (Renormalized continuity scheme). *Fix any $m = 1, \ldots, M$ and let $(\varrho_h^m, \boldsymbol{u}_h^m) \in Q_h \times \boldsymbol{V}_h$ satisfy the continuity scheme (3.1). Then $(\varrho_h^m, \boldsymbol{u}_h^m)$ also satisfies the renormalized continuity scheme*

$$\int_\Omega B(\varrho_h^m) \phi_h \, dx$$

$$- \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(B(\varrho_-^m)(\boldsymbol{u}_h^m \cdot \nu)^+ + B(\varrho_+^m)(\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h]_\Gamma \, dx$$

$$+ \Delta t \int_\Omega b(\varrho_h^m) \operatorname{div} \boldsymbol{u}_h^m \phi_h \, dx + \int_\Omega B''(\xi(\varrho_h^m, \varrho_h^{m-1})) \left[\varrho_h^{m-1}\right]^2 \phi_h \, dx$$

$$+ \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma B''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) [\varrho_h^m]_\Gamma^2 (\phi_h)_- (\boldsymbol{u}_h^m \cdot \nu)^+ \tag{5.1}$$

$$- B''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) [\varrho_h^m]_\Gamma^2 (\phi_h)_+ (\boldsymbol{u}_h^m \cdot \nu)^- \, dS(x)$$

$$= \int_\Omega B(\varrho_h^{m-1}) \phi_h \, dx, \qquad \forall \phi_h \in Q_h(\Omega),$$

for any $B \in C[0, \infty) \cap C^2(0, \infty)$ with $B(0) = 0$ and $b(\varrho) := \varrho B'(\varrho) - B(\varrho)$. Given two positive real numbers a_1 and a_2, we denote by $\xi(a_1, a_2)$ and $\xi^\Gamma(a_1, a_2)$ two numbers between a_1 and a_2 (they will be precisely defined below).

Proof. Since $x \mapsto B'(\varrho_h^m(x))\phi_h(x)$ is piecewise constant, we can take $B'(\varrho_h^m)\phi_h$ as a test function in the continuity scheme (3.3), yielding

$$\int_\Omega \left[\varrho_h^{m-1}\right] B'(\varrho_h^m)\phi_h \, dx$$

$$= -\Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(\varrho_+^m(\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m(\boldsymbol{u}_h^m \cdot \nu)^-\right) B'(\varrho_+^m)\phi_h \, dS(x)$$

$$= -\Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(B'(\varrho_+^m)\varrho_+^m(\boldsymbol{u}_h^m \cdot \nu) - [\varrho_h^m]_{\partial E} B'(\varrho_+^m)(\boldsymbol{u}_h^m \cdot \nu)^-\right) \phi_h \, dS(x).$$
(5.2)

A Taylor expansion yields

$$B'(z)(y - z) = B(y) - B(z) - B''(z^*)(y - z)^2,$$

for some number z^* between z and y. Consequently,

$$[\varrho_h^m]_{\partial E} B'(\varrho_+^m) = [B(\varrho_h^m)]_{\partial E} - B''(\xi^{\partial E}(\varrho_+^m, \varrho_-^m)) [\varrho_h^m]_{\partial E}^2,$$

$$\left[\varrho_h^{m-1}\right] B'(\varrho_h^m) = \left[B(\varrho_h^{m-1})\right] - B''(\xi(\varrho_h^m, \varrho_h^{m-1})) \left[\varrho_h^{m-1}\right]^2,$$

where

$$\xi^{\partial E}(\varrho_+^m, \varrho_-^m)(x) \in [\varrho_-^m(x), \varrho_+^m(x)], \qquad x \in \partial E,$$

and

$$\xi(\varrho_h^m, \varrho_h^{m-1})(x) \in [\varrho_h^{m-1}(x), \varrho_h^m(x)], \qquad x \in \Omega.$$

Inserting these identities in (5.2), recalling the definition of b, and applying Green's theorem, we achieve

$$\int_\Omega \left(B(\varrho_h^m) - B(\varrho_h^{m-1})\right) \phi_h \, dx + \int_\Omega B''(\xi(\varrho_h^m, \varrho_h^{m-1})) \left[\varrho_h^{m-1}\right]^2 \phi_h \, dx$$

$$= -\Delta t \int_\Omega b(\varrho_+^m) \operatorname{div} \boldsymbol{u}_h^m \phi_h \, dx$$

$$\quad - \Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(B(\varrho_+^m)(\boldsymbol{u}_h^m \cdot \nu)^+ + B(\varrho_-^m)(\boldsymbol{u}_h^m \cdot \nu)^-\right) \phi_h \, dS(x)$$
(5.3)

$$\quad + \Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} B''(\xi^{\partial E}(\varrho_+^m, \varrho_-^m)) [\varrho_h^m]_{\partial E}^2 (\boldsymbol{u}_h^m \cdot \nu)^- \phi_h \, dS(x).$$

Denote by I the second term on the right-hand side of the equality sign. Then, as in Remark 3.2, we have the identity

$$I = \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(B(\varrho_+^m)(\boldsymbol{u}_h^m \cdot \nu)^+ + B(\varrho_-^m)(\boldsymbol{u}_h^m \cdot \nu)^-\right) [\phi_h]_\Gamma \, dS(x).$$
(5.4)

Recalling that for each $\Gamma \in \Gamma_h$ we let E_+ and E_- be the two elements sharing the face Γ and such that the normal component associated with Γ points from E_+ to E_-, we can write

$$\Delta t \sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} B''(\xi^{\partial E}(\varrho_+^m, \varrho_-^m)) [\varrho_h^m]_{\partial E}^2 (\boldsymbol{u}_h^m \cdot \nu)^- \phi_h \, dS(x)$$

$$= \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma -B''(\xi^{\partial E_-}(\varrho_+^m, \varrho_-^m)) [\varrho_h^m]_\Gamma^2 (\phi_h)_- (\boldsymbol{u}_h^m \cdot \nu)^+$$
(5.5)

$$\quad + B''(\xi^{\partial E_+}(\varrho_+^m, \varrho_-^m)) [\varrho_h^m]_\Gamma^2 (\phi_h)_+ (\boldsymbol{u}_h^m \cdot \nu)^- \, dS(x).$$

Once we introduce into (5.5) the notations

$$\xi^\Gamma(\varrho_+^m, \varrho_-^m) := \xi^{\partial E_+}(\varrho_+^m, \varrho_-^m), \qquad \xi^\Gamma(\varrho_-^m, \varrho_+^m) := \xi^{\partial E_-}(\varrho_+^m, \varrho_-^m),$$

inserting (5.4), (5.5) into (5.3) yields the final result (5.1). $\qquad\square$

In what follows we shall need a discrete Hodge decomposition. The following lemma is a consequence of (2.5).

Lemma 5.2. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.4) and Definition 3.1. For each fixed $h > 0$, there exist unique functions $\boldsymbol{\zeta}_h^m \in \boldsymbol{W}_h^{0,\perp}$ and $\boldsymbol{z}_h^m \in \boldsymbol{V}_h^{0,\perp}$ such that*

$$\boldsymbol{u}_h^m = \operatorname{curl} \boldsymbol{\zeta}_h^m + \boldsymbol{z}_h^m, \qquad m = 1, \dots, M. \tag{5.6}$$

Moreover, if we let $\boldsymbol{\zeta}_h(t,x)$, $\boldsymbol{z}_h(t,x)$ denote the functions obtained by extending, as in (3.4), $\{\boldsymbol{\zeta}_h^m\}_{m=1}^M$, $\{\boldsymbol{z}_h^m\}_{m=1}^M$ to the whole of $(0,T] \times \Omega$, then

$$\boldsymbol{u}_h(t,\cdot) = \operatorname{curl} \boldsymbol{\zeta}_h(\cdot,t) + \boldsymbol{z}_h(\cdot,t), \qquad t \in (0,T).$$

We now state a basic stability estimate satisfied by any solution of the discrete problem given in Definition 3.1.

Lemma 5.3. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.4) and Definition 3.1. For $\varrho > 0$, set $P(\varrho) := \frac{a}{\gamma-1}\varrho^\gamma$. For any $m = 1, \dots, M$, we have*

$$\begin{aligned}
\int_\Omega &P(\varrho_h^m) \, dx \\
&+ \sum_{k=1}^m \int_\Omega P''(\xi(\varrho_h^k, \varrho_h^{k-1})) \left[\varrho_h^{k-1}\right]^2 \, dx \\
&\quad + \sum_{k=1}^m \sum_{\Gamma \in \Gamma_h^I} \Delta t \int_\Gamma P''(\varrho_\dagger^k) \left[\varrho_h^k\right]_\Gamma^2 \left|\boldsymbol{u}_h^k \cdot \nu\right| \, dx \\
&\quad + \sum_{k=1}^m \Delta t \int_\Omega \left|\boldsymbol{u}_h^k\right|^2 \, dx + \sum_{k=1}^m \Delta t \int_\Omega \left|\operatorname{div}\boldsymbol{u}_h^k\right|^2 \, dx \\
&\quad + \sum_{k=1}^m \Delta t \int_\Omega \left|\boldsymbol{w}_h^k\right|^2 \, dx + \sum_{k=1}^m \Delta t \int_\Omega \left|\operatorname{curl}\boldsymbol{w}_h^k\right|^2 \, dx \\
&\leq \int_\Omega P(\varrho_0) \, dx + C \sum_{k=1}^m \Delta t \int_\Omega \left|\boldsymbol{f}_h^k\right|^2 \, dx.
\end{aligned} \tag{5.7}$$

Consequently, $\varrho_h \in_b L^\infty(0,T; L^\gamma(\Omega))$.

Proof. Since $P'(\varrho)\varrho - P(\varrho) = p(\varrho)$ and $\varrho_h > 0$, it follows by taking $\phi_h \equiv 1$ in the renormalized scheme (5.1) that

$$\begin{aligned}
\int_\Omega &P(\varrho_h^k) \, dx + \Delta t \int_\Omega p(\varrho_h^k) \operatorname{div}\boldsymbol{u}_h^k \, dx + \int_\Omega P''(\xi(\varrho_h^k, \varrho_h^{k-1})) \left[\varrho_h^{k-1}\right]^2 \, dx \\
&+ \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) \left[\varrho_h^k\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)^+ \\
&\qquad - P''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) \left[\varrho_h^k\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)^- \, dS(x) = \int_\Omega P(\varrho^{k-1}) \, dx.
\end{aligned} \tag{5.8}$$

For $k = 1, \dots, M$ and $x \in \bigcup_{\Gamma \in \Gamma_h^I} \Gamma$, set

$$\varrho_\dagger^k(x) := \begin{cases} \max\{\varrho_+^k(x), \varrho_-^k(x)\}, & 1 < \gamma \leq 2, \\ \min\{\varrho_+^k(x), \varrho_-^k(x)\}, & \gamma \geq 2, \end{cases} \tag{5.9}$$

and note that

$$\Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\xi^\Gamma(\varrho_+^m, \varrho_-^m)) \left[\varrho_h^k\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)^+$$

$$- P''(\xi^\Gamma(\varrho_-^m, \varrho_+^m)) \left[\varrho_h^k\right]_\Gamma^2 (\boldsymbol{u}_h^k \cdot \nu)^- \; dS(x), \qquad (5.10)$$

$$\geq \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\varrho_\dagger^k) \left[\varrho_h^k\right]_\Gamma^2 \left|\boldsymbol{u}_h^k \cdot \nu\right| \; dS(x).$$

