An Energy-Momentum Interaction of Fluid Elements

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Abstract

This macroscopic approach is based on the conception of fluid elements which represent (mass)-points and (point)-masses at the same time. As individual point the fluid element owns a momentum and as collective domain a potential of energy. This dualistic view has the advantage that the basic equations of fluid mechanics can be simplified. If the system is reduced to a balance of momentum and energy the continuity equation does not explicitly become part of it. Then, the number of dynamic variables in the classical equations will be reduced to four: the momentum vector and a quantity of energy leading to a kinematic representation in the end. In case of incompressible flows agreement is reached with the Navier-Stokes equations. Computational results concern unsteady three-dimensional flows, particularly showing the formation of Taylor vortices in a turning cylinder and of spiral vortices behind a backwards-facing step in a channel flow. As a remarkable result of this approach a deeper insight into the phenomenon of turbulence could be achieved.

1. Introduction

The conservation laws of fluid mechanics can be derived in integral or differential form. The former assumes constancy of mass of identifiable particles inside a finite, time-dependent fluid volume being transported with the flow, while the differential form applies to a fixed volume passed through. Each modus operandi leads to the continuity equation concerning conservation of

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mass, where the integral definition corresponds to a concept of the continuum that consists of basic particles not being connected without gap.

However, both definitions of the continuum - the "granular" interpretation and its literal meaning "continuous" - are deficient. Thus, in a strictly continuous domain no border will be definable. No sub-domain may be cut out of a "cloud" which will endure with respect to time. But even the assumption of a fluid element which consists of identifiable material particles is insufficient. A cohesion of these particles, i.e. a spatiotemporal closed system, do not exist normally - particularly not in a whirling compressible flow. For instance, Howard Brenner [1] realized that "because individual molecules freely enter and leave such a body through its permeable surface due to their respective "Brownian" motions, a material fluid particle is not permanently composed of the same matter". However, Brenner is not going thus far to deny the relevance of constancy of mass in the context of the basic equations. Consequently Brenner in his comprehensive publication has to postulate two different kinds of velocity in compressible flows: the *mass velocity*, appearing in the continuity equation, and the *momentum velocity* of the Navier-Stokes Equations. By the way, an indication of this problem may be found in the "Hydrodynamics" of Landau-Lifschitz [2]. As will be seen, this problem of two velocities disappears when the condition of constancy of mass of a fluid element becomes irrelevant.

In spite of such objections the construct *fluid element* cannot be relinquished. This holds for visualizations (e.g. colored volumes) and in the same degree for numerical simulations. Therefore the only option to agree with physical reality would be to adopt a *duality of the fluid element as particle and field* of user-defined spatiotemporal dimensions. As individual point it owns a momentum and as collective domain a potential of energy whichever origin. On the one hand the physics of trajectories (Euler, Navier-Stokes) does apply and on the other hand - below a limit of resolution - the thermodynamic state. Both classifications are coupled in the fluid element and accordingly the basic equations. Though subliminally any modeling of dissipative phenomena in detail is impossible and could be considered last but not least as *reductionist* these phenomena are implicitly involved by the macroscopic interaction of convection, friction and diffusion. In this view only a physical effect like pressure on the surrounding fluid elements is detectable, inevitably scaled down to a scalar quantity as necessary and sufficient. Then in respect of the dualistic nature of fluid elements the number of dynamic variables in the classical equations is reduced to four: the momentum vector and a quantity of energy. Because these terms connect mass with velocity they are more universal than the concept "density of mass". Consequently the fields of momentum and energy already determine the distribution of mass depending on initial conditions and temporally accumulated divergences of velocities. Altogether, mass can be considered as "passive" and only obtains dynamic quality as momentum. As far as the basic equations are concerned then the variable *density of mass has to be replaced by the individual fluid element* which interacts with its neighbors. Furthermore, because mass is only involved linearly both in momentum and energy, these quantities may be reduced to kinematic variables.

After all a mathematical aspect should be mentioned. Though for reasons of clearness the fluid element has been considered as spatiotemporal *finite*, the basic equations should be regarded as differential equations and not as relations of differences. Because starting out from the insight that it is not decidable whether infinite decimal numbers are zero or not, the existence of infinitely small elements is ensured (s.[3]). By the way an analogy to chemical reaction kinetics may be mentioned. Reaction rates and concentration gradients are alike acceleration potentials of fluid mechanics, where the flux of forces also is essential. Without going into detail these analogies may be extended on the production of entropy in fluid flows, steady or unsteady, and near or far from equilibrium. In the latter case a negative entropy production is possible leading to instabilities like turbulence.

