A Domain Decomposition Method for Semilinear Hyperbolic Systems with Two-scale Relaxations

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Abstract

We present a domain decomposition method on a semilinear hyperbolic system with multiple relaxation times. In the region where the relaxation time is small, an asymptotic equilibrium equation can be used for computational efficiency. An interface condition is provided to couple the two systems in a domain decomposition setting. A rigorous analysis, based on the Laplace Transform, on the L^2 error estimate is presented for the linear case, which shows how the error of the domain decomposition method depends on the smaller relaxation time, and the boundary and interface layer effects. The given convergence rate is optimal. We present a numerical implementation of this domain decomposition method, and present some numerical results in order to study the performance of this method.

1 Introduction

Consider the hyperbolic system

$$\begin{cases} u_t^\epsilon + v_x^\epsilon = 0, & \text{(1.1a)} \\ v_t^\epsilon + u_x^\epsilon = -\dfrac{1}{\epsilon(x)}(v^\epsilon - f(u^\epsilon)), & \text{(1.1b)} \end{cases}$$

where $\epsilon(x)$ is the relaxation time and $f(x)$ satisfies the sub-characteristic condition:

$$|f'(x)| < 1. \tag{1.2}$$

The problem is posed for $x \in [-L, L]$ and $t > 0$ with initial data

$$u^\epsilon(x,0) = u_0(x), \; v^\epsilon(x,0) = v_0(x) \tag{1.3}$$

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and the order of the relaxation time varies considerably over the domain $[-L, L]$. In this paper, we consider the case when $\epsilon(x)$ is given by:

$$\epsilon(x) = 1, \quad x \in [-L, 0); \qquad \epsilon(x) = \epsilon, \quad x \in (0, L], \tag{1.4}$$

where $\epsilon \ll 1$ is a small parameter. For the boundary condition, we simply choose the Dirichlet condition for u, i.e:

$$u^\epsilon(x_L, t) = b_L(t), \; u^\epsilon(x_R, t) = b_R(t). \tag{1.5}$$

More general boundary conditions can also be analyzed by the methods of the present paper. The initial data and boundary data are required to be compatible, i.e: $b_1(0) = u_0(x_L)$, $b_2(0) = u_0(x_R)$. Since the relaxation time is small in the region $(0, L]$, numerical computation of this system becomes very costly. On the other hand, in $(0, L]$, the solution is, to leading order in ϵ, governed by the equilibrium equation

$$u_t + f(u)_x = 0\,, \tag{1.6}$$

which can be more efficiently solved numerically. Thus a domain decomposition method, which couples the relaxation system (1.1) for $x \in [-L, 0)$, where $\epsilon(x) = O(1)$, with the equilibrium equation (1.6) for $x \in (0, L]$, is computationally competitive. Interface conditions at $x = 0$ must be provided for this coupling.

System (1.1) was first proposed by Jin-Xin [14] for numerical purpose, which supplies a new and powerful approximation to equilibrium conservation law (1.6). There have been many works concerning the asymptotic convergence of the relaxation systems (1.1) to the corresponding conservation laws (1.6) as the relaxation time tends to zero. Most of the results dealt with the Cauchy problem. In particular, Natalini [22] gave a rigorous proof that the solution to Cauchy problem (1.1) with initial condition (1.3) converges strongly in $C([0, \infty), L^1_{Loc}(\mathbb{R}))$ to the unique entropy solution of (1.6) when $\epsilon \to 0$. See also [23] for a review in this direction, and results for larger systems [2] and on more general hyperbolic systems with relaxations [7].

In the presence of physical boundary conditions, Kriess and some others first gave the suitability of boundary conditions for linear hyperbolic systems when the source term is not stiff, see, for examples [15], [13], [21], [25]. Wang and Xin [29] later gave a similar result of the system (1.1) (1.3) with boundary condition (1.5). They proved that when the initial and boundary data satisfy a strict version of the subcharacteristic condition (1.2), the solution of the relaxation system converges as $\epsilon \to 0$ to a unique week solution of the conservation law (1.6) which satisfies the boundary-entropy condition. [32] and [31] then gave an explicit necessary and sufficient condition (the so-called "Stiff Kriess Condition") on the boundary that guarantees the uniform well-posedness of the IBVP, and also revealed the boundary layer structures. [31] dealt with the linear cases while [32] considered the nonlinear one.

Domain decomposition methods connecting kinetic equation and its hydrodynamic or diffusion limit have received a lot of attention in the past 20 years. Our paper is strongly motivated by [12], others can refer to [1], [27], [3], [11], [34], [16], [17], [8], [10]. A thorough study on the problem of this paper provides a better understanding of the more general coupling problem of kinetic and hydrodynamic equations, since indeed the Jin-Xin relaxation system (1.1) can be viewed as a discrete-velocity kinetic model, while (1.6) resembles some important features of hydrodynamic (compressible Euler) equations.

Relaxation systems themselves are important in many physical situations, such as kinetics theories [5], gases not in thermodynamic equilibrium [28], phase transitions with small transition time [19], river flows, traffic flows and more general waves [30].

Denote

$$U^\epsilon = \begin{pmatrix} u^\epsilon \\ v^\epsilon \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 0 \\ \lambda & -1 \end{pmatrix}.$$

For the linear case, when $f(u) = \lambda u$, with $|\lambda| < 1$, we have the following main theorems.

For stiff well-posedness, we have:

Theorem 1. *If $u_0(x), v_0(x), b_L(t), b_R(t) \in L^2$, and $U_0(\pm L) = 0$, $b_L(0) = b_R(0) = 0$, then the solution to the original system (1.1), with variable $\epsilon(x)$ given in (1.4), is stiffly well-posed in the sense:*

$$\begin{aligned}
&\int_0^T\int_{-L}^L |U^\epsilon(x,t)|^2 dxdt + \int_0^T |U^\epsilon(-L,t)|^2 dt + \int_0^T |U^\epsilon(L,t)|^2 dt \\
&\le K_T \int_0^T |b_L(t)|^2 dt + K_T \int_0^T |b_R(t)|^2 dt + K_T \int_{-L}^L |U_0(x)|^2 dx,
\end{aligned}$$

where K_T is a positive constant independent of ϵ. Moreover, if $u_0(x), v_0(x), b_L(t)$ and $b_R(t)$ are continuous, then the solution U^ϵ is continuous in x.

For the asymptotic convergence, we have:

Theorem 2. *Let U^ϵ be the solution of the original system (1.1). Assume $b_L(t), b_R(t) \in L^2(\mathbf{R}^+)$, $U_0(x) \in H^1([-L,L])$, with the compatibility condition $b_L(0) = b'_L(0) = b_R(0) = b'_R(0) = 0$, $U_0(\pm L) = U'_0(\pm L) = 0$, then there exists a unique solution $U = (u,v)$ of the domain decomposition system (3.2)-(3.1) or (3.3)-(3.4) such that*

$$\int_{-L}^L\int_0^\infty |U^\epsilon - U|^2 e^{-2\alpha t} dtdx \to 0$$

as $\epsilon \to 0$ for any $\alpha > 0$. Moreover, if we assume $b_L(t),\ b_R(t) \in H^2(\mathbb{R}^+)$, $U_0(x) \in C^\infty([-L,L])$, then

$$\begin{aligned}
&\int_{-L}^L\int_0^\infty |U^\epsilon - U|^2 e^{-2\alpha t}\, dtdx \\
\le\ & O(1)\epsilon\|b_L\|_{L^2}^2 + O(1)\epsilon\|b_R\|_{L^2}^2 + O(1)\epsilon^2\|b_L\|_{H^2}^2 \\
&+ O(1)\epsilon^2\|b_R\|_{H^2}^2 + O(1)\epsilon\|v_0 - \lambda u_0\|_{L^2[0,L]}^2 \\
&+ \begin{cases} O(1)\epsilon^2\|U_0\|_{H^2}^2(1+o(\epsilon)), & \textit{for } \lambda > 0, \\ O(1)\epsilon\|U_0\|_{L^2}^2 + O(1)\epsilon^2\|U_0\|_{H^2}^2(1+o(\epsilon)), & \textit{for } \lambda < 0. \end{cases}
\end{aligned}$$

Here the term $o(\epsilon)$ will depend on the norm $\|(u_0)^{(2k)}\|_{L^2}$, $(k \ge 2)$.