Next, by using $\boldsymbol{v}_h = \boldsymbol{u}_h^k$ as a test function in the first equation of (3.2) and then using the second equation of (3.2) with $\boldsymbol{\eta}_h = \boldsymbol{w}_h^k$, we obtain the identity

$$\int_\Omega p(\varrho_h^k) \operatorname{div} \boldsymbol{u}_h^k \, dx = (\mu + \lambda) \int_\Omega \left|\operatorname{div} \boldsymbol{u}_h^k\right|^2 \, dx + \mu \int_\Omega \left|\boldsymbol{w}_h^k\right|^2 \, dx - \int_\Omega \boldsymbol{f}_h^k \boldsymbol{u}_h^k \, dx. \quad (5.11)$$

Similarly, specifying $\boldsymbol{v}_h = \operatorname{curl} \boldsymbol{w}_h^k$ in the first equation of (3.2) yields

$$\mu \int_\Omega \left|\operatorname{curl} \boldsymbol{w}_h^k\right|^2 \, dx = \int_\Omega \boldsymbol{f}_h^k \operatorname{curl} \boldsymbol{w}_h^k \, dx.$$

An application of Cauchy's inequality (with epsilon) then yields

$$\int_\Omega \left|\operatorname{curl} \boldsymbol{w}_h^k\right|^2 \, dx \leq C \int_\Omega \left|\boldsymbol{f}_h^k\right|^2 \, dx. \qquad (5.12)$$

Thanks to (5.6), we can write $\boldsymbol{u}_h^k = \operatorname{curl} \boldsymbol{\zeta}_h^k + \boldsymbol{z}_h^k$ for with $\boldsymbol{\zeta}_h^k \in \boldsymbol{W}_h^{0,\perp}$ and $\boldsymbol{z}_h^k \in \boldsymbol{V}_h^{0,\perp}$. Choosing $\boldsymbol{\eta}_h = \operatorname{curl} \boldsymbol{\zeta}_h^k$ in the second equation of (3.2) gives

$$\int_\Omega \left|\operatorname{curl} \boldsymbol{\zeta}_h^k\right|^2 \, dx = \int_\Omega \boldsymbol{w}_h^k \boldsymbol{\zeta}_h^k \, dx \leq \left(\int_\Omega \left|\boldsymbol{w}_h^k\right|^2 \, dx\right)^{\frac{1}{2}} \left(\int_\Omega \left|\boldsymbol{\zeta}_h^k\right|^2 \, dx\right)^{\frac{1}{2}}.$$

Thus, since the discrete Poincaré inequality (2.7) tells us that

$$\int_\Omega \left|\boldsymbol{\zeta}_h^k\right|^2 \, dx \leq C \int_\Omega \left|\operatorname{curl} \boldsymbol{\zeta}_h^k\right|^2 \, dx,$$

we arrive at the estimate

$$\int_\Omega \left|\boldsymbol{\zeta}_h^k\right|^2 + \left|\operatorname{curl} \boldsymbol{\zeta}_h^k\right|^2 \, dx \leq C \int_\Omega \left|\boldsymbol{w}_h^k\right|^2 \, dx. \qquad (5.13)$$

In view of the discrete Poincaré inequality (2.6), we also have

$$\int_\Omega \left|\boldsymbol{z}_h^k\right|^2 \, dx \leq C \int_\Omega \left|\operatorname{div} \boldsymbol{u}_h^k\right|^2 \, dx,$$

which, together with (5.13), allow us to conclude

$$\int_\Omega \left|\boldsymbol{u}_h^k\right|^2 \, dx = \int_\Omega \left|\boldsymbol{\zeta}_h^k\right|^2 + \left|\boldsymbol{z}_h^k\right|^2 \, dx \leq C \left(\int_\Omega \left|\boldsymbol{w}_h^k\right|^2 \, dx + \int_\Omega \left|\operatorname{div} \boldsymbol{u}_h^k\right|^2 \, dx\right). \quad (5.14)$$

Now, by first inserting (5.11) into (5.8) and subsequently utilizing (5.10), (5.12), (5.14) and Cauchy's inequality (with epsilon) to treat the last integral appearing

in (5.11), we acquire the estimate

$$\int_\Omega P(\varrho_h^k) - P(\varrho_h^{k-1}) \ dx$$

$$+ \int_\Omega P''(\xi(\varrho_h^k, \varrho_h^{k-1})) \left[\varrho_h^{k-1}\right]^2 \ dx + \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma P''(\varrho_\dagger^k) \left[\varrho_h^k\right]_\Gamma^2 \left|\boldsymbol{u}_h^k \cdot \nu\right| \ dS(x)$$

$$+ \Delta t \int_\Omega \left|\boldsymbol{u}_h^k\right|^2 \ dx + \Delta t \int_\Omega \left|\operatorname{div} \boldsymbol{u}_h^k\right|^2 \ dx$$

$$+ \Delta t \int_\Omega \left|\boldsymbol{w}_h^k\right|^2 \ dx + \Delta t \int_\Omega \left|\operatorname{curl} \boldsymbol{w}_h^k\right|^2 \ dx \le C\Delta t \int_\Omega \left|\boldsymbol{f}_h^k\right|^2 \ dx. \tag{5.15}$$

Finally, summing (5.15) over k we conclude that (5.7) holds. $\qquad\square$

The stability estimate only provides the bound $p(\varrho_h) \in_b L^\infty(0, T; L^1(\Omega))$. Hence, it is not clear that $p(\varrho_h)$ converges weakly to an integrable function. Moreover, the subsequent analysis relies heavily on the pressure having higher integrability. In the ensuing lemma we establish that the pressure is in fact bounded in $L^2(0, T; L^2(\Omega))$, independently of h.

To simplify notation, we denote the effective viscous flux by

$$P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h) = p(\varrho_h) - (\mu + \lambda) \operatorname{div} \boldsymbol{u}_h. \tag{5.16}$$

We will also continue to use this notation in the subsequent sections.

Lemma 5.4 (Higher integrability on the pressure). *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.4) and Definition 3.1. Then*

$$p(\varrho_h) \in_b L^2((0, T) \times \Omega).$$

Proof. For all $m = 1, \ldots, M$, let $\boldsymbol{v}_h^m \in \boldsymbol{V}_h^{0,\perp}$ be such that

$$\operatorname{div} \boldsymbol{v}_h^m = P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) - \frac{1}{|\Omega|} \int_\Omega P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx.$$

Now, since the momentum scheme (3.2) gives

$$\int_\Omega P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \operatorname{div} \boldsymbol{v}_h \ dx = - \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h \ dx, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^{0,\perp},$$

we can use $\boldsymbol{v}_h^m$ as test function to obtain

$$\int_\Omega |P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m)|^2 \ dx = \frac{1}{|\Omega|} \left(\int_\Omega P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx\right)^2 - \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h^m \ dx.$$

Hence, with $\epsilon > 0$,

$$\int_\Omega \left|P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) - \frac{1}{|\Omega|} \int_\Omega P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx\right|^2 \ dx$$

$$= - \int_\Omega \boldsymbol{f}_h^m \boldsymbol{v}_h^m \ dx \le \frac{1}{4\epsilon} \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx + \epsilon \int_\Omega |\boldsymbol{v}_h^m|^2 \ dx. \tag{5.17}$$

By the Poincaré inequality (2.6)

$$\int_\Omega |\boldsymbol{v}_h^m|^2 \ dx \le C \int_\Omega \left|P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) - \frac{1}{|\Omega|} \int_\Omega P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx\right|^2 \ dx.$$

Consequently, by fixing ϵ small enough in (5.17),

$$\int_\Omega \left|P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) - \frac{1}{|\Omega|} \int_\Omega P_{\text{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx\right|^2 \ dx \le C \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx,$$

and thus

$$\int_\Omega |P_{\mathrm{eff}}(\varrho_h^m, \boldsymbol{u}_h^m)|^2 \ dx \le \frac{1}{|\Omega|} \left(\int_\Omega P_{\mathrm{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx \right)^2 + C \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx. \quad (5.18)$$

Now, due to the boundary conditions,

$$\int_\Omega P_{\mathrm{eff}}(\varrho_h^m, \boldsymbol{u}_h^m) \ dx = \int_\Omega p(\varrho_h^m) \ dx \le C,$$

where we also have put into use Lemma 5.3 and subsequently our assumptions on the source term $\boldsymbol{f}$ and the initial data ϱ_0. Hence, (5.18) allows us to conclude

$$\int_\Omega |P_{\mathrm{eff}}(\varrho_h^m, \boldsymbol{u}_h^m)|^2 \ dx \le C \left(1 + \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx \right), \qquad m = 1, \dots, M,$$

from which we obtain

$$\sum_{m=1}^M \Delta t \int_\Omega |P_{\mathrm{eff}}(\varrho_h^m, \boldsymbol{u}_h^m)|^2 \ dx \le C \left(T + \sum_{m=1}^M \Delta t \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx \right).$$

In view of the definition of P_{eff}, this immediately yields

$$\sum_{m=1}^M \Delta t \int_\Omega |p(\varrho_h)|^2 \ dx$$

$$\le C \left(T + \sum_{m=1}^M \Delta t \int_\Omega |\mathrm{div}\, \boldsymbol{u}_h^m|^2 \ dx + \sum_{m=1}^M \Delta t \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx \right),$$

which, due to Lemma 5.3 and $\boldsymbol{f} \in L^2((0,T) \times \Omega)$, concludes the proof. $\qquad\square$

We conclude this section by establishing a weak time continuity of the density approximation. For this purpose we shall need the following technical lemma, which provides a bound on the artificial diffusion introduced by the upwind discretization of the continuity equation.

Lemma 5.5. *Let* $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ *be a sequence of numerical solutions constructed according to* (3.4) *and Definition 3.1. There exists a constant* $C > 0$, *depending only on the shape regularity of* E_h, *the size of* $|\Omega|$, *and the final time* T, *such that*

$$\left| \sum_{E \in E_h} \int_0^T \int_{\partial E \setminus \partial \Omega} [\varrho_h]_{\partial E} \, (\boldsymbol{u}_h \cdot \nu)^- (\Pi_h^Q \phi - \phi) \ dS(x) dt \right| \quad (5.19)$$

$$\le C \, \|D\phi\|_{L^\infty(0,T; \boldsymbol{L}^\infty(\Omega))} \, h^{\frac{1}{2}},$$

for any $\phi \in L^\infty(0,T; W^{1,\infty}(\Omega))$.

Proof. We shall need the auxiliary function

$$B(z) = \begin{cases} z^2, & \gamma > 2, \\ z^\gamma, & \gamma \le 2. \end{cases}$$

Moreover, set

$$\phi^m(x) = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \phi(s, x) \ ds, \quad \phi_h^m = \Pi_h^Q \phi^m, \quad m = 1, \dots, M. \quad (5.20)$$

Using $B''(z) > 0$ for $z > 0$ and Hölder's inequality, we obtain

$$I^2 := \left| \sum_{m=1}^M \sum_{E \in E_h} \Delta t \int_{\partial E \setminus \partial \Omega} [\varrho_h^m]_{\partial E} \, (\boldsymbol{u}_h^m \cdot \nu)^- (\phi_h^m - \phi^m) \ dS(x) \right|^2$$

$$\leq \left(\sum_{m=1}^{M} \sum_{E \in E_h} \Delta t \int_{\partial E \setminus \partial \Omega} B''(\varrho_\dagger^m) \, [\varrho_h^m]^2 \, |\boldsymbol{u}_h^m \cdot \nu| \ dS(x) \right)$$

$$\times \left(\sum_{m=1}^{M} \sum_{E \in E_h} \Delta t \int_{\partial E \setminus \partial \Omega} \left(B''(\varrho_\dagger^m) \right)^{-1} |\boldsymbol{u}_h^m \cdot \nu| \, |\phi_h^m - \phi^m|^2 \ dS(x) \right),$$

$$=: I_1 \times I_2,$$

where the "intermediate" numbers $\varrho_\dagger^m$ are defined in (5.9).