The present approach has a previous history. Already in 1983 a mathematical model was proposed in order to simplify numerical calculations concerning fluid flows of high Reynolds-numbers [4]. Though the applied equations led to remarkable results in qualitative as well as in quantitative respect (e.g. [5]), the physical background left questions open. In literature three publications were concerned with this model: A comparison with the Navier-Stokes-Equations using symmetry methods [6], an alternative mathematical approach to non-equilibrium phenomena [7] and a doctoral thesis [8].

2. Basic Equations

In the kinetic theory of gases the number of elementary particles (m_e) $n \times m_e$ per unit of volume (V) substitutes *density of mass*. Then the (internal) kinetic energy of a fluid element is determined by the product *number of particles* × *average value of the square of microscopic velocities*, identical to the sum of these squared terms. In this context it has to be kept in mind that it is irrelevant whether there are *many particles of low speed or a few particles of high speed*. Thus the density of the (internal) kinetic energy of a fluid element is given by $e_f = \frac{m_e}{V} \sum_{i=1}^{n} \frac{q_i^2}{2} = \frac{m_e}{V} \times \phi$, where the positive quantity ϕ does not include a *detailed* knowledge of microscopic processes. The dimension of $\phi = \sum_{i=1}^{n} \frac{q_i^2}{2}$ is (length/time)^2 , while the dimension of mass density does not occur in ϕ. Which modes of energy are actually present may remain open at first, whether identified as internal energy, pressure or thermodynamic potential, and depends last not least on the fluid and flow parameters. Thus A. Sommerfeld in his classical textbook [9] denotes fluid pressure as "quasi potential energy of forces being effective within the fluid element".

Based on the assumption that fluid elements will cover the total subliminal energy of the flow field, the universal law of conservation is valid concerning the term $\frac{m_e}{V}\,\phi$. Then, eliminating $\frac{m_e}{V}$ the conservation equation reads in differential form:

$$\frac{\partial \phi}{\partial t} \; + \; \text{div}\, \vec{\mathbf{J}} \; = \; 0 \tag{1}$$

where the energy flux $\vec{\mathbf{J}}$ is composed of a convective and a "non-convective" (diffusion) fraction. If the latter is assumed to be proportional to the gradient of ϕ itself - comparable to the term of heat conduction in the hydrodynamic energy equation, eq.(1) becomes:

$$\boxed{\frac{\partial \phi}{\partial t} \; + \; \text{div}\, (\phi \vec{v}) \; = \; \epsilon \, \Delta \phi} \qquad \epsilon \; = \text{kinematic coefficient of substance} \tag{2}$$

The kinetic energy of a fluid element as an entirety is on the contrary not subject to a conservation law. This results directly from the dual character of the fluid element as particle and field. In case of single component fluids the term $\frac{m_e}{V}$ may be eliminated as multiplication factor. Moreover, formula

(2) corresponds to the well known relations which describe transport of heat in the continuum if convection is neglected. These are:

$$\frac{\partial \mathrm{q}}{\partial t} = -\,\mathrm{div}\,\vec{j} \qquad \text{and} \quad \vec{j} = -\kappa\,\mathrm{grad}\,\mathrm{T} \tag{3}$$

(q = heat quantity, κ = heat conduction, T = temperature)

Eq. (2) is different from the hydrodynamic energy equation in so far as the dissipation of energy of motion into internal energy (heat) does not explicitly appear. However, implicitly the powers of friction and pressure forces are taken into account by means of the interaction of fluid elements on the macroscopic level. Thus the relevant terms of the equation of momentum influence for their part convection and diffusion of potential energy. For instance, a small loss of energy by diffusion will mean an increase of irreversible energy production provided that transport by convection is unchanged. Consequently additional terms of power are not required as in the case of the much more complicate hydrodynamic energy equation with three scalar variables, e.g. pressure, temperature and density, instead of one.