Remark 3. (1) In the $\lambda < 0$ case, there is an interface layer near $x = 0^+$, while in the $\lambda > 0$ case, there is a boundary layer near $x = L^-$, so in both cases, the optimal convergence rate due to the boundary data is $O(1)\epsilon$, which is where the terms $O(1)\epsilon\|b_L(t)\|^2_{L^2} + O(1)\epsilon\|b_R(t)\|^2_{L^2}$ come from.

(2) The lower convergence rate in the case of $\lambda < 0$ is due to the presence of an interface layer near $x = 0^+$ generated by the initial data.

(3) $O(1)\epsilon\|v_0 - \lambda u_0\|^2_{L^2[0,L]}$ comes from the initial layer in v.

The paper is organized as follows. In Section 2 we show the formal expansion of the initial boundary value problem (1.1) in the upper half plane $\{x > 0,\ t > 0\}$ in which the boundary layer may exist. We also refer to the theorems in [32] which validate this expansion. Section 3 is devoted to give the domain decomposition method, and the interface condition is given. We then prove the stiff well-posedness and asymptotic convergence for the linear case. The theorems are proved in two parts: one for homogeneous initial data (Section 4) and the other the inhomogeneous one (Section 5). For the homogeneous one, we simply use the Laplace transform to represent the solution, while for the inhomogeneous case, we construct several auxiliary systems to decompose the solution into two parts, one generated by the initial data, and the other by the interface condition. With this decomposition, we are able to use some existent result for the Cauchy problem to avoid the difficulties raised by the Laplace transform. Finally in Section 6, we present the corresponding numerical algorithms and give some numerical examples to validate the theoretical analysis.

2 The local equilibrium limit

In this section, we recall the asymptotic analysis proposed in [32]. Here we only consider the boundary layer effect, and let

$$v_0(x) = f(u_0(x))$$

in order to avoid the initial layer effect. When $x \in [0, L]$ where ϵ is small, one can use the hyperbolic conservation law (1.6) to approximate the relaxation system. Away from $x = 0$ and $t = 0$, use the expansion

$$u^\epsilon(x,t) \sim u^0(x,t) + \epsilon u^1(x,t) + \epsilon^2 u^2(x,t) + \dots,$$
$$v^\epsilon(x,t) \sim v^0(x,t) + \epsilon v^1(x,t) + \epsilon^2 v^2(x,t) + \dots,$$

then matching the orders of ϵ, one obtains:

$$\begin{aligned} v^0 &= f(u^0), \\ \partial_t u^0 + \partial_x v^0 &= 0, \\ \partial_t v^0 + \partial_x u^0 &= -(v^1 - f'(u^0)u^1). \\ &\cdots \end{aligned}$$

Thus the leading order of the expansion gives

$$\partial_t u^0 + \partial_x f(u^0) = 0,\ v^0 = f(u^0), \tag{2.1}$$

which is the equilibrium limit (the zero relaxation limit) (1.6).

Near $x = 0$, introduce the stretched variable $\zeta = x/\epsilon$,

$$
\begin{aligned}
u^\epsilon(x,t) &\sim u(x,t) + \epsilon u_1(x,t) \ + \ldots + \Gamma_u^0(\zeta,t) + \epsilon\Gamma_u^1(\zeta,t) + \ldots, \\
v^\epsilon(x,t) &\sim v(x,t) + \epsilon v_1(x,t) + \ldots + \Gamma_v^0(\zeta,t) + \epsilon\Gamma_v^1(\zeta,t) + \ldots,
\end{aligned}
$$

so the boundary layer equations to the leading order is:

$$\partial_\zeta \Gamma_v^0 = 0, \tag{2.2}$$

$$\partial_\zeta \Gamma_u^0 + \Gamma_v^0 + f(u(0,t)) = f(\Gamma_u^0 + u(0,t)). \tag{2.3}$$

(2.2) implies $\Gamma_v^0 \equiv 0$ because the boundary layer $\Gamma_v^0(\xi, 0)$ should decay as $\zeta \to 0$. Also, (2.3) can be written as

$$(\Gamma_u^0)_\zeta = -(v^0 - f(u^0 + \Gamma_u^0)) \simeq f'(u^0(0,t))\Gamma_u^0(\zeta,t),$$

thus one gets the behavior of the boundary layer in u:

$$\Gamma_u^0(\zeta,t) = \exp(f'(u^0(0,t))\zeta)\Gamma_u^0(0,t). \tag{2.4}$$

Since the boundary layer has to decay exponentially fast, one needs $f'(u^0(0,t)) < 0$. In other words, if $f'(u^0(0,t)) < 0$, there will be a boundary layer, otherwise there will not be a boundary layer.

The above analysis was rigorously validated in [32].

3 A domain decomposition method

In section 2, one sees that when ϵ goes to 0, the hyperbolic system can be approximated by the equilibrium equation that does not have any stiff term. But the interface condition that connects the two regions should be provided. In this section, we will give the detailed algorithm that approximates the solution of the two-scale problem. We will consider the case with $f'(u) < 0$ and $f'(u) > 0$ separately.

3.1 $f'(u) < 0$

In this case, there will be an interface layer in u near the interface $x = 0$, so one can not simply use u obtained from $(0, L]$ to solve (1.6) in domain $[-L, 0)$. Instead we can use the information of v at $x = 0$ directly from the equation in $(0, L]$ since there is no $O(1)$ interface layer in v. Here is the the coupling algorithm.

- **Step 1.** For $x \in (0, L]$, solve

$$
\begin{cases}
u_t^0 + f(u^0)_x = 0, & \text{(3.1a)} \\
u^0(x,0) = u_0(x), & \text{(3.1b)} \\
u^0(L,t) = b_R(t), & \text{(3.1c)} \\
v^0(x,t) = f(u^0(x,t)). & \text{(3.1d)}
\end{cases}
$$

Note in this case one can solve (3.1) first to get $v^0(0,t)$, and then solve (3.2).

- **Step 2.** For $x \in [-L, 0)$, solve

$$\begin{cases} \overline{u}_t + \overline{v}_x = 0, & (3.2a) \\ \overline{v}_t + \overline{u}_x = -\dfrac{1}{c_1}(\overline{v} - f(\overline{u}), & (3.2b) \\ \overline{u}(x,0) = u_0(x),\ \overline{v}(x,0) = v_0(x), & (3.2c) \\ \overline{u}(-L,t) = b_L(t), & (3.2d) \\ \overline{v}(0,t) = v^0(0,t); & (3.2e) \end{cases}$$

  where $v^0(0,t)$ is obtained from Step 1.

3.2 $f'(u) > 0$

In this case, at the interface $x = 0$ there is no $O(1)$ interface layer in u and v. In other words, u and v are in local equilibrium $v = f(u)$, and we can just this as the interface condition. We give the following algorithm.