If $1 < \gamma \leq 2$, then Lemma 5.3 can be applied:

$$I_1 \leq C \int_\Omega B(\varrho_0) \ dx + \sum_{m=1}^{M} \Delta t \int_\Omega |\boldsymbol{f}_h^m|^2 \ dx. \tag{5.21}$$

However, (5.21) continues to hold in the case $\gamma \geq 2$. This follows directly from the renormalized scheme (5.1), with $\phi_h := 1$, together with the fact that

$$\sum_{m=1}^{M} \Delta t \int_\Omega b(\varrho_h^m) \operatorname{div} \boldsymbol{u}_h^m \ dx dt$$

$$\leq \left(\sum_{m=1}^{M} \Delta t \int_\Omega |\varrho_h^m|^4 \ dx \right)^{\frac{1}{2}} \left(\sum_{m=1}^{M} \Delta t \int_\Omega |\operatorname{div} \boldsymbol{u}_h^m|^2 \ dx \right)^{\frac{1}{2}},$$

which is bounded by Lemmas 5.3 and 5.4.

Next, using Lemma 2.9, we have that

$$I_2 \leq h^2 \, \|D\phi\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))}^2 \left(\sum_{m=1}^{M} \sum_{E \in E_h} \Delta t \int_{\partial E \setminus \partial \Omega} \left[\left(B''(\varrho_\dagger^m) \right)^{-1} \right]^2 \ dS(x) \right)^{\frac{1}{2}}$$

$$\times \left(\sum_{m=1}^{M} \sum_{E \in E_h} \Delta t \int_{\partial E \setminus \partial \Omega} |\boldsymbol{u}_h^m \cdot \nu|^2 \ dS(x) \right)^{\frac{1}{2}}. \tag{5.22}$$

Thanks to Lemma 2.11 and Lemma 2.10,

$$\int_{\partial E} |\boldsymbol{u}_h^m \cdot \nu|^2 \ dS(x) \leq c \, h^{-1} \int_E |\boldsymbol{u}_h^m|^2 \ dx. \tag{5.23}$$

Moreover, since

$$\left(B''(\varrho_\dagger^m) \right)^{-1} \leq \left| \varrho_+^m + \varrho_-^m \right|^{2-\gamma} \leq C(1 + \varrho_+^m + \varrho_-^m),$$

whenever $1 < \gamma \leq 2$, and $\left(B''(\varrho_\dagger^m) \right)^{-1} = \frac{1}{2}$, whenever $\gamma > 2$, Lemma 2.11 also gives

$$\int_{\partial E} \left[\left(B''(\varrho_\dagger^m) \right)^{-1} \right]^2 \ dS(x) \leq C h^{-1} \left(|E| + \int_{E \cup \mathcal{N}(E)} |\varrho_h^m|^2 \ dx \right),$$

where $\mathcal{N}(E)$ denotes the union of the neighboring elements of E. Observe that

$$
\sum_{E \in E_h} h^{-1} \left(|E| + \sum_{F \in \mathcal{N}(E)} \int_F |\varrho_h^m|^2 \, dx \right)
$$
$$
\leq h^{-1} \left(|\Omega| + \int_{\bigcup_{E \in E_h} (E \cup \mathcal{N}(E))} |\varrho_h^m|^2 \, dx \right) \tag{5.24}
$$
$$
\leq h^{-1} \left(|\Omega| + (N+2) \int_\Omega |\varrho_h^m|^2 \, dx \right),
$$

where we have utilized the fact that

$$
\left| \bigcup_{E \in E_h} (E \cup \mathcal{N}(E)) \right| \leq (N+2)|\Omega|,
$$

which is true since the maximal cardinality of the set $\{(\mathcal{N}(E) \cup E) \cap F\}_{E \in E_h}$ is $N + 2$ for any $F \in E_h$. Inserting (5.23) and (5.24) into (5.22), we have arrived at

$$
I_2 \leq h^2 \, \|D\phi\|^2_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))} \, h^{-\frac{1}{2}} h^{-\frac{1}{2}}
$$
$$
\times \left(T + \sum_{m=1}^M \Delta t \int_\Omega |\varrho_h^m|^2 \, dx \right)^{\frac{1}{2}} \left(\sum_{m=1}^M \Delta t \int_\Omega |\boldsymbol{u}_h^m|^2 \, dx \right)^{\frac{1}{2}}
$$
$$
\leq C \, h \, \|D\phi\|^2_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))},
$$

where Lemmas 5.3 and 5.4 have been used to work out the last inequality. This concludes the proof of (5.19). $\qquad\square$

To simplify the notation, let us introduce the interpolation operator

$$
(\Pi_{\mathcal{L}} f)(t) = f^{m-1} + \frac{t - t^{m-1}}{\Delta t}(f^m - f^{m-1}), \qquad t \in (t^{m-1}, t^m). \tag{5.25}
$$

Lemma 5.6. *Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to* (3.4) *and Definition 3.1. Then*

$$
\frac{\partial}{\partial t} (\Pi_{\mathcal{L}} \varrho_h) \in_b L^1(0, T; W^{-1,1}(\Omega)).
$$

Proof. Fix $\phi \in L^\infty(0, T; W^{1,\infty}(\Omega))$, and recall the definitions of ϕ^m, ϕ_h^m, cf. (5.20). The continuity scheme (3.1) with ϕ_h^m as test function reads

$$
\Delta t \int_\Omega \frac{d}{dt}(\Pi_{\mathcal{L}} \varrho_h) \phi^m \, dxdt
$$
$$
= \Delta t \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h^m]_\Gamma \, dS(x). \tag{5.26}
$$

Since the traces of ϕ^m taken from either side of a face are equal, we can write

$$
\sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_-^m(\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m(\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h^m]_\Gamma \, dx
$$
$$
= \sum_{\Gamma \in \Gamma_h^I} \int_\Gamma \left(\varrho_+^m(\boldsymbol{u}_h^m \cdot \nu)^- + \varrho_-^m(\boldsymbol{u}_h^m \cdot \nu)^+ \right) [\phi_h^m - \phi^m] \, dS(x),
$$
$$
= -\sum_{E \in E_h} \int_{\partial E \setminus \partial \Omega} \left(\varrho_+^m(\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_-^m(\boldsymbol{u}_h^m \cdot \nu)^- \right) (\phi_h^m - \phi^m) \, dS(x),
$$

$$= \int_\Omega - \mathrm{div}(\varrho_h^m \boldsymbol{u}_h^m (\phi_h^m - \phi^m)) \ dx + \sum_{E \in E_h} \int_{\partial E \backslash \partial \Omega} [\varrho_h^m]_{\partial E} (\boldsymbol{u}_h^m \cdot \nu)^- (\phi_h^m - \phi^m) \ dS(x)$$

$$= \int_\Omega \varrho_h^m \boldsymbol{u}_h^m \cdot D\phi^m \ dx + \sum_{E \in E_h} \int_{\partial E \backslash \partial \Omega} [\varrho_h^m]_{\partial E} (\boldsymbol{u}_h^m \cdot \nu)^- (\phi_h^m - \phi^m) \ dS(x).$$

By summing (5.26) over m, taking absolute values, and using the above identity, we find

$$\left| \sum_{m=1}^M \Delta t \int_\Omega \frac{d}{dt} (\Pi_{\mathcal{L}} \varrho_h) \phi^m \ dxdt \right|$$

$$\leq \left| \sum_{m=1}^M \Delta t \int_\Omega \varrho_h^m \boldsymbol{u}_h^m D\phi^m \ dx \right|$$

$$+ \left| \sum_{m=1}^M \sum_{E \in E_h} \Delta t \int_{\partial E \backslash \partial \Omega} [\varrho_h^m]_{\partial E} (\boldsymbol{u}_h^m \cdot \nu)^- (\phi_h^m - \phi^m) \ dS(x) \right|.$$

Using Lemma 5.5, together with an application of Hölder's inequality, we deduce

$$\left| \sum_{m=1}^M \Delta t \int_\Omega \frac{d}{dt} (\Pi_{\mathcal{L}} \varrho_h) \phi^m \ dx \right|$$

$$\leq \left(\sum_{m=1}^M \Delta t \int_\Omega |\varrho_h^m|^2 \ dx \right)^{\frac{1}{2}} \left(\sum_{m=1}^M \Delta t \int_\Omega |\boldsymbol{u}_h^m|^2 \right)^{\frac{1}{2}} \|D\phi\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))}$$

$$+ C \, h^{\frac{1}{2}} \, \|D\phi\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))} \, .$$

By Lemmas 5.3 and 5.4, the first two factors on the right–hand side is bounded, so we conclude that

$$\left| \int_{\Delta t}^T \int_\Omega \frac{d}{dt} (\Pi_{\mathcal{L}} \varrho_h) \phi \ dxdt \right|$$

$$= \left| \sum_{m=1}^M \Delta t \int_\Omega \frac{d}{dt} (\Pi_{\mathcal{L}} \varrho_h) \phi^m \ dx \right| \leq C \left(1 + h^{\frac{1}{2}} \right) \|D\phi\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))} \, .$$

$\square$

6. Convergence

Let $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ be a sequence of numerical solutions constructed according to (3.4) and Definition 3.1. In this section we establish that a subsequence of $\{(\varrho_h, \boldsymbol{w}_h, \boldsymbol{u}_h)\}_{h>0}$ converges to a weak solution of the semi–stationary Stokes system, thereby proving Theorem 3.3. The proof is divided into several steps:

(1) Convergence of the continuity scheme.
(2) Weak sequential continuity of the discrete viscous flux.
(3) Strong convergence of the density.
(4) Convergence of the velocity scheme.

Our starting point is that the results of Section 5 assure us that the approximate solutions $(\boldsymbol{w}_h, \boldsymbol{u}_h, \varrho_h)$ satisfy the following h–independent bounds:

$$\varrho_h \in_b L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T) \times \Omega)$$

and

$$\boldsymbol{w}_h \in_b L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)), \quad \boldsymbol{u}_h \in_b L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega)).$$

Consequently, we may assume that there exist functions $\varrho, \boldsymbol{w}, \boldsymbol{u}$ such that

$$\varrho_h \overset{h\to 0}{\rightharpoonup} \varrho, \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega),$$

$$\boldsymbol{w}_h \overset{h\to 0}{\rightharpoonup} \boldsymbol{w}, \quad \text{in } L^2(0,T;\boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)), \tag{6.1}$$

$$\boldsymbol{u}_h \overset{h\to 0}{\rightharpoonup} \boldsymbol{u}, \quad \text{in } L^2(0,T;\boldsymbol{W}_0^{\mathrm{div},2}(\Omega)).$$

Moreover,

$$\varrho_h^\gamma \overset{h\to 0}{\rightharpoonup} \overline{\varrho^\gamma}, \quad \varrho_h^{\gamma+1} \overset{h\to 0}{\rightharpoonup} \overline{\varrho^{\gamma+1}}, \quad \varrho_h \log \varrho_h \overset{h\to 0}{\rightharpoonup} \overline{\varrho \log \varrho},$$

where each $\overset{h\to 0}{\rightharpoonup}$ signifies weak convergence in a suitable L^p space with $p > 1$.

Finally, ϱ_h, $\varrho_h \log \varrho_h$ converge respectively to ϱ, $\overline{\varrho \log \varrho}$ in $C([0,T];L^p_{\mathrm{weak}}(\Omega))$ for some $1 < p < \gamma$, cf. Lemma 2.2 and also [9, 15]. In particular, ϱ, $\varrho \log \varrho$, and $\overline{\varrho \log \varrho}$ belong to $C([0,T];L^p_{\mathrm{weak}}(\Omega))$.