Furthermore, in order to get a closed system of equations an equation of motion is required based on the Navier-Stokes equations. However, it has to be taken into account that this approach replaces the continuous mass density ρ by single fluid elements. Accordingly the fluid element *represents one pseudo-mass point* in the macroscopic flow field. Disregarding gravity effects the momentum equation then reads:

$$\rho\,\frac{D}{Dt}\vec{v} = -\,\mathrm{grad}\,\mathrm{p} + \rho\,\vec{R}$$

$$\text{passing into :} \quad \frac{m_e n}{\mathrm{V}}\,\frac{D}{Dt}\vec{v} = -\,\mathrm{grad}\frac{m_e}{\mathrm{V}}\phi + \frac{m_e n}{\mathrm{V}}\vec{R}$$

With the number of particles reduced to one (n=1) and the constant term m_e/V eliminated we get:

$$\boxed{\frac{D}{Dt}\vec{v} = -\,\mathrm{grad}\,\phi + \vec{R}} \qquad \vec{R} = \text{friction} \tag{4}$$

where $\frac{m_e}{\mathrm{V}}\phi$ represents pressure as assumed and ϕ its kinematic reduction.

For reasons of symmetry the mathematical approach [4] only considered bulk viscosity, a restriction that will not be maintained here. A universally valid notation of friction terms reads - with kinematic coefficients of substance α/β

$$\vec{R} \;=\; \alpha \text{ grad div } \vec{v} \;- \beta \text{ curl curl } \vec{v} \tag{5}$$

and in the Stokes-approximation

$$\vec{R} \;=\; \nu \; (\Delta \vec{v} \;+\; \frac{1}{3} \text{ grad div } \vec{v}) \qquad \nu \;=\; \text{kinematic viscosity} \tag{6}$$

In the cause-and-effect chain these friction forces influence the energy field by the convective flux div $(\phi\vec{v})$, while the energy distribution governs the macroscopic velocity field. It is this feedback between particle and field that links mechanical work with heat production in agreement with the first law of thermodynamics. Therefore, there is no need to add the *power* of friction forces to the hydrodynamic energy equation explicitly.

3. Some flow simulations

In the following some computational results are presented concerning 3D-flow configurations, particularly with regard to a turning cylinder filled with a fluid and the flow behind a backwards facing step. Because of the complex flow configurations the calculations confine themselves to qualitative results. As characteristic numbers the Reynolds-, Prandtl- and Sommerfeld-Number come into consideration. The latter is related to the pressure loss in a channel flow. The ratio of shear to bulk viscosity is determined by the Stokes Approximation. The temperature dependency of material parameters has been disregarded as well as gravity effects. The reproducible generation of vortices and their disintegration, whether by experiment or numerical simulation, requires well-defined time dependent boundary conditions which are difficult to obtain in general.
Rather simple to realize, however, is the configuration of a turning cylinder (see e.g. [10]. Here, the start-up procedure (spin-up) differs from the spin-down procedure of the cylinder coming to rest. Fig. (1) shows the evolution of Taylor vortices in a spin-up- process (axial-radial cross section) which are azimuthal of helical structure. The calculation has been done using cylindrical coordinates applying the boundary conditions: spin-up: $\vec{v}_{shell}(t_0) = 0,\ \vec{v}_{shell}(t > t_0) > 0, \nabla\phi_{shell} = 0.$

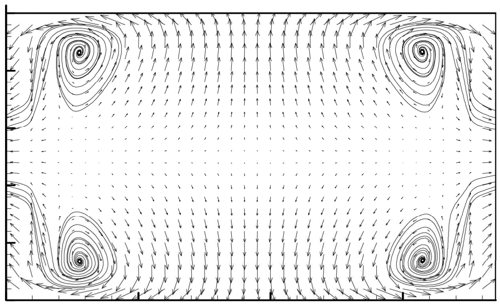

Fig.1

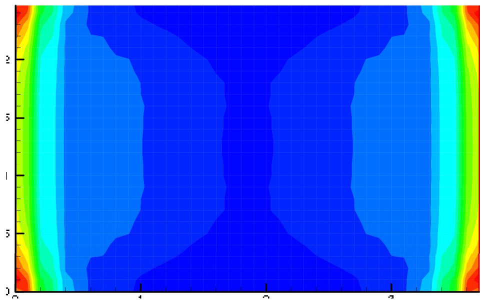

Fig.2

These vortices develop because of the frictional grip at the walls of a turning cylinder. In this context it is interesting that the Taylor vortices seem to have little influence on the pressure distribution (Fig.2). Helical vortices are presumably a very "economic" flow configuration.

The disintegration of Taylor vortices depending on increasing Reynolds Numbers (700 and 1000 respectively) concerning a Couette Flow between rotating cylinders is shown in Fig.3.