- **Step 1.** For $x \in [-L, 0)$, solve

$$\begin{cases} \overline{u}_t + \overline{v}_x = 0, & (3.3a) \\ \overline{v}_t + \overline{u}_x = -\dfrac{1}{c_1}(\overline{v} - f(\overline{u})), & (3.3b) \\ \overline{u}(x,0) = u_0(x),\ \overline{v}(x,0) = v_0(x), & (3.3c) \\ \overline{u}(-L,t) = b_1(t), & (3.3d) \\ f(\overline{u}(0,t)) = \overline{v}(0,t); & (3.3e) \end{cases}$$

- **Step 2.** For $x \in (0, L]$, solve

$$\begin{cases} u^0_t + f(u^0)_x = 0, & (3.4a) \\ u^0(x,0) = u_0(x), & (3.4b) \\ v^0(x,t) = f(u^0(x,t)), & (3.4c) \\ u^0(0,t) = \overline{u}(0,t)\,, & (3.4d) \end{cases}$$

  where $\overline{u}(0,t)$ is obtained from Step 1.

Remark 4. In this case there will be a boundary layer in u near $x = L^-$, which is why in Theorem 2 that the convergence rate is $O(\epsilon)$.

In both cases, we define the solution to the domain decomposition system as follows:

$$\begin{cases} u(x,t) = \overline{u}(x,t),\ v(x,t) = \overline{v}(x,t), \quad (x,t) \in [-L,0) \times [0,T], & (3.5a) \\ u(x,t) = u^0(x,t),\ v(x,t) = v^0(x,t), \quad (x,t) \in (0,L] \times [0,T]. & (3.5b) \end{cases}$$

The detailed numerical implementation of this domain decomposition method is given in section 6.

4 Error estimate for the domain decomposition method for the linear case: the Homogeneous initial data

In this and the next section, we will give a rigorous justification of the domain decomposition system. The main results are given in Theorems 1 and 2. Here the basic idea is to represent the exact solution by the Laplace transform, then the stiff wellposedness and asymptotic convergence are followed by direct calculations.

Here we consider system (1.1) with zero initial data (1.3), i.e., $u_0(x) = 0$, $v_0(x) = 0$ and nonzero boundary data (1.5). In this case one can focus on the boundary layer effects and avoid the interactions between the initial and boundary layers.

4.1 Solution by the Laplace transform

Let

$$\tilde{U^\epsilon}(x,\xi) = \mathcal{L}(U^\epsilon) = \int_0^\infty e^{-\xi t} U^\epsilon(x,t)dt, \quad \mathrm{Re}(\xi) > 0.$$

Here $\xi = \alpha + i\beta$, then $\mathcal{L}(\partial_t U^\epsilon) = \xi\tilde{U^\epsilon} - U^\epsilon(x,0) = \xi\tilde{U^\epsilon}(x,\xi)$. With the homogeneous initial condition, system (1.1)-(1.5) becomes

$$\partial_x \tilde{U^\epsilon} = \frac{1}{\epsilon(x)} A^{-1}(S - \epsilon(x)\xi I)\tilde{U^\epsilon} = \frac{1}{\epsilon(x)} M(\epsilon(x)\xi)\tilde{U^\epsilon}, \tag{4.1}$$

$$\tilde{u^\epsilon}(-L,\xi) = \tilde{b}_L(\xi), \ \tilde{u^\epsilon}(L,\xi) = \tilde{b}_R(\xi), \tag{4.2}$$

where matrix $M(\xi)$ has two eigenvalues

$$\mu_\pm(\xi) = \frac{\lambda \pm \sqrt{\lambda^2 + 4\xi(1+\xi)}}{2}, \tag{4.3}$$

and two corresponding eigenvectors

$$\begin{pmatrix} 1 \\ \frac{\mu_\mp(\xi)}{1+\xi} \end{pmatrix} = \begin{pmatrix} 1 \\ g_\mp(\xi) \end{pmatrix}. \tag{4.4}$$

Thus the solution of (4.1) (4.2) can be written as:
For $x < 0$, $\epsilon(x) = 1$,

$$\tilde{U^\epsilon}(x,\xi) = c_1 e^{\mu_-(\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + c_2 e^{\mu_+(\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \tag{4.5}$$

for $x > 0$, $\epsilon(x) = \epsilon$,

$$\tilde{U^\epsilon}(x,\xi) = c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} \begin{pmatrix} 1 \\ g_-(\epsilon\xi) \end{pmatrix}, \tag{4.6}$$

where the coefficient c_1, c_2, c_3, c_4 are determined by the boundary conditions:

$$c_1 e^{-\mu_-(\xi)L} + c_2 e^{-\mu_+(\xi)L} = \tilde{b}_L(\xi), \tag{4.7}$$

$$c_3 e^{\mu_-(\xi\epsilon)\frac{L}{\epsilon}} + c_4 e^{\mu_+(\xi\epsilon)\frac{L}{\epsilon}} = \tilde{b}_R(\xi). \tag{4.8}$$

By continuity at the interface, one has

$$
\begin{aligned}
c_1 + c_2 &= c_3 + c_4, &(4.9)\\
c_1 g_+(\xi) + c_2 g_-(\xi) &= c_3 g_+(\epsilon\xi) + c_4 g_-(\epsilon\xi). &(4.10)
\end{aligned}
$$

From (4.7)–(4.10), one sees that $c_1, \cdots, c_4$ are uniquely determined. Denote

$$
\begin{aligned}
&c_3 = Ec_1 + Fc_2, &(4.11)\\
&c_4 = Gc_1 + Hc_2, &(4.12)
\end{aligned}
$$

where

$$
E = \frac{g_+(\xi) - g_-(\epsilon\xi)}{g_+(\epsilon\xi) - g_-(\epsilon\xi)},\ F = \frac{g_-(\xi) - g_-(\epsilon\xi)}{g_+(\epsilon\xi) - g_-(\epsilon\xi)},\ G = \frac{g_+(\xi) - g_+(\epsilon\xi)}{g_-(\epsilon\xi) - g_+(\epsilon\xi)},\ H = \frac{g_-(\xi) - g_+(\epsilon\xi)}{g_-(\epsilon\xi) - g_+(\epsilon\xi)}.
$$

Plugging (4.11) (4.12) into (4.7) (4.8), gives

$$
c_1 = \frac{\tilde{b}_R(\xi)e^{-\mu_+(\xi)L} - \tilde{b}_L(\xi)(Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})}{(Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_+(\xi)L} - (Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_-(\xi)L}}, \tag{4.13}
$$

$$
c_2 = \frac{\tilde{b}_R(\xi)e^{-\mu_-(\xi)L} - \tilde{b}_L(\xi)(Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})}{(Fe^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + He^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_-(\xi)L} - (Ee^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} + Ge^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}})e^{-\mu_+(\xi)L}}. \tag{4.14}
$$

4.2 Stiff well-posedness

Before detailed calculations, we summarize some properties of the important functions in the expression of the solutions in the following lemma, which will be used many times later.

Lemma 5. *Under the subcharacteristic condition $|\lambda| < 1$, one has*

$$
(1)\ |\lambda|(1 + 2\alpha) \le Re\ \sqrt{\lambda^2 + 4\xi(1 + \xi)} \le 1 + 2\alpha,\ \textit{for } Re\ (\xi) = \alpha \ge 0; \tag{4.15}
$$

$$
(2)\ Re\ \mu_+(\xi) > 0,\ \ Re\ \mu_-(\xi) < 0; \tag{4.16}
$$

$$
\begin{aligned}
(3)\qquad &\textit{when } \lambda < 0,\ 2Re\ \mu_-(\epsilon\xi) \le -2|\lambda|,\ 2Re\ \mu_+(\epsilon\xi) \ge -2\epsilon\lambda\alpha; &(4.17)\\
&\textit{when } \lambda > 0,\ 2Re\ \mu_-(\epsilon\xi) \le -2\epsilon\lambda\alpha,\ 2Re\ \mu_+(\epsilon\xi) \ge 2\lambda. &(4.18)
\end{aligned}
$$

For the proof of the lemma, please refer to [31].