6.1. Density scheme.

Lemma 6.1 (Convergence of $\varrho_h \boldsymbol{u}_h$). *Given* (6.1),

$$\varrho_h \boldsymbol{u}_h \overset{h\to 0}{\rightharpoonup} \varrho \boldsymbol{u} \quad \text{in the sense of distributions on } (0,T)\times\Omega.$$

Proof. By virtue of Lemma 5.2, there exist sequences $\{\boldsymbol{\zeta}_h\}_{h>0}$, $\{\boldsymbol{z}_h\}_{h>0}$ satisfying

$$\boldsymbol{u}_h(\cdot,t) = \operatorname{curl}\boldsymbol{\zeta}_h(\cdot,t) + \boldsymbol{z}_h(\cdot,t),$$

$$\boldsymbol{\zeta}_h(\cdot,t) \in \boldsymbol{W}_h^{0,\perp}, \quad \boldsymbol{z}_h(t,\cdot) \in \boldsymbol{V}_h^{0,\perp},$$

for all $t \in (0,T)$. In Lemma 6.2 below we prove that

$$\operatorname{curl}\boldsymbol{\zeta}_h \to \operatorname{curl}\boldsymbol{\zeta} \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)).$$

As a consequence, $\operatorname{curl}\boldsymbol{\zeta}_h\,\varrho_h \rightharpoonup \operatorname{curl}\boldsymbol{\zeta}\,\varrho$ in the sense of distributions.

It remains to prove that

$$\varrho_h \boldsymbol{z}_h \rightharpoonup \varrho \boldsymbol{z} \quad \text{in the sense of distributions.}$$

To this end, we adapt the proof of [15, Lemma 5.1] to our specific discrete setting. We begin by introducing the regularized field $\boldsymbol{z}_h^\epsilon = \kappa^\epsilon \underset{(x)}{\star} \boldsymbol{z}_h$, where κ^ϵ is a standard regularizing kernel and $\underset{(x)}{\star}$ denotes the convolution product (in x). Lemma 6.3 guarantees that

$$\|\boldsymbol{z}_h^\epsilon - \boldsymbol{z}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \to 0 \quad \text{as } \epsilon \to 0, \text{ uniformly in } h.$$

In addition, for any k and p, since $\boldsymbol{z}_h^\epsilon \in L^2(0,T;\boldsymbol{W}^{k,p}(\Omega))$ we have that $\boldsymbol{z}_h^\epsilon \overset{h\to 0}{\rightharpoonup} \boldsymbol{z}^\epsilon$ in $L^2(0,T;\boldsymbol{W}^{k,p}(\Omega))$. Moreover, $\boldsymbol{z}^\epsilon \overset{\epsilon\to 0}{\rightarrow} \boldsymbol{z}$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$. Hence, by writing $\varrho_h \boldsymbol{z}_h = \varrho_h(\boldsymbol{z}_h - \boldsymbol{z}_h^\epsilon) + \varrho_h \boldsymbol{z}_h^\epsilon$ it suffices to prove $\varrho_h \boldsymbol{z}_h^\epsilon \overset{h\to 0}{\rightharpoonup} \varrho \boldsymbol{z}^\epsilon$ for each fixed $\epsilon > 0$.

Next, let us introduce auxiliary functions $\boldsymbol{Z}_h^{\epsilon,m} \in \boldsymbol{V}_h$, $m = 1,\ldots,M$, defined by

$$\boldsymbol{Z}_h^{\epsilon,m}(x) = \Delta t \sum_{k=0}^m \boldsymbol{z}_h^{\epsilon,m}(x), \qquad \boldsymbol{z}_h^{\epsilon,m} = \kappa^\epsilon \star \boldsymbol{z}_h^m.$$

We extend $\{\boldsymbol{Z}_h^{\epsilon,m}\}_{m=1}^M$ to a function $\boldsymbol{Z}_h^\epsilon$ defined on $(-\Delta t,T]\times\Omega$ by setting

$$\boldsymbol{Z}_h^\epsilon(t,\cdot) = \boldsymbol{Z}_h^{\epsilon,m}(\cdot), \qquad t \in (t^{m-1},t^m], \quad m = 1,\ldots,M,$$

and $\boldsymbol{Z}_h^\epsilon(t,\cdot) = \boldsymbol{Z}_h^{\epsilon,0}$, for $t \in (-\Delta t,0]$. In view of the regularity of $\boldsymbol{z}_h^\epsilon$,

$$\boldsymbol{Z}_h^\epsilon(t,\cdot) \to \boldsymbol{Z}^\epsilon(t,\cdot) = \int_0^t \boldsymbol{z}^\epsilon(s,\cdot)\,ds \quad \text{in } C^k(\Omega) \text{ for any } k \geq 0, \tag{6.2}$$

uniformly in t on $[0,T]$.

Now, we write

$$\varrho_h^m \boldsymbol{z}_h^{\epsilon,m} = \frac{\varrho_h^m \boldsymbol{Z}_h^{\epsilon,m} - \varrho_h^{m-1} \boldsymbol{Z}_h^{\epsilon,m-1}}{\Delta t} - \boldsymbol{Z}_h^{\epsilon,m-1} \frac{\varrho_h^m - \varrho_h^{m-1}}{\Delta t},$$

which alternatively can be written as

$$\varrho_h \boldsymbol{z}_h^{\epsilon} = \frac{\partial}{\partial t} \Pi_{\mathcal{L}} \left(\varrho_h \boldsymbol{Z}_h^{\epsilon} \right) - \boldsymbol{Z}_h^{\epsilon}(\cdot - \Delta t, \cdot) \frac{\partial}{\partial t} \left(\Pi_{\mathcal{L}} \varrho_h \right),$$

on $(t_{m-1}, t_m] \times \Omega$, $m = 1, \dots, M$.

Fix $\phi \in C_c^\infty((0, T) \times \Omega)$. Summation by parts gives

$$\int_0^T \int_\Omega \frac{\partial}{\partial t} \Pi_{\mathcal{L}} \left(\varrho_h \boldsymbol{Z}_h^{\epsilon} \right) \phi \; dx dt$$
$$= - \int_{\Delta t}^T \int_\Omega \varrho_h(t - \Delta t, x) \boldsymbol{Z}_h^{\epsilon}(t - \Delta t, x) \frac{\partial}{\partial t} \left(\Pi_{\mathcal{L}} \phi_h \right) \; dx dt,$$

where $\phi_h(t, \cdot) = \frac{1}{\Delta t} \int_{t_{m-1}}^{t^m} \phi(s, \cdot) \; ds$ for $t \in (t_{m-1}, t_m)$.

Thanks to (6.1) and (6.2), $\varrho_h \boldsymbol{Z}_h^{\epsilon} \overset{h \to 0}{\rightharpoonup} \varrho \boldsymbol{Z}^{\epsilon}$ in $L^{2\gamma}(0, T; L^{2\gamma}(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega))$, and hence

$$\frac{\partial}{\partial t} \Pi_{\mathcal{L}} \left(\varrho_h \boldsymbol{Z}_h^{\epsilon} \right) \overset{h \to 0}{\rightharpoonup} \frac{\partial}{\partial t} \left(\varrho \boldsymbol{Z}^{\epsilon} \right) \quad \text{in the sense of distributions on } (0, T) \times \Omega.$$

In addition, Lemma 5.6 tells us that $\frac{\partial}{\partial t} \left(\Pi_{\mathcal{L}} \varrho_h \right) \in_b L^1(0, T; W^{-1,1}(\Omega))$, and thus

$$\boldsymbol{Z}_h^{\epsilon}(\cdot - \Delta t, \cdot) \frac{\partial}{\partial t} \left(\Pi_{\mathcal{L}} \varrho_h \right) \overset{h \to 0}{\rightharpoonup} \boldsymbol{Z}^{\epsilon} \frac{\partial}{\partial t} \varrho$$

in the sense of distributions on $(0, T) \times \Omega$.

We conclude observing that $\varrho \boldsymbol{z}^{\epsilon} = \frac{d}{dt} \left(\varrho \boldsymbol{Z}^{\epsilon} \right) - \boldsymbol{Z}^{\epsilon} \frac{\partial \varrho}{\partial t}$. $\square$

In the proof of the previous lemma we utilized

Lemma 6.2. *Given* (6.1), *define* $\{(\boldsymbol{\zeta}_h, \boldsymbol{z}_h)\}_{h>0}$ *in terms of the decomposition* $\boldsymbol{u}_h(t, \cdot) = \operatorname{curl} \boldsymbol{\zeta}_h(t, \cdot) + \boldsymbol{z}_h(t, \cdot)$ *with* $\boldsymbol{\zeta}_h(t, \cdot) \in \boldsymbol{W}_h^{0,\perp}$, $\boldsymbol{z}_h(t, \cdot) \in \boldsymbol{V}_h^{0,\perp}$, $t \in (0, T)$. *Then*

$$\boldsymbol{w}_h \overset{h \to 0}{\to} \boldsymbol{w}, \quad \operatorname{curl} \boldsymbol{\zeta}_h \overset{h \to 0}{\to} \operatorname{curl} \boldsymbol{\zeta} \quad \text{in } L^2(0, T; \boldsymbol{L}^2(\Omega)). \tag{6.3}$$

Proof. Subtract the first equation of (3.2) with $\boldsymbol{v}_h = \operatorname{curl} \boldsymbol{\xi}_h^m$ from μ times the second equation of (3.2). Multiplying the result with Δt and summing over all $m = 1, \dots, M$ yields

$$\int_0^T \int_\Omega \mu \operatorname{curl} \boldsymbol{\eta}_h \operatorname{curl} \boldsymbol{\zeta}_h - \mu \operatorname{curl} \boldsymbol{w}_h \operatorname{curl} \boldsymbol{\xi}_h \; dx dt$$
$$= \int_0^T \int_\Omega \mu \boldsymbol{w}_h \boldsymbol{\eta}_h - \boldsymbol{f}_h \operatorname{curl} \boldsymbol{\xi}_h \; dx dt, \tag{6.4}$$

for all $\boldsymbol{\eta}_h, \boldsymbol{\xi}_h$ that are piecewise constant in time with values in $\boldsymbol{W}_h(\Omega)$. Fixing $\boldsymbol{\eta}, \boldsymbol{\xi} \in C_c^\infty((0, T) \times \Omega)$, we use in (6.4) the test functions

$$\boldsymbol{\xi}_h(t, \cdot) = \boldsymbol{\xi}_h^m(\cdot) := \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \Pi_h^W \boldsymbol{\xi}(\cdot, s) \; ds, \quad t \in (t_{m-1}, t_m), \; m = 1, \dots, M.$$

$$\boldsymbol{\zeta}_h(t, \cdot) = \boldsymbol{\zeta}_h^m(\cdot) := \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \Pi_h^W \boldsymbol{\zeta}(\cdot, s) \; ds, \quad t \in (t_{m-1}, t_m), \; m = 1, \dots, M.$$

Due to Lemma 2.9, $\operatorname{curl}\boldsymbol{\xi}_h \to \operatorname{curl}\boldsymbol{\xi}$ and $\operatorname{curl}\boldsymbol{\eta}_h \to \operatorname{curl}\boldsymbol{\eta}$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$. As a consequence, keeping in mind (6.1), we let $h \to 0$ in (6.4) to obtain

$$\int_0^T \int_\Omega \mu \operatorname{curl}\boldsymbol{\eta} \operatorname{curl}\boldsymbol{\zeta} - \mu \operatorname{curl}\boldsymbol{w} \operatorname{curl}\boldsymbol{\xi} \, dxdt$$
$$= \int_0^T \int_\Omega \mu \boldsymbol{w}\boldsymbol{\eta} - \boldsymbol{f}\operatorname{curl}\boldsymbol{\zeta} \, dxdt, \quad \forall \boldsymbol{\eta},\boldsymbol{\xi} \in \boldsymbol{C}_c^\infty((0,T) \times \Omega). \tag{6.5}$$