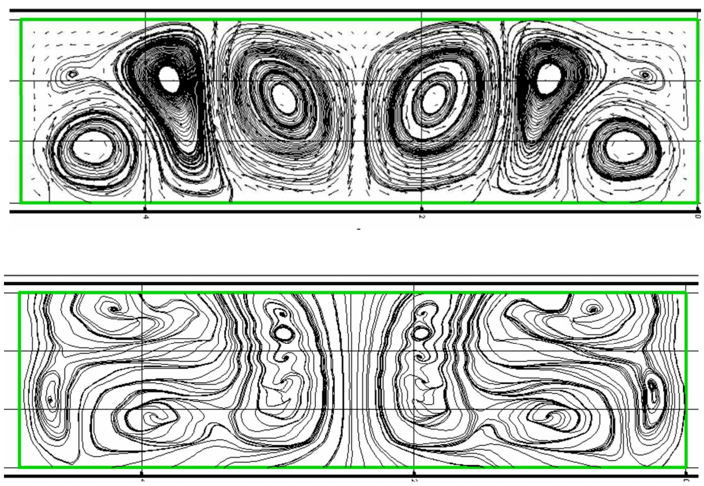

Fig.3

Less clear is the flow behind a step in a 3D-channel flow (Fig.4-6). Then, two effects will overlap: the development of vortices behind the step and on the other hand vortices caused by friction at the walls of the channel. Even here visualization shows helical vortices which, however, are more difficult to realize because they develop being interconnected in three dimensions.

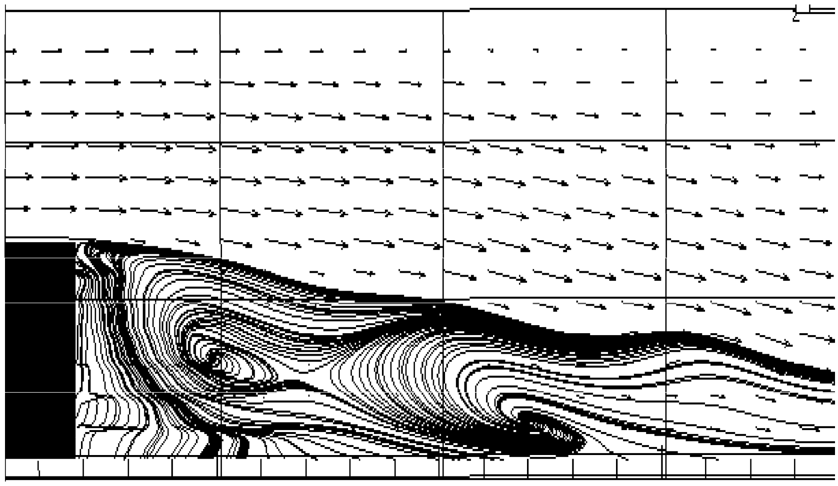

Fig.4

Local eddies melt together and extensive flow patterns disintegrate. Actually these mechanisms remind of the motion of gyroscopes which influence each other.

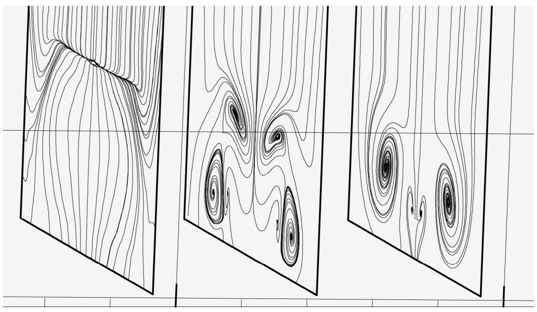

Fig.5

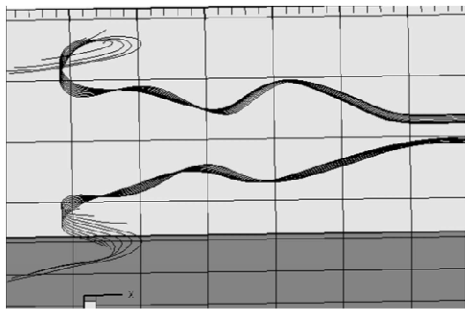

Fig.6

Fig.4/5 show vector fields and stream lines respectively in different cross sections of the channel. Finally Fig.6 illustrates a single result of such interaction leading to helical patterns in three dimensions.

4. On the Origin of Turbulence

There are two fundamental forms of motion resulting from the interaction of fields of velocity and energy: *spiral patterns* and *waves* which may be considered as the main sources of turbulence, depending on dimensionless numbers, but independent of scales of space and time. A vivid example of the development of helical vortices is given by the simulation of the flow over a vertical oriented plate (Fig.7), where the image plane is the plane of symmetry in the midst of a channel, while the distance of the plate to the walls above and below should be different. Then the mechanism of vortex separation, leading to a vortex-street in the end, can be visualized showing the phenomenon of reverse rotating vortices which will influence each other in such a way that *one eats up the other* alternately. Furthermore the *growing vortex will separate* - carried along with the outer flow. Of course, single vortices could be produced e.g. by plates of definite angles of attack.