Lemma 6. *Under the subcharacteristic condition $|\lambda < 1$, one has*
(1) For $\lambda > 0$, $g_-(\epsilon\xi) = O(1)\epsilon\xi$, and $0 < C_1 \le |g_+(\epsilon\xi)| \le C_2$, here C_1 and C_2 are two positive constants, and $g_+(\epsilon\xi) - \lambda = O(1)\epsilon\xi$;
(2) For $\lambda < 0$, $g_+(\epsilon\xi) = O(1)\epsilon\xi$, and $0 < C_3 \le |g_-(\epsilon\xi)| \le C_4$, here C_3 and C_4 are two positive constants, and $g_-(\epsilon\xi) - \lambda = O(1)\epsilon\xi$;
(3)$g_\pm(\xi) - g_\pm(\epsilon\xi)$, $g_+(\xi) - g_-(\epsilon\xi)$, $g_+(\epsilon\xi) - g_-(\xi)$ are uniformly bounded with respect to both ϵ and ξ, and they are bounded away from zero for $Re\xi = \alpha > 0$.

Proof. (1). When $\lambda > 0$, from the definition (4.4), one sees

$$g_-(\epsilon\xi) = \frac{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)} = \frac{-2\epsilon\xi}{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}} = O(1)\epsilon\xi,$$

and

$$g_+(\epsilon\xi) = \frac{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)}.$$

In order to prove that $g_+(\epsilon\xi)$ is uniformly bounded with respect to $\epsilon\xi$, and the denominator is nonzero, one just needs to check what happens when $|\epsilon\xi|$ goes to 0 or ∞. Let $\epsilon\xi = re^{i\theta}$, one sees that when $|\epsilon\xi| \to 0$, i.e., when $r \to 0$, $|g_+(\epsilon\xi)| \to \lambda$; when $|\epsilon\xi| \to \infty$, i.e., when $|r| \to \infty$, one has

$$|g_+(\epsilon\xi)| \to |\frac{\sqrt{\lambda^2 + 4re^{i\theta}(1+re^{i\theta})}}{2(1+re^{i\theta})}| \to (\cos^2 2\theta + \sin^4\theta)^{\frac{1}{4}},$$

which is bounded and nonzero. Moreover,

$$g_+(\epsilon\xi) - \lambda = \frac{2(1-\lambda^2)\epsilon\xi}{\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)} + \lambda(1+2\epsilon\xi)} = O(1)\epsilon\xi.$$

(2). When $\lambda < 0$, similarly one has

$$g_+(\epsilon\xi) = \frac{\lambda + \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)} = \frac{-2\epsilon\xi}{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}} = O(1)\epsilon\xi,$$

and

$$g_-(\epsilon\xi) = \frac{\lambda - \sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{2(1+\epsilon\xi)}.$$

In the same way as in (1), one can prove $g_-(\epsilon\xi)$ is uniformly bounded in $\epsilon\xi$.
(3). Note

$$g_+(\xi)-g_-(\epsilon\xi)=\frac{\lambda\xi(\epsilon-1) + (1+\epsilon\xi)\sqrt{\lambda^2 + 4\xi(1+\xi)} + (1+\xi)\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{(1+\xi)(1+\epsilon\xi)}.$$

Let $\xi = re^{\theta}$, then when $\epsilon \to 0$, and $|r| \to 0$, one has $|g_+(\xi)-g_-(\epsilon\xi)| \to 2|\lambda|$. When $\epsilon \to 0$, $|r| \to \infty$, and $\epsilon|r| \to 0$, one has

$$|g_+(\xi)-g_-(\epsilon\xi)| \to |\frac{\lambda + \sqrt{\lambda^2 + 4\xi(1+\xi)}}{1+\xi}| \to \frac{1}{2}(\cos^2 2\theta + \sin^4\theta)^{\frac{1}{4}},$$

which is bounded and nonzero. When $\epsilon \to 0$, $|r| \to \infty$, and $\epsilon|r| \to \infty$, one can still prove $|g_+(\xi)-g_-(\epsilon\xi)|$ is uniformly bounded away from 0, but the detailed calculation will be omitted. Similarly, one can prove the same result for $g_+(\epsilon\xi)-g_-(\xi)$. As for $g_+(\xi) - g_+(\epsilon\xi)$, notice

$$g_+(\xi)-g_+(\epsilon\xi) = \frac{\lambda\xi(\epsilon-1) + (1+\epsilon\xi)\sqrt{\lambda^2 + 4\xi(1+\xi)} - (1+\xi)\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)}}{(1+\xi)(1+\epsilon\xi)},$$

then following the same procedure as above, it is not hard to check that it is uniformly bounded as $\epsilon \to 0$, and $|\xi| \to \infty$. Moreover, when $\epsilon \to 0$, $|\xi| \to \alpha$,

$$|g_+(\xi) - g_+(\epsilon\xi)| \to |\frac{-\lambda\alpha + \sqrt{\lambda^2 + 4\alpha(1+\alpha)} - (1+\alpha)\lambda}{1+\alpha}|,$$

which is nonzero, one can arrive at the same conclusion for $g_-(\xi) - g_-(\epsilon\xi)$. $\square$

Remark 7. (1) We will fix Re $\xi = \alpha > 0$ from now on.
(2) From the definition (4.6), one sees that, when $\lambda > 0$, by (4.18), there is a boundary layer near $x = L$, and on the other hand, when $\lambda < 0$, by (4.17), there is an interface layer near $x = 0$. This observation will play an important role in the following proof.

Now we go back to the proof of Theorem 1. Consider the integral:

$$\begin{aligned}
\int_{-L}^{L} dx \int_{-\infty}^{\infty} |\tilde{U}^\epsilon(x,\xi)|^2 d\beta \;=\; & \int_{-L}^{0} e^{2Re\mu_-(\xi)x} dx \int_{-\infty}^{\infty} |c_1|^2 (1 + |g_+(\xi)|^2) d\beta \\
& + \int_{-L}^{0} e^{2Re\mu_+(\xi)x} dx \int_{-\infty}^{\infty} |c_2|^2 (1 + |g_-(\xi)|^2) d\beta \\
& + \int_{0}^{L} e^{2Re\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx \int_{-\infty}^{\infty} |c_3|^2 (1 + |g_+(\epsilon\xi)|^2) d\beta \\
& + \int_{0}^{L} dx \int_{-\infty}^{\infty} |c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}|^2 (1 + |g_-(\epsilon\xi)|^2) d\beta.
\end{aligned}$$

By Lemma 4 one sees E, F, G, and H in (4.11) (4.12) are uniformly bounded away from 0. And from (4.13) (4.14) (4.11) and (4.12) one gets

$$c_1,\ c_2,\ c_3,\ c_4 = O(1)(\tilde{b}_L(\xi) + \tilde{b}_R(\xi)),$$

and moreover, from (4.8),

$$e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} c_4 = (\tilde{b}_R(\xi) - c_3 e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}), \tag{4.19}$$

so $e^{\mu_+(\epsilon\xi)\frac{L}{\epsilon}} c_4 = O(1) e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \tilde{b}_L(\xi) + O(1)\tilde{b}_R(\xi)$. Therefore

$$\int_{-L}^{L} dx \int_{-\infty}^{\infty} |\tilde{U}^\epsilon(x,\xi)|^2 d\beta \le O(1) \int_{-\infty}^{\infty} (|\tilde{b}_L(\xi)|^2 + |\tilde{b}_R(\xi)|^2) d\beta. \tag{4.20}$$

Then by Parseval's identity:

$$\int_{0}^{\infty} e^{-2\alpha t} |U^\epsilon(x,t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\tilde{U}^\epsilon(x, \alpha + i\beta)|^2 d\beta, \tag{4.21}$$

the stiff well-posedness, as stated in Theorem 1, now follows.