Since $\boldsymbol{C}_c^\infty((0,T) \times \Omega)$ is dense in $L^2(0,T;\boldsymbol{W}_0^{\operatorname{curl},2}(\Omega))$ ([11]), we conclude that (6.5) holds for all $\boldsymbol{\eta},\boldsymbol{\xi} \in L^2(0,T;\boldsymbol{W}_0^{\operatorname{curl},2}(\Omega))$. Hence, taking $\boldsymbol{\eta} = \boldsymbol{w}$, $\boldsymbol{\xi} = \boldsymbol{\zeta}$ in (6.5),

$$0 = \int_0^T \int_\Omega \mu |\boldsymbol{w}|^2 - \boldsymbol{f}\operatorname{curl}\boldsymbol{\xi} \, dxdt. \tag{6.6}$$

Next, setting $\boldsymbol{\eta}_h = \boldsymbol{w}_h$ and $\boldsymbol{\xi}_h = \boldsymbol{\zeta}_h$ in (6.4), we observe that

$$0 = \int_0^T \int_\Omega \mu |\boldsymbol{w}_h|^2 - \boldsymbol{f}_h \operatorname{curl}\boldsymbol{\xi}_h \, dxdt.$$

Letting $h \to 0$ and comparing the result with (6.6) reveals that

$$\lim_{h \to 0} \int_0^T \int_\Omega \mu |\boldsymbol{w}_h|^2 \, dxdt = \int_0^T \int_\Omega \mu |\boldsymbol{w}|^2 \, dxdt,$$

which implies the first part of (6.3):

$$\boldsymbol{w}_h \to \boldsymbol{w} \quad \text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{6.7}$$

To prove the second part of (6.3), we make use of $\boldsymbol{\eta}_h = \boldsymbol{\zeta}_h^m$ as a test function in the second equation of (3.2), sum the result over $m = 1,\dots,M$, and subsequently send h to zero:

$$\lim_{h \to 0} \int_0^T \int_\Omega |\operatorname{curl}\boldsymbol{\zeta}_h|^2 \, dxdt = \lim_{h \to 0} \int_0^T \int_\Omega \boldsymbol{w}_h\boldsymbol{\zeta}_h \, dxdt$$
$$\overset{(6.7)}{=} \int_0^T \int_\Omega \boldsymbol{w}\boldsymbol{\zeta} \, dxdt = \int_0^T \int_\Omega |\operatorname{curl}\boldsymbol{\zeta}|^2 \, dxdt,$$

where the last equality follows by arguing along the lines leading up to (6.6). $\quad\square$

During the proof of Lemma 6.1 we made use of a spatial compactness property stated in the next lemma.

Lemma 6.3. *Given* (6.1), *define* $\{(\boldsymbol{\zeta}_h,\boldsymbol{z}_h)\}_{h>0}$ *in terms of the decomposition* $\boldsymbol{u}_h(\cdot,t) = \operatorname{curl}\boldsymbol{\zeta}_h(t,\cdot) + \boldsymbol{z}_h(\cdot,t)$ *with* $\boldsymbol{\zeta}_h(\cdot,t) \in \boldsymbol{W}_h^{0,\perp}$, $\boldsymbol{z}_h(t,\cdot) \in \boldsymbol{V}_h^{0,\perp}$, *for* $t \in (0,T)$. *Then, for any* $\xi \in \mathbb{R}^N$,

$$\|\boldsymbol{z}_h(t,\cdot) - \boldsymbol{z}_h(t,\cdot - \xi)\|_{L^2(0,T;\boldsymbol{L}^2(\Omega_\xi))}^2 \leq C\left(|\xi|^{\frac{4-N}{2}} + |\xi|^2\right) \|\operatorname{div}\boldsymbol{z}_h\|_{L^2(0,T;L^2(\Omega))}^2,$$

where $\Omega_\xi = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > \xi\}$.

Proof. For each $t \in (0,T)$ we know that $\boldsymbol{z}_h(t,\cdot) \in \boldsymbol{V}_h^{0,\perp}(\Omega)$, so Theorem A.1 can be applied to give

$$\|\boldsymbol{z}_h(t,\cdot) - \boldsymbol{z}_h(t,\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 \leq C\left(|\xi|^{\frac{4-N}{2}} + |\xi|^2\right) \|\operatorname{div}\boldsymbol{z}_h\|_{L^2(\Omega)}^2,$$

where $C > 0$ is independent of h,ξ,t. We conclude by integrating over $(0,T)$. $\quad\square$

Lemma 6.4 (Continuity equation). *The limit pair* $(\varrho,\boldsymbol{u})$ *constructed in* (6.1) *is a weak solution of the continuity equation* (1.1) *in the sense of Definition 2.5.*

Proof. Fix a test function $\phi \in C_c^\infty([0,T) \times \overline{\Omega})$, and introduce the piecewise constant approximations $\phi_h := \Pi_h^Q \phi$, $\phi_h^m := \Pi_h^Q \phi^m$, and $\phi^m := \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \phi(t, \cdot) \, dt$.

Let us employ ϕ_h^m as test function in the continuity scheme (3.1) and sum over $m = 1, \ldots, M$. The resulting equation reads

$$\sum_{m=1}^M \Delta t \int_\Omega \frac{d}{dt} \left(\Pi_\mathcal{L} \varrho_h \right) \phi_h^m \, dxdt$$

$$= \sum_{\Gamma \in \Gamma_h^I} \sum_{m=1}^M \Delta t \int_\Gamma \left(\varrho_-^m (\boldsymbol{u}_h^m \cdot \nu)^+ + \varrho_+^m (\boldsymbol{u}_h^m \cdot \nu)^- \right) [\phi_h^m]_\Gamma \, dS(x).$$

As in the proof of Lemma 5.6 we can rewrite this as

$$\sum_{m=1}^M \Delta t \int_\Omega \frac{d}{dt} \left(\Pi_\mathcal{L} \varrho_h \right) \phi_h^m \, dxdt$$

$$= \sum_{m=1}^M \Delta t \int_\Omega \varrho_h^m \boldsymbol{u}_h^m D\phi^m \, dx$$

$$+ \sum_{E \in E_h} \sum_{m=1}^M \Delta t \int_{\partial E \setminus \partial \Omega} [\varrho_h^m]_{\partial E} \, (\boldsymbol{u}_h^m \cdot \nu)^- (\phi_h^m - \phi^m) \, dS(x) \qquad (6.8)$$

$$= \int_0^T \int_\Omega \varrho_h \boldsymbol{u}_h D\phi \, dxdt$$

$$+ \sum_{E \in E_h} \int_0^T \int_{\partial E \setminus \partial \Omega} [\varrho_h]_{\partial E} \, (\boldsymbol{u}_h \cdot \nu)^- (\phi_h - \phi) \, dS(x)dt.$$

Lemma 5.5 tells us that

$$\left| \sum_{E \in E_h} \int_0^T \int_{\partial E \setminus \partial \Omega} [\varrho_h]_{\partial E} \, (\boldsymbol{u}_h \cdot \nu)^- (\phi_h - \phi) \, dS(x)dt \right| \leq C \, h^{\frac{1}{2}} \, \|D\phi\|_{L^\infty(0,T;\boldsymbol{L}^\infty(\Omega))} \cdot$$

In view of Lemma 6.1,

$$\lim_{h \to 0} \int_0^T \int_\Omega \varrho_h \boldsymbol{u}_h D\phi \, dxdt = \int_0^T \int_\Omega \varrho \boldsymbol{u} D\phi \, dxdt.$$

Summation by parts gives

$$\sum_{m=1}^M \Delta t \int_\Omega \frac{d}{dt} \left(\Pi_\mathcal{L} \varrho_h \right) \phi_h^m \, dxdt$$

$$= - \int_{\Delta t}^T \int_\Omega \varrho_h(t - \Delta t, x) \frac{\partial}{\partial t} \left(\Pi_\mathcal{L} \phi_h \right) \, dxdt - \int_\Omega \varrho_h^0 \phi_h^1 \, dx$$

$$\overset{h \to 0}{\to} - \int_0^T \int_\Omega \varrho \phi_t \, dxdt - \int_\Omega \varrho_0 \phi(0, x) \, dx.$$

where (6.1), together with the strong convergence $\varrho_h^0 \overset{h \to 0}{\to} \varrho_0$, was used to pass to the limit. Summarizing, letting $h \to 0$ in (6.8) delivers the desired result (2.2) $\square$

6.2. **Strong convergence of density approximations.** The instrument used to establish the strong convergence of the density approximations ϱ_h is a weak continuity property of the quantity $P_{\text{eff}}(\varrho_h, \boldsymbol{u}_h)$ defined in (5.16). To derive this property we exploit our choice of numerical method and the boundary conditions; specifically, the finite element spaces, which are chosen such that (6.10) below holds.

Lemma 6.5 (Discrete effective viscous flux). *Given the convergences in* (6.1),

$$\lim_{h\to 0}\int_0^t\int_\Omega P_{eff}(\varrho_h,\boldsymbol{u}_h)\,\varrho_h\;dxds=\int_0^t\int_\Omega\overline{P_{eff}(\varrho,\boldsymbol{u})}\,\varrho\;dxds,\quad\forall t\in(0,T).$$

Proof. For each $m=1,\dots,M$, consider the problem

$$\operatorname{div}\boldsymbol{v}_h^m=q_h^m-\frac{1}{|\Omega|}\int_\Omega q_h^0\;dx,\qquad q_h^m:=\varrho_h-\frac{1}{\Delta t}\int_{t^{m-1}}^{t^m}\Pi_h^Q\varrho\;dt,\qquad(6.9)$$

where $q_h^0=\varrho_h^0-\varrho_0$. Observe that $\int_\Omega q_h^m\;dx=0$. Indeed, using the continuity scheme (3.1) and the continuity equation satisfied by the limit ϱ, cf. Lemma 6.4,

$$\int_\Omega\varrho_h\;dx=\int_\Omega\varrho_h^0\;dx,\qquad\int_\Omega\Pi_h^Q\varrho\;dx=\int_\Omega\varrho\;dx=\int_\Omega\varrho_0\;dx$$

Thus, there exists a unique solution $\boldsymbol{v}_h^m\in\boldsymbol{V}_h^{0,\perp}$ of (6.9). We denote by $\boldsymbol{v}_h(t,\cdot)$, $q_h(t,\cdot)$ the usual "piecewise constant" extensions of $\{v_h^m\}_{m=1}^M$, $\{q_h^m\}_{m=1}^M$ to $(0,T)$.

Utilizing $\boldsymbol{v}_h^m$ as test function, the velocity scheme (3.2) reads

$$\int_\Omega P_{\text{eff}}(\varrho_h^m,\boldsymbol{u}_h^m)q_h^m\;dx=\frac{1}{|\Omega|}\int_\Omega q_h^0\;dx\left(\int_\Omega P_{\text{eff}}(\varrho_h^m,\boldsymbol{u}_h^m)\;dx\right)-\int_\Omega\boldsymbol{f}_h^m\boldsymbol{v}_h^m\;dx.$$
$$(6.10)$$

Multiplying by Δt, summing over m, and using the definition of q_h^m, we arrive at

$$\int_0^t\int_\Omega P_{\text{eff}}(\varrho_h,\boldsymbol{u}_h)(\varrho_h-\varrho)\;dxds=\frac{1}{|\Omega|}\int_\Omega q_h^0\;dx\left(\int_0^t\int_\Omega P_{\text{eff}}(\varrho_h,\boldsymbol{u}_h)\;dxdt\right)$$
$$-\int_0^t\int_\Omega\boldsymbol{f}_h\boldsymbol{v}_h\;dxds,$$

for any $t\in(0,T)$.