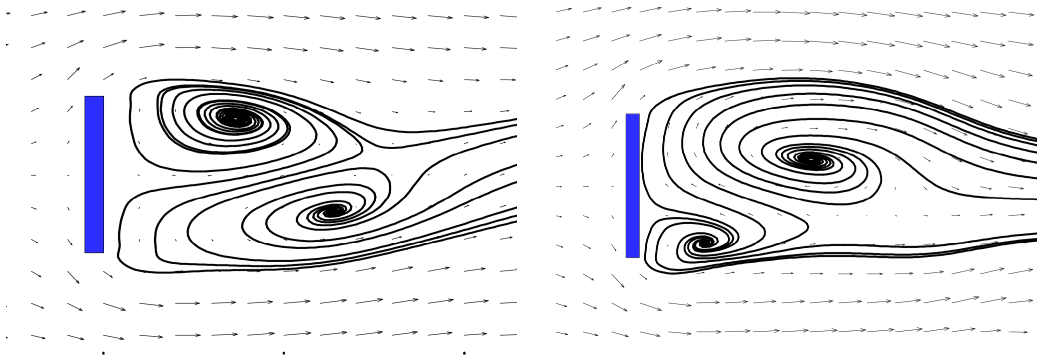

Fig.7
Closed-loop line integrals (*circulation*) will only exist in the borderline case when symmetric configurations like torus-shaped smoke rings are generated. Depending on adequate flow conditions such *vortex rings* can stick to solid surfaces or separate. In order to illustrate this two examples are given: Fig.8 shows the cross section of a simulated pipe flow around a disk (Reynolds number 100), which will not separate in steady flow - corresponding to [11]). On the other hand an unsteady shock wave is obtained starting from a high pressure gradient in the pipe. Then flow separation occurs (see Fig.9), associated with a small vortex ring which will disappear by and by (see the blue points in the low pressure phase).
In this context it has turned out that to go without the continuity equation the basic equations will avoid discontinuity surfaces. As is well known shock waves have a small but finite thickness - conditions which are met by the equations of momentum and energy which involve terms of viscosity and diffusion (see [12]). The equation of continuity only includes a convective transport of mass, so that discontinuity surfaces would require the Rankine-Hugoniot-Relations if necessary. Discontinuous distributions of uniform mass are only conceivable under outer space conditions.

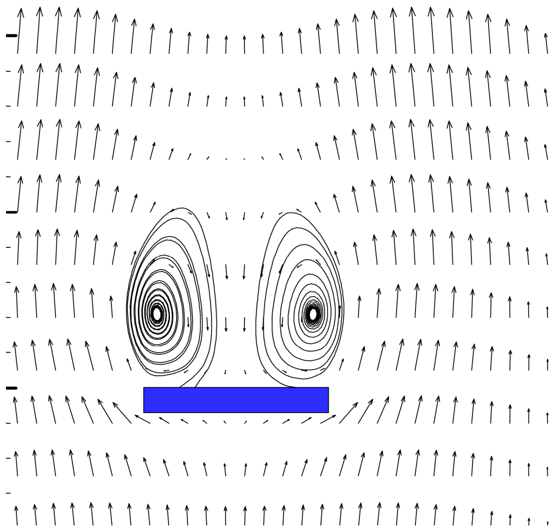

Fig.8

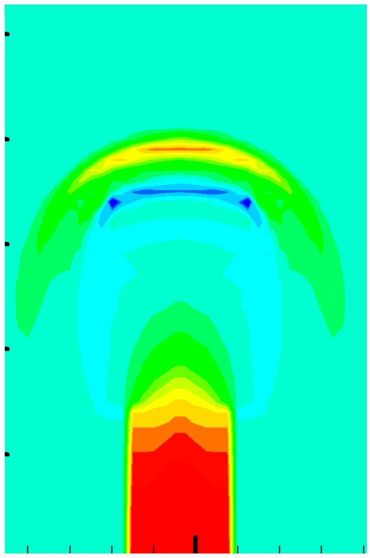

Fig.9

The influence of *waves* on complex flow configurations will essentially be due to the phenomenon of *interferences*. A simple simulation of a flow through a double-slit (Fig.10) shows the difference between the velocity field (no interference) and the energy potential (with interference) in a cross section downstream.