4.3 Asymptotic convergence

Next we turn to the proof of the asymptotic convergence. Still we compare the analytical solution with the help of the Laplace transform. Consider the case $\lambda < 0$ first. Compare system (1.1) with (3.2)-(3.1). The solution of (3.1) is

$$u^0(x,t) = \begin{cases} 0, & x - L \le \lambda t, \\ b_R(t + \frac{1}{\lambda}(L-x)), & x - L \ge \lambda t,\ 0 \le x \le L. \end{cases}$$

After the Laplace transform, it becomes

$$\tilde{u}^0(x,\xi) = \tilde{b}_R(\xi) e^{\frac{\xi}{\lambda}(L-x)}, \quad \text{for } x > 0. \tag{4.22}$$

The solution of (3.2) is

$$\tilde{\tilde{U}}(x,\xi) = d_1 e^{\mu_-(\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + d_2 e^{\mu_+(\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \tag{4.23}$$

where d_1 and d_2 are determined by

$$d_1 e^{-\mu_-(\xi)L} + d_2 e^{-\mu_+(\xi)L} = \tilde{b_L}(\xi), \tag{4.24}$$

$$d_1 g_+(\xi) + d_2 g_-(\xi) = \lambda \tilde{b_R}(\xi) e^{\frac{\xi}{\lambda}L}. \tag{4.25}$$

Now compare (4.5) with (4.23), (4.6) with (4.22) respectively. For $x \in [0, L]$, using (4.6) and (4.22), one gets

$$\begin{aligned} &\int_0^L dx \int_{-\infty}^{\infty} |\tilde{\bar{u}} - \tilde{u^\epsilon}|^2 d\beta = \int_0^L dx \int_{-\infty}^{\infty} |c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} - \tilde{b}_R e^{\frac{\xi}{\lambda}(L-x)}|^2 d\beta \\ \le\ & \int_0^L dx \int_{-\infty}^{\infty} |c_3 (e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}})|^2 d\beta + \int_0^L dx \int_{-\infty}^{\infty} |\tilde{b}_R(\xi)|^2 |e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - e^{\frac{\xi}{\lambda}(L-x)}|^2 d\beta \\ =\ & I_1 + I_2. \end{aligned}$$

Here the first inequality was derived by substituting c_4 in (4.19). For I_1, it is easy to see:

$$I_1 \le O(1) \int_{-\infty}^{\infty} |c_3(\xi)|^2 d\beta \Big(\int_0^L e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx + e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{L}{\epsilon}} \int_0^L e^{2\mathrm{Re}\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} dx \Big).$$

Then by (4.17) one gets the estimate for I_1 as:

$$\begin{aligned} I_1 &\le O(1)\epsilon \int_{-\infty}^{\infty} |c_3(\xi)|^2 d\beta \\ &= O(1)\epsilon \int_{-\infty}^{\infty} (|\tilde{b}_L|^2 + |\tilde{b}_R|^2) d\beta \le O(1)\epsilon (\|b_L\|^2_{L^2} + \|b_R\|^2_{L^2}). \end{aligned} \tag{4.26}$$

Note here in I_1, the term that contains $e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}$ is the result of interface layer, which drops the L^2 convergence rate down to $\epsilon^{\frac{1}{2}}$.
For I_2, one has

$$\begin{aligned} I_2 &\le \int_{-\infty}^{\infty} O(1) \Big|\frac{\mu_+(\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda}\Big|^2 |\tilde{b}_R(\xi)|^2 d\beta = O(1)\epsilon^2 \int_{-\infty}^{\infty} |\xi|^4 |\tilde{b}_R(\xi)|^2 d\beta \\ &\le O(1)\epsilon^2 \|b_R\|^2_{H^2}. \end{aligned} \tag{4.27}$$

Here we use the fact

$$\frac{\mu_+(\epsilon\xi)}{\epsilon} + \frac{\xi}{\lambda} = \frac{2\epsilon\xi^2(1-\lambda^2)}{\lambda(\lambda^2 + 2\epsilon\xi - \lambda\sqrt{\lambda^2 + 4\epsilon\xi(1+\epsilon\xi)})} = O(1)\epsilon\xi^2, \tag{4.28}$$

and also we assume that $b_R(t) \in H^2(\mathbb{R}^+)$ and $b_R(t)$ satisfies the compatibility condition $b_R(0) = b'_R(0) = 0$. Adding I and II yields we have

$$\int_0^L dx \int_{-\infty}^{\infty} |\tilde{\bar{u}} - \tilde{u^\epsilon}|^2 d\beta \leq O(1)\epsilon(\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2) + O(1)\epsilon^2 \, \|b_R\|_{H^2}^2 \, . \tag{4.29}$$

When $x \in [-L, 0]$ the difference between (4.5) and (4.23) is the difference between the coefficients, i.e.

$$\int_{-L}^0 dx \int_{-\infty}^{\infty} |\tilde{\bar{U}} - \tilde{U^\epsilon}|^2 d\beta \quad = \quad O(1) \int_{-\infty}^{\infty} (|d_1 - c_1|^2 + |d_2 - c_2|^2) d\beta.$$

Compare (4.7)–(4.10) with (4.24)(4.25), one finds

$$|c_1 - d_1| = O(1)\epsilon\xi(\tilde{b}_L + \tilde{b}_R), \quad |c_2 - d_2| = O(1)\epsilon\xi(\tilde{b}_L + \tilde{b}_R),$$

after using Lemma 4 and some basic calculations. The details are omitted.
Therefore,

$$\begin{aligned} \int_{-L}^0 dx \int_{-\infty}^{\infty} |\tilde{\bar{U}} - \tilde{U^\epsilon}|^2 d\beta \quad &\leq \quad O(1)\epsilon^2 \int_{-\infty}^{\infty} (|\xi\tilde{b}_L(\xi)|^2 + |\xi\tilde{b}_R(\xi)|^2) d\xi \\ &\leq \quad O(1)\epsilon^2(\|b_L\|_{H^1}^2 + \|b_R\|_{H^1}^2). \end{aligned} \tag{4.30}$$

Here we used the assumption that $b_L(t) \in H^1(\mathbb{R}^+)$, and $b_L(t)$ satisfies $b_L(0) = 0$. Now we are done with the $\lambda < 0$ case.

For $\lambda > 0$, the proof is similar. First the solution to (3.3) is

$$\tilde{\bar{U}}(x,\xi) = k_1 e^{\mu_-(\xi)x} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} + k_2 e^{\mu_+(\xi)x} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix}, \; -L \leq x \leq 0, \tag{4.31}$$

where k_1 and k_2 are determined by

$$k_1 e^{-\mu_-(\xi)L} + k_2 e^{-\mu_+(\xi)L} \quad = \quad \tilde{b}_L(\xi), \tag{4.32}$$

$$k_1(\lambda - g_+(\xi)) + k_2(\lambda - g_-(\xi)) \quad = \quad 0. \tag{4.33}$$

When $0 \leq x \leq L$, the solution to (3.4) is

$$u^0(x,t) = \begin{cases} 0, & \lambda t \leq x \leq L, \\ \bar{u}(0, t - \frac{x}{\lambda}), & 0 \leq x \leq \lambda t, 0 \leq x \leq L. \end{cases}$$

So after Laplace transform, one gets:

$$\tilde{u}^0(x,\xi) = e^{-\xi\frac{x}{\lambda}} \tilde{\bar{u}}(0^-, \xi) = e^{-\xi\frac{x}{\lambda}}(k_1 + k_2). \tag{4.34}$$