In view of Theorem A.1, we have that $\boldsymbol{v}_h\rightharpoonup 0$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$. Since $\boldsymbol{f}_h\to\boldsymbol{f}$ in $L^2((0,T)\times\Omega)$ and $\int_\Omega q_h^0\;dx\to 0$, we conclude the desired result

$$\lim_{h\to 0}\int_0^t\int_\Omega P_{\text{eff}}(\varrho_h,\boldsymbol{u}_h)(\varrho_h-\varrho)\;dxds=0.$$

$\square$

We are now in a position to infer the sought-after strong convergence of the density approximations.

Lemma 6.6 (Strong convergence of ϱ_h). *Suppose that* (6.1) *holds. Then, passing to a subsequence if necessary,*

$$\varrho_h\to\varrho\quad\text{a.e. in }(0,T)\times\Omega.$$

Proof. In view of Lemma 6.4, the limit $(\varrho,\boldsymbol{u})$ is a weak solution of the continuity equation and hence, by Lemma 2.7, also a renormalized solution. In particular,

$$(\varrho\log\varrho)_t+\operatorname{div}((\varrho\log\varrho)\,\boldsymbol{u})=\varrho\operatorname{div}\boldsymbol{u}\quad\text{in the weak sense on }[0,T)\times\overline{\Omega}.$$

Since $t\mapsto\varrho\log\varrho$ is continuous with values in some Lebesgue space equipped with the weak topology, we can use this equation to obtain for any $t>0$

$$\int_\Omega(\varrho\log\varrho)\,(t)\;dx-\int_\Omega\varrho_0\log\varrho_0\;dx=-\int_0^t\int_\Omega\varrho\operatorname{div}\boldsymbol{u}\;dxds\qquad(6.11)$$

Next, we specify $\phi_h\equiv 1$ as test function in the renormalized scheme (5.1), multiply by Δt, and sum the result over m. Making use of the convexity of $z\log z$, we infer for any $m=1,\dots,M$

$$\int_\Omega\varrho_h^m\log\varrho_h^m\;dx-\int_\Omega\varrho_h^0\log\varrho_h^0\;dx\leq-\sum_{k=1}^m\Delta t\int_\Omega\varrho_h^m\operatorname{div}\boldsymbol{u}_h^m\;dxdt.\qquad(6.12)$$

In view of the convergences stated at the beginning of this section and strong convergence of the initial data, we can send $h \to 0$ in (6.12) to obtain

$$\int_\Omega \left(\overline{\varrho \log \varrho} \right)(t) \; dx - \int_\Omega \varrho_0 \log \varrho_0 \; dx \leq - \int_0^t \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}} \; dxds. \qquad (6.13)$$

Subtracting (6.11) from (6.13) gives

$$\int_\Omega \left(\overline{\varrho \log \varrho} - \varrho \log \varrho \right)(t) \; dx \leq - \int_0^t \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}} - \varrho \operatorname{div} \boldsymbol{u} \; dxds,$$

for any $t \in (0, T)$. Lemma 6.5 tells us that

$$\int_0^t \int_\Omega \overline{\varrho \operatorname{div} \boldsymbol{u}} - \varrho \operatorname{div} \boldsymbol{u} \; dxds = \frac{a}{\mu + \lambda} \int_0^t \int_\Omega \overline{\varrho^{\gamma+1}} - \overline{\varrho^\gamma} \varrho \; dxds \geq 0,$$

where the last inequality follows as in [9, 15], so the following relation holds:

$$\overline{\varrho \log \varrho} = \varrho \log \varrho \quad \text{a.e. in } (0, T) \times \Omega.$$

Now an application of Lemma 2.1 brings the proof to an end. $\qquad \square$

6.3. Velocity scheme.

Lemma 6.7 (Velocity equation). *The limit triple $(\boldsymbol{w}, \boldsymbol{u}, \varrho)$ constructed in (6.1) is a weak solution of the velocity equation (1.2) in the sense of (2.3).*

Proof. Fix $(\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{C}_c^\infty((0, T) \times \Omega)$, and introduce the projections $\boldsymbol{v}_h = \Pi_h^V \boldsymbol{v}$, $\boldsymbol{\eta}_h = \Pi_h^W \boldsymbol{\eta}$ and $\boldsymbol{v}_h^m = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \boldsymbol{v}_h \; dt$, $\boldsymbol{\eta}_h^m = \frac{1}{\Delta t} \int_{t^{m-1}}^{t^m} \boldsymbol{\eta}_h \; dt$.

Utilizing $\boldsymbol{v}_h^m$ and $\boldsymbol{\eta}_h^m$ as test functions in the velocity scheme (3.2), multiplying by Δt, and summing the result over m, we gather

$$\int_0^T \int_\Omega \mu \operatorname{curl} \boldsymbol{w}_h \boldsymbol{v}_h + [(\mu + \lambda) \operatorname{div} \boldsymbol{u}_h - p(\varrho_h)] \operatorname{div} \boldsymbol{v}_h \; dxdt = \int_0^T \int_\Omega \boldsymbol{f}_h \boldsymbol{v}_h \; dxdt,$$

$$\int_0^T \int_\Omega \boldsymbol{w}_h \boldsymbol{\eta}_h - \boldsymbol{u}_h \operatorname{curl} \boldsymbol{\eta}_h \; dxdt = 0.$$

$$\qquad (6.14)$$

From Lemma 2.9 we have $\boldsymbol{v}_h \overset{h \to 0}{\to} \boldsymbol{v}$ in $L^2(0, T; \boldsymbol{W}_0^{\operatorname{div},2}(\Omega))$ and $\boldsymbol{\eta}_h \overset{h \to 0}{\to} \boldsymbol{\eta}$ in $L^2(0, T; \boldsymbol{W}_0^{\operatorname{curl},2}(\Omega))$. Furthermore, by Lemma 6.6 and the first part of (6.1), $p(\varrho_h) \overset{h \to 0}{\to} p(\varrho)$ in $L^2((0, T) \times \Omega)$. Hence, we can send $h \to 0$ in (6.14) to obtain that the limit $(\boldsymbol{w}, \boldsymbol{u}, \varrho)$ constructed in (6.1) satisfies (2.3) for all test functions $(\boldsymbol{v}, \boldsymbol{\eta}) \in \boldsymbol{C}_c^\infty((0, T) \times \Omega)$. Since $\boldsymbol{C}_c^\infty((0, T) \times \Omega)$ is dense in both $L^2(0, T; \boldsymbol{W}_0^{\operatorname{div},2}(\Omega))$ and $L^2(0, T; \boldsymbol{W}_0^{\operatorname{curl},2}(\Omega))$ [11], this concludes the proof. $\qquad \square$

APPENDIX A. COMPACTNESS OF FUNCTIONS IN $V_h^{0,\perp}(\Omega)$

In this appendix we prove that discrete weakly curl free approximations in $\boldsymbol{V}^{0,\perp}(\Omega)$ with L^2 bounded divergence possesses an $\boldsymbol{L}^2$ space translation estimate, which was previously needed to conclude the weak convergence of the product $\varrho_h \boldsymbol{u}_h$ to the product of the corresponding weak limits $\varrho \boldsymbol{u}$. As part of the proof, in Lemma A.5 we show that if a sequence $\{\boldsymbol{v}_h\}_{h>0}$ belongs to $\boldsymbol{V}_h^{0,\perp}(\Omega)$ and besides satisfies $\operatorname{div} \boldsymbol{v}_h \in_b L^2(\Omega)$, then $\{\operatorname{curl} \boldsymbol{v}_h\}_{h>0}$ and $\{\operatorname{div} \boldsymbol{v}_h\}_{h>0}$ are actually compact in $W^{-1,2}(\Omega)$. Thus, strong $L^2(\Omega)$ convergence of a subsequence of $\{\boldsymbol{v}_h\}_{h>0}$ follows directly from the div–curl lemma. However, this is not sufficient to conclude the sought after convergence of $\varrho_h \boldsymbol{u}_h$ (cf. Subsection 6.1). The problem is a lack of temporal control of the velocity approximations $\boldsymbol{u}_h$.

A.1. **Space translation estimate.** The argument is inspired by Brenner's work [2] on Poincaré–Friedrich inequalities for piecewise H^1 vector fields. The basic idea is to project the relevant function into the Crouzeix–Raviart element space and then use the standard translation estimate satisfied by functions in this space (cf. Stummel [20]). Then, since the relevant function is discrete weakly curl free, we have sufficient control on the curl to suitably bound the projection error.

The Crouzeix–Raviart element space is defined as a non–conforming $\mathbb{P}_1$ element for each component of the vector field. That is, on each element $E \in E_h$ the Crouzeix–Raviart polynomial space $\boldsymbol{R}(E)$ is given by

$$\boldsymbol{R}(E) = [\mathbb{P}^1(E)]^N,$$

where $\mathbb{P}_1(E)$ is the space of linear scalar fields on E and N is the spatial dimension. The degrees of freedom of $\boldsymbol{R}(E)$ are the average integrals over the faces of E. The Crouzeix–Raviart element space, denoted $\boldsymbol{R}_h(\Omega)$, is formed on E_h by matching the degrees of freedom on each face $\Gamma \in \Gamma_h$. Hence, $\boldsymbol{R}_h(\Omega)$ is discontinuous across element faces and thus leads to non–conforming discretizations of $\boldsymbol{H}^1(\Omega)$. However, it has the property that for any $\boldsymbol{v}_h \in \boldsymbol{R}_h$, $\int_\Gamma [\boldsymbol{v}_h] \, dS(x) = 0$ for all $\Gamma \in \Gamma_h$.

Theorem A.1. *Given $\boldsymbol{z}_h \in \boldsymbol{V}_h^{0,\perp}$, there exists a constant $C > 0$, depending only on Ω and the shape regularity of E_h, such that for every vector $\xi \in \mathbb{R}^N$,*

$$\|\boldsymbol{z}_h(\cdot) - \boldsymbol{z}_h(\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega_\xi)} \le C \left(|\xi|^{\frac{4-N}{2}} + |\xi|^2 \right)^{\frac{1}{2}} \|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)}, \tag{A.1}$$

where $\Omega_\xi = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \xi\}$.

Proof. Let us introduce an interpolation operator $\Pi_h^R : \boldsymbol{V}_h \to \boldsymbol{R}_h$ by specifying

$$\frac{1}{|\Gamma|} \int_\Gamma \Pi_h^R \boldsymbol{z}_h \, dS(x) = \frac{1}{|\Gamma|} \int_\Gamma \{\boldsymbol{z}_h\} \, dS(x), \qquad \forall \Gamma \in \Gamma_h^I,$$

where $\{\cdot\}$ denotes the average of the traces from the two sides of Γ. According to Brenner [2], we have the following error estimate:

$$\left\| \boldsymbol{z}_h - \Pi_h^R \boldsymbol{z}_h \right\|_{\boldsymbol{L}^2(\Omega)}^2 \le C \left(\sum_{E \in E_h} h^2 \|D\boldsymbol{z}_h\|_{\boldsymbol{L}^2(\Omega)}^2 + \sum_{\Gamma \in \Gamma_h^I} h^{2-N} \left| \int_\Gamma [\boldsymbol{z}_h] \, dS(x) \right|^2 \right). \tag{A.2}$$

By a standard decomposition of vector fields,

$$[\boldsymbol{z}_h]_\Gamma = [(\boldsymbol{z}_h \cdot \nu)\nu]_\Gamma - [(\boldsymbol{z}_h \times \nu) \times \nu]_\Gamma = -[(\boldsymbol{z}_h \times \nu) \times \nu]_\Gamma,$$

where the last equality follows for the reason that $[\boldsymbol{z}_h \cdot \nu]_\Gamma = 0$ for all $\Gamma \in \Gamma_h^I$. To have the above decomposition well–defined in two dimensions, we set $(\boldsymbol{z}_h \times \nu) \times \nu = -(\boldsymbol{z}_h \times \nu)\tau$, where τ is the tangential vector.