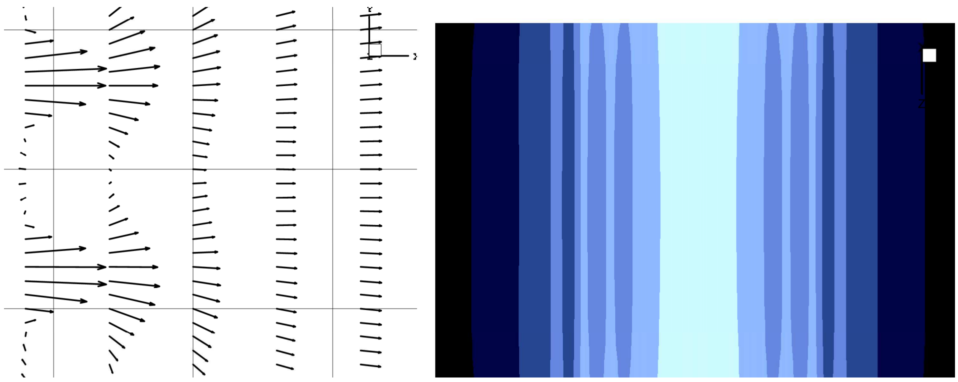

Fig.10

Whether a flow will turn out to be *laminar or turbulent* thus depends on a great variety of local and regional area interactions, flow and fluid parameters like initial- and boundary conditions and material coefficients. For this reason it seems almost impossible to predict these highly complex processes by means of *reductionist approaches* like incompressibility, plane flow or special disturbance functions which may trigger a transition to turbulence under certain conditions. Therefore, a chance to achieve a realistic forecast of unsteady compressible flows will only result from considerable computational effort. Two examples, obtained with minimal computational equipment, may provide an indication of future possibilities. So Fig.11 shows a photo of ascending smoke[1] and Fig.12 similar swirls in a channel flow behind a step. Furthermore Fig.13 reproduces *vorticity* in a boundary layer due to transverse gradients.