Now compare (4.31) with (4.5), and (4.34) with (4.6). The difference between (4.31) and (4.5) is again the difference between the coefficients. Thus

$$\int_{-L}^{0} dx \int_{-\infty}^{\infty} |\tilde{\bar{U}} - \tilde{U}^\epsilon|^2 d\beta \;=\; O(1)\int_{-\infty}^{\infty} (|k_1 - c_1|^2 + |k_2 - c_2|^2) d\beta.$$

Compare (4.7)–(4.10) with (4.32) (4.33), one finds

$$|c_1 - k_1| = O(1)\epsilon\xi\tilde{b}_L,\; |c_2 - k_2| = O(1)\epsilon\xi\tilde{b}_L.$$

Therefore,

$$\begin{aligned}\int_{-L}^{0} dx \int_{-\infty}^{\infty} |\tilde{\bar{U}} - \tilde{U}^\epsilon|^2 d\beta &\le O(1)\epsilon^2 \int_{-\infty}^{\infty} |\xi\tilde{b}_L(\xi)|^2 d\xi \\ &\le O(1)\epsilon^2 \|b_L\|_{H^1}^2. \end{aligned} \tag{4.35}$$

The difference between (4.34) and (4.6) is estimated as follows:

$$\begin{aligned} & \int_0^L dx \int_{-\infty}^{\infty} |\tilde{u}^0 - \tilde{u}^\epsilon|^2 d\beta \\ = & \int_0^L dx \int_{-\infty}^{\infty} |c_3 e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} + c_4 e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ = & \int_0^L dx \int_{-\infty}^{\infty} |c_3 (e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}) + \tilde{b}_R e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} - (k_1 + k_2) e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ \le & J_1 + J_2 + J_3. \end{aligned}$$

To get the second equality, we again use (4.19). First,

$$\begin{aligned} J_1 &= \int_{-\infty}^{\infty} |\tilde{b}_R(\xi)|^2 d\beta \int_0^L e^{2\mathrm{Re}\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} dx \\ &\le O(1)\epsilon \|b_R(t)\|_{L^2}^2, \end{aligned} \tag{4.36}$$

where the inequalities (4.16), (4.17) and (4.18) were used. For J_2, one has:

$$\begin{aligned} J_2 &= \int_0^L dx \int_{-\infty}^{\infty} |[c_3 - (k_1 + k_2)] e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - c_3 e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}} e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}}|^2 \\ &\le \int_0^L dx \int_{-\infty}^{\infty} |c_3 - k_1 - k_2|^2 e^{2Re\mu_-(\epsilon\xi)\frac{x}{\epsilon}} d\beta + O(1)\epsilon \int_{-\infty}^{\infty} |c_3|^2 d\beta. \end{aligned}$$

Since $c_3 + c_4 = c_1 + c_2 = k_1 + k_2 + O(1)\epsilon\xi\tilde{b}_L(\xi)$, $c_4 = e^{-\mu_+(\epsilon\xi)\frac{L}{\epsilon}}(\tilde{b}_R(\xi) - c_3 e^{\mu_-(\epsilon\xi)\frac{L}{\epsilon}})$, one has $|c_3 - k_1 - k_2|^2 = O(1)\epsilon^2 |\xi\tilde{b}_L(\xi)|^2$. Therefore,

$$J_2 \le O(1)\epsilon^2 \|b_L\|_{H^1}^2 + O(1)\epsilon \|b_L\|_{L^2}^2. \tag{4.37}$$

Note here the convergence rate is ϵ, which is caused by the boundary layer effect of $e^{\mu_+(\epsilon\xi)\frac{x-L}{\epsilon}}$ in J_1 and J_2. The remaining part J_3 is

$$\begin{aligned} J_3 &= \int_0^L dx \int_{-\infty}^{\infty} |(k_1+k_2)e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - (k_1+k_2)e^{-\xi\frac{x}{\lambda}}|^2 d\beta \\ &\le O(1)\int_0^{\infty} |e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} - e^{-\xi\frac{x}{\lambda}}|^2 dx \int_{-\infty}^{\infty} |k_1+k_2|^2 d\beta \\ &\le O(1)\epsilon^2 \|b_L\|_{H^2}^2. \end{aligned} \tag{4.38}$$

The calculation here is similar to (4.27). In total, one gets

$$\int_0^L dx \int_{-\infty}^{\infty} |\tilde{u}^0 - \tilde{u}^\epsilon|^2 d\beta \le O(1)\epsilon(\|b_L\|_{L^2}^2 + \|b_R\|_{L^2}^2) + O(1)\epsilon\ \|b_L\|_{H^2}^2\ . \tag{4.39}$$

To this end, we have proved Theorem 2 with zero initial data.

5 Error estimate for the domain decomposition method for the linear case: the Inhomogeneous initial data

The case with inhomogeneous initial data is much more complicated. To clarify the idea in the proof, we consider instead the Cauchy problem here, that is, $x \in (-\infty, \infty)$ instead of $[-L, L]$. A new idea here is to construct some related initial value problem and make use of the existent results of them to overcome the difficulties arisen in the Laplace transform. With these two results, the problem with both boundary and initial data is straightforward, and details will be omitted.

5.1 Solution by the Laplace transform

Again, we solve system (1.1) by the Laplace transform. Then (1.1) (1.3) becomes:

$$\partial_x \tilde{U^\epsilon} = \frac{1}{\epsilon(x)} M(\epsilon(x)\xi)\tilde{U^\epsilon} + A^{-1}U_0(x), \tag{5.1}$$

where M is the same as before. Then the general solution is:
For $x < 0$, $\epsilon(x) = 1$,

$$\tilde{U^\epsilon}(x,\xi) = e^{M(\xi)x}(\tilde{U}_L + \int_0^x e^{-M(\xi)y} A^{-1} U_0(y)dy); \tag{5.2}$$

For $x > 0$, $\epsilon(x) = \epsilon$,

$$\tilde{U^\epsilon}(x,\xi) = e^{M(\epsilon\xi)\frac{x}{\epsilon}}(\tilde{U}_R + \int_0^x e^{-M(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y)dy), \tag{5.3}$$

where one can denote $e^{M(\xi)x}$ by

$$e^{M(\xi)x} = e^{\mu_+(\xi)x}\Phi_+(\xi) + e^{\mu_-(\xi)x}\Phi_-(\xi). \tag{5.4}$$

Here $\Phi_\pm$ are defined by:

$$\Phi_+(\xi) = \frac{1}{g_+(\xi) - g_-(\xi)} \begin{pmatrix} 1 \\ g_-(\xi) \end{pmatrix} (g_+(\xi) \ \ -1), \tag{5.5}$$

$$\Phi_-(\xi) = \frac{1}{g_+(\xi) - g_-(\xi)} \begin{pmatrix} 1 \\ g_+(\xi) \end{pmatrix} (-g_-(\xi) \ 1). \tag{5.6}$$

Then (5.2) (5.3) can be rewritten as:
For $x < 0$, $\epsilon(x) = 1$,

$$\begin{aligned} \tilde{U^\epsilon}(x,\xi) &= e^{\mu_+(\xi)x}\Phi_+(\xi)(\tilde{U}_L(\xi) + \int_0^x e^{-\mu_+(\xi)y} A^{-1} U_0(y)dy) \\ &\quad + e^{\mu_-(\xi)x}\Phi_-(\xi)(\tilde{U}_L(\xi) + \int_0^x e^{-\mu_-(\xi)y} A^{-1} U_0(y)dy); \end{aligned} \tag{5.7}$$