Since ν is constant on each $\Gamma \in \Gamma_h$ with $|\nu| = 1$,

$$\left| \int_\Gamma [\boldsymbol{z}_h] \, dS(x) \right|^2 = \left| \left(\int_\Gamma [\boldsymbol{z}_h \times \nu] \, dS(x) \right) \times \nu \right|^2 \le \left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \, dS(x) \right|^2, \quad \forall \Gamma \in \Gamma_h^I.$$

As a result, applying (A.11) of Lemma A.2 below to (A.2) yields the error estimate

$$\left\| \boldsymbol{z}_h - \Pi_h^R \boldsymbol{z}_h \right\|_{\boldsymbol{L}^2(\Omega)}^2 \le C \left(\sum_{E \in E_h} h^2 \|D\boldsymbol{z}_h\|_{\boldsymbol{L}^2(\Omega)}^2 + h^{\frac{4-N}{2}} \|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)}^2 \right). \tag{A.3}$$

To continue, fix an arbitrary $\xi \in \mathbb{R}^N$. By the triangle inequality, we write

$$
\begin{aligned}
\|\boldsymbol{z}_h(\cdot) &- \boldsymbol{z}_h(\cdot - \xi)\|_{L^2(\Omega_\xi)}^2 \\
&\leq C \Big(\left\| \boldsymbol{z}_h(\cdot) - \Pi_h^R \boldsymbol{z}_h(\cdot) \right\|_{L^2(\Omega_\xi)}^2 \\
&\qquad + \left\| \Pi_h^R \boldsymbol{z}_h(\cdot) - \Pi_h^R \boldsymbol{z}_h(\cdot - \xi) \right\|_{L^2(\Omega_\xi)}^2 \\
&\qquad\qquad + \left\| \Pi_h^R \boldsymbol{z}_h(\cdot - \xi) - \boldsymbol{z}_h(\cdot - \xi) \right\|_{L^2(\Omega_\xi)}^2 \Big),
\end{aligned}
\tag{A.4}
$$

which transfers the translation onto the projected function $\Pi_h^R \boldsymbol{z}$.

Since $\Pi_h^R \boldsymbol{z}_h$ is a function in the Crouzeix–Raviart element space, Stummel's work [20, Theorem 2.1] can be applied, yielding

$$
\begin{aligned}
\left\| \Pi_h^R \boldsymbol{z}_h(\cdot) \right. &\left. - \Pi_h^R \boldsymbol{z}_h(\cdot - \xi) \right\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 \\
&\leq C \left(h^2 + |\xi|^2 \right) \sum_{E \in E_h} \left\| D\Pi_h^R \boldsymbol{z}_h \right\|_{\boldsymbol{L}^2(E)}^2 \\
&\leq C \left(h^2 + |\xi|^2 \right) \sum_{E \in E_h} \left\| D\boldsymbol{z}_h \right\|_{\boldsymbol{L}^2(E)}^2,
\end{aligned}
\tag{A.5}
$$

where the constant C only depends on Ω and the shape regularity of E_h.

Utilizing (A.3) and (A.5) in (A.4) gives

$$
\begin{aligned}
\|\boldsymbol{z}_h(\cdot) &- \boldsymbol{z}_h(\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 \\
&\leq C \left(h^2 + |\xi|^2 \right) \sum_{E \in E_h} \left\| D\boldsymbol{z}_h \right\|_{\boldsymbol{L}^2(E)}^2 + h^{\frac{4-N}{2}} \left\| \operatorname{div} \boldsymbol{z}_h \right\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Since $|D\boldsymbol{z}_h| = |\operatorname{div} \boldsymbol{z}_h|$ on each $E \in E_h$, this immediately yields

$$
\|\boldsymbol{z}_h(\cdot) - \boldsymbol{z}_h(\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega_\xi)} \leq C \left(h^{\frac{4-N}{2}} + h^2 + |\xi|^2 \right)^{\frac{1}{2}} \left\| \operatorname{div} \boldsymbol{z}_h \right\|_{L^2(\Omega)},
\tag{A.6}
$$

Finally, let us argue that (A.6) implies (A.1). To this end, fix any $\xi \in \mathbb{R}^N$ and $h > 0$. There exists a shape regular partition G_h of Ω into triangles/tetrahedrals such that $\cup_{E \in G_h} E = \cup_{E \in E_h} E$, each $E \in G_h$ has a non–empty intersection with at most one element $E \in E_h$, and such that

$$
\max_{E \in G_h} \operatorname{diam}(E) < \frac{|\xi|}{3}.
$$

Next, let $\boldsymbol{V}_{|\xi|}(\Omega)$ denote the first order div conforming Nédélec element space of first kind formed on the mesh G_h, and let $\Pi_{|\xi|}^V : \boldsymbol{V}_h(\Omega) \to \boldsymbol{V}_{|\xi|}(\Omega)$ denote the usual projection into this space.

Now, for any $\boldsymbol{z}_h \in \boldsymbol{V}_h^{0,\perp}(\Omega)$, we calculate

$$
\begin{aligned}
\|\boldsymbol{z}_h(\cdot) &- \boldsymbol{z}_h(\cdot - \xi)\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 \\
&\leq \left\| \boldsymbol{z}_h(\cdot) - \Pi_{|\xi|}^V \boldsymbol{z}_h(\cdot) \right\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 + \left\| \Pi_{|\xi|}^V \boldsymbol{z}_h(\cdot) - \Pi_{|\xi|}^V \boldsymbol{z}_h(\cdot - \xi) \right\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 \\
&\qquad + \left\| \Pi_{|\xi|}^V \boldsymbol{z}_h(\cdot - \xi) - \boldsymbol{z}_h(\cdot - \xi) \right\|_{\boldsymbol{L}^2(\Omega_\xi)}^2.
\end{aligned}
\tag{A.7}
$$

By Theorem A.1,

$$
\left\| \Pi_{|\xi|}^V \boldsymbol{z}_h(\cdot) - \Pi_{|\xi|}^V \boldsymbol{z}_h(\cdot - \xi) \right\|_{\boldsymbol{L}^2(\Omega_\xi)}^2 \leq C \left(|\xi|^{\frac{4-N}{2}} + |\xi|^2 \right)^{\frac{1}{2}} \left\| \operatorname{div} \boldsymbol{z}_h \right\|_{L^2(\Omega)}.
\tag{A.8}
$$

By Lemma 2.9,

$$\left\| \boldsymbol{z}_h(\cdot) - \Pi^V_{|\xi|} \boldsymbol{z}_h(\cdot) \right\|^2_{\boldsymbol{L}^2(\Omega_\xi)} \leq C \left| \xi \right|^2 \sum_{E \in E_h} \| D\boldsymbol{z}_h \|^2_{\boldsymbol{L}^2(\Omega)} = C \left| \xi \right|^2 \| \operatorname{div} \boldsymbol{z}_h \|^2_{L^2(\Omega)} .$$

(A.9)

Inserting (A.8) and (A.9) into (A.7) completes the proof of (A.1). □

A.2. **Tangential jumps.** In the proof of Theorem A.1 we harnessed

Lemma A.2. *Given* $\boldsymbol{z}_h \in \boldsymbol{V}^{0,\perp}_h$, *there exists a constant* $C > 0$, *independent of* h, *such that*

$$\left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \, dS(x) \right| \leq C h^{\frac{N}{2}} \| \operatorname{div} \boldsymbol{z}_h \|_{L^2(\Omega)} , \qquad \forall \Gamma \in \Gamma_h, \qquad \text{(A.10)}$$

and

$$\sum_{\Gamma \in \Gamma^I_h} \left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \, dS(x) \right|^2 \leq C \left| \Omega \right|^{\frac{1}{2}} h^{\frac{N}{2}} \| \operatorname{div} \boldsymbol{z}_h \|^2_{L^2(\Omega)} . \qquad \text{(A.11)}$$

Proof. Let $\boldsymbol{\phi} \in \boldsymbol{W}^{1,2}_0(\Omega)$. In virtue of Lemma A.5 below,

$$\left| \int_\Omega \boldsymbol{z}_h \operatorname{curl} \boldsymbol{\phi} \, dx \right| \leq C h \| \boldsymbol{\phi} \|_{\boldsymbol{W}^{1,2}(\Omega)} \| \operatorname{div} \boldsymbol{z}_h \|_{L^2(\Omega)} .$$

Applying integration by parts (2.4), keeping in mind that $\operatorname{curl} \boldsymbol{z}_h |_E = 0$ for all $E \in E_h$, yields

$$\sum_{\Gamma \in \Gamma^I_h} \int_\Gamma \boldsymbol{\phi} [\boldsymbol{z}_h \times \nu] \, dS(x) = \int_\Omega \boldsymbol{z}_h \operatorname{curl} \boldsymbol{\phi} \, dx,$$

and so

$$\left| \sum_{\Gamma \in \Gamma^I_h} \int_\Gamma \boldsymbol{\phi} [\boldsymbol{z}_h \times \nu] \, dS(x) \right| \leq C h \| \boldsymbol{\phi} \|_{\boldsymbol{W}^{1,2}(\Omega)} \| \operatorname{div} \boldsymbol{z}_h \|_{L^2(\Omega)} . \qquad \text{(A.12)}$$

The bound (A.12) serves as the starting point for proving (A.10) and (A.11); the remaining objective is to construct a suitable test function $\boldsymbol{\phi}$. Fix $\Gamma \in \Gamma^I_h$. Let E_-, E_+ denote the two elements in E_h sharing the egde/face Γ, where E_-, E_+ are chosen so that ν points from E_- to E_+. In view of Lemma A.3, we can choose a continuous piecewise linear (scalar) function $\tilde{\phi}$ on Γ such that

$$\int_\Gamma \tilde{\phi} f \, dx = \frac{1}{N} \int_\Gamma f \, dx, \qquad \forall f \in \mathbb{P}_1(\Gamma).$$

Denote by $\phi_{\partial E}$ the extension by zero of $\tilde{\phi}$ to $(\partial E_- \setminus \Gamma) \bigcup (\partial E_+ \setminus \Gamma)$, and fix a piecewise affine function ϕ_E on $E_- \cup E_+$ such that $\phi_E \big|_{\partial E_+ \cup \partial E_-} = \phi_{\partial E}$. Clearly, ϕ_E can be chosen such that

$$|D\phi_E| \leq C h^{-1} \text{ in the interior of } E_- \cup E_+.$$

Finally, let ϕ_Γ denote the extension by zero of ϕ_E to all of Ω.