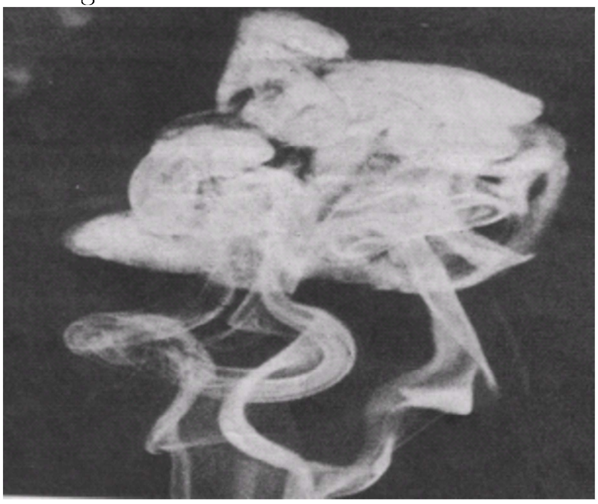

Fig.11

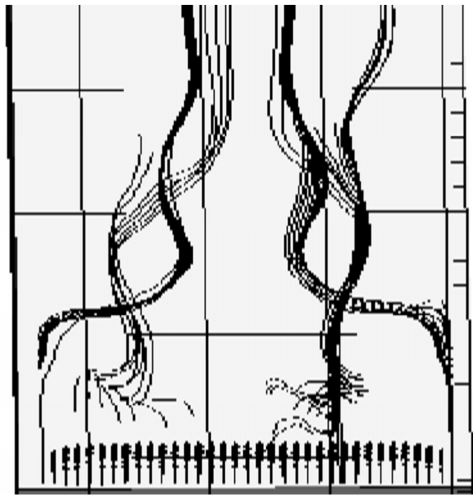

Fig.12

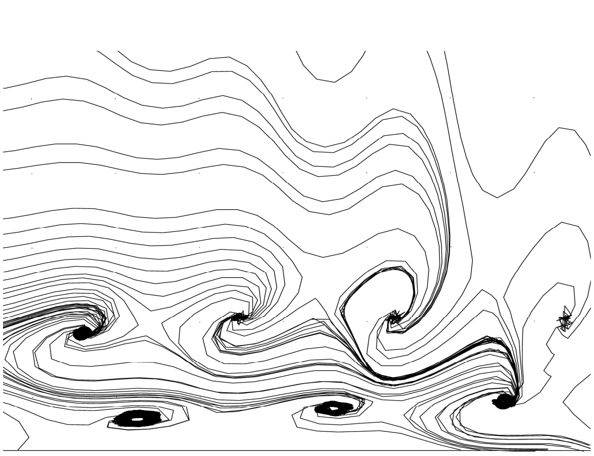

Fig.13

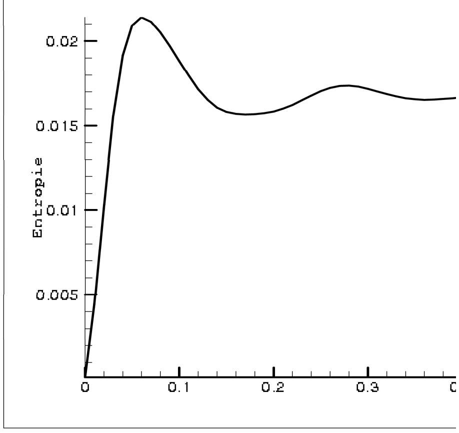

Fig.14

Finally the irreversible dynamics of fluid flows is associated with thermody-

[1] Frankfurter Allgemeine Zeitung /11.14.2010

namics and in particular the *dissipation of motion* to internal energy. In the hydrodynamic energy equation this function is described by the dissipation function which is deduced from mechanical stresses. However, as far as thermodynamics is concerned it seems more reasonable to relate the irreversible processes, i.e. the production of entropy, to the product *flux* $\times$ *force*, equivalent to *velocity* $\times$ *gradient of potential energy*. Fig.14 shows the temporal development of this product starting from a system at rest but from a non-equilibrium state of potential energy. Far from equilibrium fluctuations occur which contain negative entropy production.

5. Extension to Magnetohydrodynamics

The variable ϕ has been introduced here as the (internal) energy per fluid element - without specifying the mode of energy. The underlying cause has been that parameters of substance, boundary conditions and generally reductionistic simplifications will affect the mode of energy. Thus W.Moehring[2] identified the variable ϕ as *enthalpy* starting from the Euler equations of the isotropic flow of an ideal gas.
Consequently the potential of an electric field may be part of ϕ as well. In this case the equations of fluid mechanics are insufficient and have to be extended to magnetohydrodynamics. Accordingly a relation to the Maxwell equations of electrodynamics has to be established. In this context it is useful that electric charges are always bound to elementary particles, a situation similar to the hydrodynamic approach, given here, where continuous mass density is replaced by particle density. The coupling of the equations of fluid mechanics and electrodynamics is based on two requirements: the consideration of Lorentz forces in an equation of motion and the idea of abstracting the different kinds of electric currents, particularly the convective and inductive modes, to a more general term of energy flow. As to the latter Ohm's law indicates the current to be proportional to the electric field and hence to the gradient of an electric potential. The feedback to a potential also has the advantage that the gradient will be independent of a frame of reference. With current $\vec{i} \propto \mathrm{grad}\phi$ from the Maxwell-equations then follows:

$$-\frac{\partial \vec{E}}{\partial t} + c^2 \,\mathrm{curl}\, \hat{B} \quad = \quad \mathbf{f}\, \nabla\phi$$

[2]Max Planck Institut für Strömungsforschung, Goettingen: private communication

$$\frac{\partial \hat{B}}{\partial t} + \text{curl } \vec{E} = 0$$

Coefficient **f** has the dimension of a frequency [1/t] resulting from a non-dimensional representation with the electric field strength $\vec{E}[1/t^2]$ and the magnetic field $\hat{B}[1/t]$. This assumption also seems to be justified because the influence of the electric field may be related to microscopic plasma vibrations of definite frequency and separation of charges. Consequently the fluid flow is not effected in the case of zero frequency and vanishing Lorentz forces. Only the vector product $\vec{v} \times \hat{B}$ of the Lorentz term will explicitly be involved in the resulting equation of motion because the contribution of the electric field strength is already covered by grad ϕ.
The term c^2 in the Maxwell-equations concerns the speed of light, which - following the mass-energy relation - may be replaced by $\phi_0 = \text{const}$ as a reference value. A non-dimensional representation of the total system of differential equations, which includes the electromagnetic field, then reads:

$$\begin{aligned} \frac{D}{Dt}\vec{v} - \vec{v} \times \hat{B} &= -\text{ grad } \phi + \frac{1}{\text{Re}} \vec{R} \qquad \vec{R} = \text{friction forces} \\ \frac{\partial \phi}{\partial t} + \text{div } (\phi \vec{v}) &= \frac{1}{\text{Re Pr}} \Delta \phi \qquad (7) \\ -\frac{\partial \vec{E}}{\partial t} + \text{curl } \hat{B} &= \frac{1}{\text{NN}} \text{ grad } \phi \\ \frac{\partial \hat{B}}{\partial t} + \text{curl } \vec{E} &= 0 \end{aligned}$$

with the relations to the dimensional quantities (*)
$\phi = \phi_0\ \phi^*$; $\vec{x} = \text{L } \vec{x}^*$; $t = \frac{\text{L}}{\sqrt{\phi_0}} t^*$; $\vec{v} = \sqrt{\phi}\ \vec{v}^*$; $\vec{E} = \frac{\phi_0}{\text{L}}\ \vec{E}^*$; $\hat{B} = \frac{\sqrt{\phi_0}}{\text{L}}\ \hat{B}^*$,
the characteristic length L, the flow parameter $U_\infty = \sqrt{\phi_0}$ and the non-dimensional characteristic numbers
$\text{Re} = \frac{\text{L}\sqrt{\phi_0}}{\nu}$; $\text{RePr} = \frac{\text{L}\sqrt{\phi_0}}{\epsilon}$ and $\text{NN} = \frac{\sqrt{\phi_0}}{\mathbf{f}\ \text{L}}$.

A verification of this approach including the influence of electromagnetic fields requires considerable effort which cannot be provided here. Merely a simple numerical simulation of a starting 3D-channel flow has been done. The enclosed figures (Fig.15/16) show

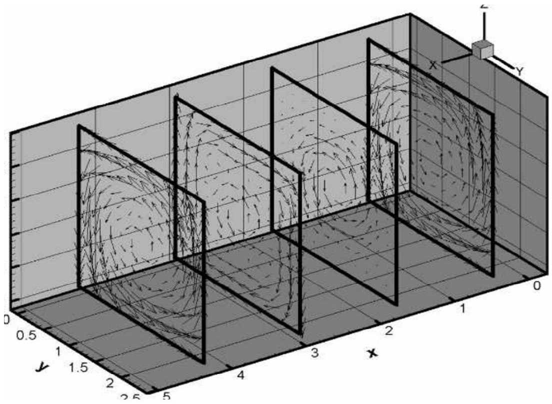

Fig.15

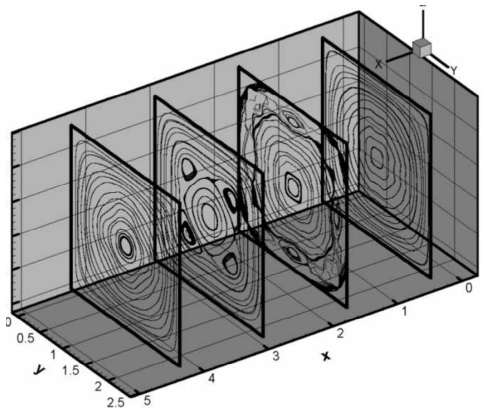

Fig.16

the magnetic field in different cross sections with the remarkable result that during the start-up period the sense of direction will change because the inflow and outflow of the conductive fluid means charge reversal at the same time. Finally, an influence of the term $\vec{v} \times \hat{B}$ on the boundary layer thickness might be verified.

6. Conclusion

The construct fluid element is inherently of dualistic nature and generally not of material entirety. It may be considered as point and collective domain at the same time. As particle the fluid element owns a *macroscopic velocity* and as a domain it represents *thermodynamic quantities*. So the energy equation does not require terms which describe the power of friction forces, because effects of these terms are definitely taken into account by the balance equation of momentum. Friction is a phenomenon between individual fluid elements, but not detectable beyond a limit of resolution.

After all, the feedback of fluid elements "*with itself*" results in the fact that the spatiotemporal dimension of an element is insignificant, as long as the dimensionless numbers as the Reynolds Number do not change. Perhaps this is the most important result of the hypothesis proposed here. Independent of the degree of resolution then flow phenomena will be *self-similar*, meaning that phenomena like turbulence do not differ qualitatively from other unsteady flow configurations.

Finally it is worth mentioning that the basic equations (2/3) take a particularly compressed and meaningful form when pressure and friction forces are introduced as tensor divergence. Then these equations read - with the tensor divergence Div, stress tensor T and convective and diffusive energy flux $\vec{J}$:

$$\frac{D\vec{v}}{dt} = + \text{Div } T$$
$$\frac{\partial\phi}{\partial t} = - \text{div } \vec{J}$$

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