For $x > 0$, $\epsilon(x) = \epsilon$,

$$\begin{aligned} \tilde{U^\epsilon}(x,\xi) &= e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}\Phi_+(\epsilon\xi)(\tilde{U}_R(\xi) + \int_0^x e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y)dy) \\ &\quad + e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}\Phi_-(\epsilon\xi)(\tilde{U}_R(\xi) + \int_0^x e^{-\mu_-(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y)dy). \end{aligned} \tag{5.8}$$

Here $\tilde{U}_L(\xi) = \begin{pmatrix} \tilde{u}_L(\xi) \\ \tilde{v}_L(\xi) \end{pmatrix}$ and $\tilde{U}_R(\xi) = \begin{pmatrix} \tilde{u}_R(\xi) \\ \tilde{v}_R(\xi) \end{pmatrix}$, to be defined later, are two vectors independent of x.
First, when $x \to \infty$, $\tilde{U^\epsilon}(x,\xi) \to 0$, one gets

$$(g_+(\epsilon\xi) \ \ -1) \begin{pmatrix} \tilde{u}_R(\xi) \\ \tilde{v}_R(\xi) \end{pmatrix} + \int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} (g_+(\epsilon\xi) \ \ -1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y)dy = 0,$$

that is,

$$g_+(\epsilon\xi)\tilde{u}_R(\xi) - \tilde{v}_R(\xi) + \int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} (v_0(y)g_+(\epsilon\xi) - u_0(y))dy = 0. \tag{5.9}$$

When $x \to -\infty$, $\tilde{U^\epsilon}(x,\xi) \to 0$, thus

$$(-g_-(\xi) \ 1) \begin{pmatrix} \tilde{u}_L(\xi) \\ \tilde{v}_L(\xi) \end{pmatrix} + \int_0^{-\infty} e^{-\mu_-(\xi)y} (-g_-(\xi) \ 1) \begin{pmatrix} v_0 \\ u_0 \end{pmatrix} (y)dy = 0,$$

that is,

$$-g_-(\xi)\tilde{u}_L(\xi) + \tilde{v}_L(\xi) + \int_0^{-\infty} e^{-\mu_-(\xi)y} (-v_0(y)g_-(\xi) + u_0(y))dy = 0. \tag{5.10}$$

Then by continuity, $\Phi_+(\xi)U_L + \Phi_-(\xi)U_L = \Phi_+(\epsilon\xi)U_R + \Phi_-(\epsilon\xi)U_R$, it is easy to get:

$$\tilde{u}_L = \tilde{u}_R, \ \tilde{v}_L = \tilde{v}_R. \tag{5.11}$$

Plugging (5.9)–(5.11) into (5.8), one ends up with a simplified version of (5.8):

$$\begin{aligned}\tilde{U^\epsilon}(x,\xi) &= \frac{1}{g_+(\epsilon\xi)-g_-(\epsilon\xi)}\left\{\begin{pmatrix}1\\ g_-(\epsilon\xi)\end{pmatrix}\int_x^\infty e^{\mu_+(\epsilon\xi)\frac{x-y}{\epsilon}}(u_0(y)-v_0(y)g_+(\epsilon\xi))dy\right.\\ &\quad+\begin{pmatrix}1\\ g_+(\epsilon\xi)\end{pmatrix}\int_0^x e^{\mu_-(\epsilon\xi)\frac{x-y}{\epsilon}}(u_0(y)-v_0(y)g_-(\epsilon\xi))dy\\ &\quad\left.+\begin{pmatrix}1\\ g_+(\epsilon\xi)\end{pmatrix}e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}(\tilde{v}_R(\xi)-\tilde{u}_R(\xi)g_-(\epsilon\xi))\right\},\quad \text{for } x>0, \end{aligned}\tag{5.12}$$

Similarly, (5.7) can be simplified to:

$$\begin{aligned}\tilde{U^\epsilon}(x,\xi) &= \frac{1}{g_+(\xi)-g_-(\xi)}\left\{\begin{pmatrix}1\\ g_+(\xi)\end{pmatrix}\int_{-\infty}^x e^{\mu_-(\xi)(x-y)}(u_0(y)-v_0(y)g_-(\xi))dy\right.\\ &\quad+\begin{pmatrix}1\\ g_-(\xi)\end{pmatrix}\int_x^0 e^{\mu_+(\xi)(x-y)}(u_0(y)-v_0(y)g_+(\xi))dy\\ &\quad\left.+\begin{pmatrix}1\\ g_-(\xi)\end{pmatrix}e^{\mu_+(\xi)x}(-\tilde{v}_L(\xi)+\tilde{u}_L(\xi)g_+(\xi))\right\},\quad \text{for } x<0. \end{aligned}\tag{5.13}$$

5.2 The Stiff well-posedness

Due to the nonzero initial data, it is hard to estimate the L^2 norm of the solution from the expression (5.12) (5.13). So we take a detour to look at the initial value problem with initial data supported in the right (or left) half plane. Then for this initial value problem, one can solve it by the Fourier transform, thus avoid the difficulties caused by the Laplace transform. Without loss of generality, we consider $x>0$ here. The $x<0$ case is the same. First we have the following lemma.

Lemma 8. *Assume $U^\epsilon_{IVP}=\begin{pmatrix}u^\epsilon_{IVP}\\ v^\epsilon_{IVP}\end{pmatrix}$ is the solution to*

$$\begin{cases} u^\epsilon_t+v^\epsilon_x=0, & (5.14\text{a})\\ v^\epsilon_t+u^\epsilon_x=-\dfrac{1}{\epsilon}(v^\epsilon-\lambda u^\epsilon), & (5.14\text{b})\\ u^\epsilon(x,0)=u_0(x), v^\epsilon(x,0)=v_0(x), & (5.14\text{c})\end{cases}$$

here u_0 and v_0 are supported in $[0,\infty)$. Then the solution is

$$\begin{aligned}\tilde{U}^\epsilon_{IVP}(x,\xi) &= \frac{1}{g_+(\epsilon\xi)-g_-(\epsilon\xi)}\{\begin{pmatrix}1\\ g_-(\epsilon\xi)\end{pmatrix}\int_x^\infty e^{\mu_+(\epsilon\xi)\frac{x-y}{\epsilon}}(u_0(y)-v_0(y)g_+(\epsilon\xi))dy\\ &\quad+\begin{pmatrix}1\\ g_+(\epsilon\xi)\end{pmatrix}\int_0^x e^{\mu_-(\epsilon\xi)\frac{x-y}{\epsilon}}(u_0(y)-v_0(y)g_-(\epsilon\xi))dy\},\end{aligned}\tag{5.15}$$

and the following inequality holds:

$$\int_{-\infty}^\infty\int_{-\infty}^\infty|\tilde{U}^\epsilon_{IVP}(x,\xi)|^2dxd\beta\le O(1)\int_0^\infty|U_0(x)|^2dx.\tag{5.16}$$

Proof. First the solution (5.15) is obtained in the same way as (5.12), so we will omit the details. Then if Fourier transform w.r.t x is used instead of the Laplace transform w.r.t t in this case, one gets [31]

$$\int_{-\infty}^{\infty} |U^{\epsilon}_{IVP}(x,t)|^2 dx \leq O(1) \int_0^{\infty} |U_0(x)|^2 dx, \quad \forall t > 0. \tag{5.17}$$

Integrating with respect to t gives

$$\int_0^{\infty} dt \int_{-\infty}^{\infty} e^{-2\alpha t} |U^{\epsilon}_{IVP}(x,t)|^2 dx \leq O(1) \int_0^{\infty} |U_0(x)|^2 dx.$$

Then by Parseval's identity (4.21), one can prove the inequality. For more details, see [31]. $\square$

One also needs to estimate $\int_0^{\infty} e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}(u_0(y) - v_0(y)g_+(\epsilon\xi))dy$ and $\int_0^{-\infty} e^{-\mu_-(\xi)y}$ $(u_0(y) - v_0(y)g_-(\xi))dy$ which appear in (5.9) and (5.10). The estimate of these two integrals are similar by using the energy estimate. So we only estimate the first integral here.