The function ϕ_Γ possesses the following properties: $\phi_\Gamma \in W^{1,2}_0(\Omega)$, $\phi_\Gamma \big|_{\tilde{\Gamma}} = 0$ for all $\tilde{\Gamma} \in \Gamma_h$ such that $\tilde{\Gamma} \neq \Gamma$, and

$$\| \phi_\Gamma \|^2_{W^{1,2}(\Omega)} = \left\| \phi_\Gamma \big|_{E_-} \right\|^2_{W^{1,2}(E_-)} + \left\| \phi_\Gamma \big|_{E_+} \right\|^2_{W^{1,2}(E_+)} \leq C h^N \left(1 + h^{-2} \right).$$

If $N = 2$ (curl is scalar), then we opt for $\boldsymbol{\phi} = \phi_\Gamma$ in (A.12) to obtain

$$\left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \, dS(x) \right| \leq C h^{\frac{N}{2}} \| \operatorname{div} \boldsymbol{z}_h \|_{L^2(\Omega)} . \qquad \text{(A.13)}$$

If $N = 3$, (A.13) still holds. Indeed, to conclude we can in (A.12) successively take $\boldsymbol{\phi} = [\phi_\Gamma, 0, 0]^T$, $\boldsymbol{\phi} = [0, \phi_\Gamma, 0]^T$, and $\boldsymbol{\phi} = [0, 0, \phi_\Gamma]^T$. Since $\Gamma \in \Gamma_h$ was arbitrary, this concludes the proof of (A.10).

To establish (A.11), we introduce the test function

$$\boldsymbol{\phi}_\Gamma = \phi_\Gamma \int_\Gamma [\boldsymbol{z}_h \times \nu] \ dS(x), \qquad \forall \Gamma \in \Gamma_h^I,$$

where ϕ_Γ is constructed as above with the additional requirement that

$$\operatorname{supp} \phi_\Gamma \bigcap \operatorname{supp} \phi_{\tilde{\Gamma}} = \emptyset, \qquad \forall \tilde{\Gamma} \neq \Gamma.$$

We have

$$\sup_{x \in \Omega} |D\boldsymbol{\phi}_\Gamma| \leq Ch^{-1} \left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \ dS(x) \right|, \qquad \forall \Gamma \in \Gamma_h^I,$$

and, for each $\Gamma \in \Gamma_h$,

$$\int_\Gamma \boldsymbol{\phi}_\Gamma f \ dS(x) = \frac{1}{N} \left(\int_\Gamma [\boldsymbol{z}_h \times \nu] \ dS(x) \right) \left(\int_\Gamma f \ dS(x) \right), \qquad \forall f \in \mathbb{P}^1(\Gamma).$$

Finally, we set $\boldsymbol{\phi} := \sum_{\Gamma \in \Gamma_h^I} \boldsymbol{\phi}_\Gamma$; this function satisfies $\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,2}(\Omega)$ and

$$\sup_{x \in \Omega} |D\boldsymbol{\phi}| \leq Ch^{-1} \max_{\Gamma \in \Gamma_h^I} \left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \ dS(x) \right| \leq Ch^{\frac{N-2}{2}} \|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)}.$$

The last inequality follows from (A.10). A direct calculation gives

$$\|D\boldsymbol{\phi}\|_{\boldsymbol{L}^2(\Omega)} \leq Ch^{\frac{N-2}{2}} |\Omega|^{\frac{1}{2}} \|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)}.$$

Setting $\boldsymbol{\phi}$ as test function in (A.12) immediately gives the estimate

$$\sum_{\Gamma \in \Gamma_h^I} \left| \int_\Gamma [\boldsymbol{z}_h \times \nu] \ dS(x) \right|^2 \leq Ch^{\frac{N}{2}} |\Omega|^{\frac{1}{2}} \|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)}^2,$$

which is (A.11). $\qquad \square$

The next lemma provides us with the specific test function that was brought into service in the above proof.

Lemma A.3. *Fix any $\Gamma \in \Gamma_h$. There exists a continuous piecewise linear (scalar) function ϕ on Γ such that $\phi|_{\partial \Gamma} = 0$, $|\phi(x)| \leq 1 \ \forall x \in \Gamma$, and*

$$\int_\Gamma \phi f \ dx = \frac{1}{N} \int_\Gamma f \ dx, \qquad \forall f \in \mathbb{P}^1(\Gamma), \tag{A.14}$$

where N is the spatial dimension.

Proof. Let b denote the barycentric middle point with respect to the vertices of Γ. Let T_h be the triangulation of Γ obtained by setting b as a vertex in addition to the vertices of Γ. On T_h let $L_h(\Gamma)$ denote the standard finite element space of continuous piecewise linear functions. Any function $\phi_h \in L_h(\Gamma)$ is uniquely determined by it's value at the vertices.

Now the relevant test function $\phi \in L_h(\Gamma)$ is obtained by requiring

$$\phi(b) = 1 \text{ and } \phi(v_i) = 0, \text{ for all vertices } v_i \text{ at } \partial \Omega.$$

By direct calculation it can be verified that ϕ satisfies (A.14). $\qquad \square$

A.3. **Negative space compactness of the curl.** In the proof of Lemma A.2, the essential ingredient was an estimate on the $W^{-1,2}$ norm of $\operatorname{curl} \boldsymbol{z}_h$. In this subsection, we prove this result.

Theorem A.4. *Consider the mixed Laplace-type problem*

$$\begin{aligned} \operatorname{curl} \boldsymbol{w} - D \operatorname{div} \boldsymbol{u} = \boldsymbol{f}, \ \boldsymbol{w} = \operatorname{curl} \boldsymbol{u} \quad in \ \Omega, \\ \boldsymbol{u} \cdot \nu = 0, \ \boldsymbol{w} \times \nu = 0 \quad on \ \partial\Omega, \end{aligned} \tag{A.15}$$

where we assume $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$. There exists a pair

$$(\boldsymbol{w}, \boldsymbol{u}) \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega) \times \boldsymbol{W}_0^{\mathrm{div},2}(\Omega),$$

satisfying (A.15) *in the weak sense. Moreover, there exists a pair*

$$(\boldsymbol{w}_h, \boldsymbol{u}_h) \in \boldsymbol{W}_h(\Omega) \times \boldsymbol{V}_h(\Omega),$$

satisfying the corresponding mixed finite element formulation of (A.15). *Finally, the following error estimate holds:*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{V}_h} \leq Ch^s \|\boldsymbol{f}\|_{\boldsymbol{L}^2(\Omega)}, \tag{A.16}$$

where the convergence rate $s \in [1/2, 1)$ depends on the regularity of $\partial\Omega$. If $\partial\Omega$ is Lipschitz and convex, (A.16) *holds with $s = 1$.*

Proof. For example, cf. Theorem 7.9 in [1]. $\qquad\square$

Lemma A.5. *Let $\{\boldsymbol{z}_h\}_{h>0}$ be a sequence in $\boldsymbol{V}_h^{0,\perp}$ for which $\|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)} \leq C$, where the constant $C > 0$ is independent of h. Then*

$$\|\operatorname{curl} \boldsymbol{z}_h\|_{\boldsymbol{W}^{-1,2}(\Omega)} \leq Ch \|\operatorname{div} \boldsymbol{z}_h\|_{L^2(\Omega)},$$

for some constant C independent of h.

Proof. To prove this lemma, we will use the mixed system (A.15) to define a new operator. To motivate the construction, consider the problem

$$-\Delta\boldsymbol{\theta} = \operatorname{curl} \boldsymbol{\phi} \quad in \ \Omega, \quad \boldsymbol{\theta} \cdot \nu = 0, \ \operatorname{curl} \boldsymbol{\theta} \times \nu = 0 \quad on \ \partial\Omega, \tag{A.17}$$

for some given $\boldsymbol{\phi} \in \boldsymbol{W}_0^{\mathrm{curl},2}(\Omega)$. By utilizing $D\Delta^{-1} \operatorname{div} \boldsymbol{\theta}$ as test function in the weak formulation of (A.17), where Δ^{-1} is the Neuman Laplace inverse, it is easily seen that the weak solution $\boldsymbol{\theta}$ of the system (A.17) is divergence free. Furthermore, we can set $\boldsymbol{w} = \operatorname{curl} \boldsymbol{\theta}$ and integrate by parts to conclude that the pair $(\boldsymbol{w}, \boldsymbol{\theta})$ is also the unique weak solution of the mixed Laplace system (A.15) with $\boldsymbol{f} = \operatorname{curl} \boldsymbol{\phi}$.

Now we define a new operator $\Pi^h : \boldsymbol{W}_0^{\mathrm{curl},2} \to \boldsymbol{W}_h$ as the unique function $\Pi^h\boldsymbol{\phi} \in \boldsymbol{W}_h$ satisfying the finite element formulation:

$$\begin{aligned} \int_\Omega \operatorname{curl}(\Pi^h\boldsymbol{\phi})\boldsymbol{v}_h + \operatorname{div} \boldsymbol{\theta}_h \operatorname{div} \boldsymbol{v}_h \ dx = \int_\Omega \operatorname{curl} \boldsymbol{\phi}\boldsymbol{v}_h \ dx, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \int_\Omega \operatorname{curl} \boldsymbol{\eta}_h \boldsymbol{\theta}_h \ dx = \int_\Omega (\Pi^h\boldsymbol{\phi})\boldsymbol{\eta}_h \ dx, \quad \forall \boldsymbol{\eta}_h \in \boldsymbol{W}_h. \end{aligned} \tag{A.18}$$

The existence of such a function $\Pi^h\boldsymbol{\phi}$ is given by Theorem A.4. Using the fact that $\operatorname{div} \boldsymbol{\theta} = 0$, the error estimate (A.16) yields

$$\left\|\Pi^h\boldsymbol{\phi} - \operatorname{curl} \boldsymbol{\theta}\right\|_{\boldsymbol{L}^2(\Omega)} + \|\boldsymbol{\theta}_h - \boldsymbol{\theta}\|_{\boldsymbol{L}^2(\Omega)} + \|\operatorname{div} \boldsymbol{\theta}_h\|_{L^2(\Omega)} \leq Ch^s \|\operatorname{curl} \boldsymbol{\phi}\|_{\boldsymbol{L}^2(\Omega)}. \tag{A.19}$$

Let $\boldsymbol{z}_h$ be as stated in the lemma. Since $\boldsymbol{z}_h$ is orthogonal to functions in $\boldsymbol{W}_h$,

$$
\begin{aligned}
\|\operatorname{curl} \boldsymbol{z}_h\|_{\boldsymbol{W}^{-1,2}(\Omega)} &= \sup_{\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,2}} \frac{\left|\int_\Omega \boldsymbol{z}_h \operatorname{curl} \boldsymbol{\phi}\, dx\right|}{\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1,2}(\Omega)}} \\
&= \sup_{\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,2}} \frac{\left|\int_\Omega \boldsymbol{z}_h \operatorname{curl}(\boldsymbol{\phi} - \Pi^h \boldsymbol{\phi})\, dx\right|}{\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1,2}(\Omega)}} \\
&\overset{(\text{A.18})}{=} \sup_{\boldsymbol{\phi} \in \boldsymbol{W}_0^{1,2}} \frac{\left|\int_\Omega \operatorname{div} \boldsymbol{z}_h \operatorname{div} \boldsymbol{\theta}_h\, dx\right|}{\|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1,2}(\Omega)}} \\
&\leq Ch \left\|\operatorname{div} \boldsymbol{z}_h\right\|_{L^2(\Omega)},
\end{aligned}
$$

where we have used (A.19) to derive the last inequality, specifically the estimate

$$
\|\operatorname{div} \boldsymbol{\theta}_h\|_{L^2(\Omega)} \leq Ch \|\operatorname{curl} \boldsymbol{\phi}\|_{L^2(\Omega)} \leq Ch \|\boldsymbol{\phi}\|_{\boldsymbol{W}^{1,2}(\Omega)}.
$$

This concludes the proof. $\qquad\square$

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