Lemma 9. *Let*

$$\tilde{w}_{IBVP}(\epsilon\xi) = \int_0^{\infty} e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}(u_0(y) - v_0(y)g_+(\epsilon\xi))dy, \tag{5.18}$$

then

$$\int_{-\infty}^{\infty} |\tilde{w}_{IBVP}(\epsilon\xi)|^2 d\beta \leq O(1) \int_0^{\infty} |U_0(x)|^2 dx. \tag{5.19}$$

Proof. The idea of the proof followes that in [31]. We construct the following initial boundary value problem on the right half plane $x > 0$. Later one can see that $\tilde{w}_{IBVP}(\epsilon\xi)$ can be expressed by the Laplace transform of the boundary value of the following problem, thus can be bounded by the initial data. This is the key motivation of constructing the following system:

$$\begin{cases} u^{\epsilon}_t + v^{\epsilon}_x = 0, & (5.20a) \\ v^{\epsilon}_t + u^{\epsilon}_x = -\dfrac{1}{\epsilon}(v^{\epsilon} - \lambda u^{\epsilon}), & (5.20b) \\ u^{\epsilon}(x,0) = u_0(x), v^{\epsilon}(x,0) = v_0(x), & (5.20c) \\ B_u u^{\epsilon}(0,t) + B_v v^{\epsilon}(0,t) = 0. & (5.20d) \end{cases}$$

Here B_u and B_v are two constants that satisfy the so-called Stiff Kreiss Condition (SKC) [31]: $\frac{B_u}{B_v} \notin [-1, \frac{\lambda+|\lambda|}{2}]$. The solution to this system can be written as:

$$\begin{aligned} \tilde{U}_{IBVP}(x,\xi) \;=\; & e^{\mu_+(\epsilon\xi)\frac{x}{\epsilon}}\Phi_+(\epsilon\xi)(\tilde{U}_{IBVP}(0,\xi) + \int_0^x e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy) \\ & + e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}}\Phi_-(\epsilon\xi)(\tilde{U}_{IBVP}(0,\xi) + \int_0^x e^{-\mu_-(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y) dy), \end{aligned} \tag{5.21}$$

where $\tilde{U}_{IBVP}(0,\xi) = \begin{pmatrix} \tilde{u}_{IBVP} \\ \tilde{v}_{IBVP} \end{pmatrix}$ satisfies

$$\begin{cases} B_u \tilde{u}_{IBVP}(0,\xi) + B_v \tilde{v}_{IBVP}(0,\xi) = 0, & (5.22a) \\ \Phi_+(\epsilon\xi)(\tilde{U}_{IBVP}(0,\xi) + \displaystyle\int_0^\infty e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} A^{-1} U_0(y)dy) = 0. & (5.22b) \end{cases}$$

From definition (5.18), the second condition (5.22b) can be written as

$$g_+(\epsilon\xi)\tilde{u}_{IBVP}(0,\xi) - \tilde{v}_{IBVP}(0,\xi) = \tilde{w}_{IBVP}(\epsilon\xi),$$

thus

$$\tilde{U}_{IBVP}(0,\xi) = \frac{\tilde{w}_{IBVP}(\epsilon\xi)}{B_u + B_v g_+(\epsilon\xi)} \begin{pmatrix} B_v \\ -B_u \end{pmatrix}. \quad (5.23)$$

Now the energy estimate can be used to get the upper bound of $\int_0^T |U_{IBVP}(0,t)|^2 dt$. Let $H = \begin{pmatrix} 1 & -\lambda \\ -\lambda & 1 \end{pmatrix}$, multiply (5.20) by $e^{-2\alpha t}U^T H$, and integrate over $[0,T]\times[0,\infty)$, one has (here we omit the subscription for a while)

$$\begin{aligned} & \frac{1}{2}\int_0^\infty (U,HU)(x,T)e^{-2\alpha T}dx + \alpha\int_0^T\!\!\int_0^\infty (U,HU)(x,t)e^{-2\alpha t}dxdt \\ & +\frac{1}{\epsilon}\int_0^T\!\!\int_0^\infty (v-\lambda u)^2 e^{-2\alpha t}dxdt + \frac{1}{2}\int_0^T (\lambda u^2 - 2uv + \lambda v^2)(0,t)e^{-2\alpha t}dt \\ = & \frac{1}{2}\int_0^\infty (U_0(x), HU_0(x))dx. \end{aligned}$$

One needs to choose the boundary condition such that $\lambda u(0,t)^2 - 2u(0,t)v(0,t) + \lambda v(0,t)^2 \geq c|U(0,t)|^2$, where c is a bounded constant. Later we will show that this kind of boundary condition exits and it is a subclass of SKC. Then one can get

$$\int_0^T |U_{IBVP}(0,t)|^2 e^{-2\alpha t}dt \leq O(1)\int_0^\infty |U_0(x)|^2 dx. \quad (5.24)$$

Let $T \to \infty$, then

$$\int_0^\infty |U_{IBVP}(0,t)|^2 e^{-2\alpha t}dt \leq O(1)\int_0^\infty |U_0(x)|^2 dx. \quad (5.25)$$

By Parseval's identity and (5.23) (5.25), one obtains (5.19). As for the boundary condition, there are plenty of choices. Any B_u and B_v that satisfy

$$\begin{aligned} & \frac{B_u}{B_v} > -\frac{1}{\lambda}(1-\sqrt{1-\lambda^2}) \ \text{ or } \ \frac{B_u}{B_v} < -\frac{1}{\lambda}(1+\sqrt{1-\lambda^2}), \quad \text{for } \lambda > 0, \\ & -\frac{1}{\lambda}(1-\sqrt{1-\lambda^2}) < \frac{B_u}{B_v} < -\frac{1}{\lambda}(1+\sqrt{1-\lambda^2}), \quad \text{for } \lambda < 0, \\ & \frac{B_u}{B_v} > 0, \quad \text{for } \lambda = 0, \end{aligned}$$

will work, and it is not hard to see it is a subclass of the SKC. □

Similarly, we have the following corollary.

Corollary 10. *Let*

$$\tilde{w}_{IBVP2}(\xi) = \int_0^{-\infty} e^{-\mu_-(\xi)y}(u_0(y) - v_0(y)g_-(\xi))dy, \tag{5.26}$$

then

$$\int_{-\infty}^{\infty} |\tilde{w}_{IBVP2}(\xi)|^2 d\beta \leq O(1)\int_{-\infty}^{0} |U_0(x)|^2 dx. \tag{5.27}$$

By looking back to the solution (5.12) for $x > 0$, one gets

$$\begin{aligned} & \int_0^{\infty} dx \int_{-\infty}^{\infty} |\tilde{U}^{\epsilon}(x,\xi)|^2 d\beta \leq \int_0^{\infty} dx \int_{-\infty}^{\infty} |\tilde{U}^{\epsilon}_{IVP}|^2 d\beta \\ & + \int_0^{\infty} dx \int_{-\infty}^{\infty} \left| \begin{pmatrix} 1 \\ g_+(\epsilon\xi) \end{pmatrix} e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} (v_R - g_-(\epsilon\xi)u_R) \right|^2 \frac{1}{|g_+(\epsilon\xi) - g_-(\epsilon\xi)|^2} d\beta \\ = \; & I_1 + I_2. \end{aligned} \tag{5.28}$$

By (5.16) I_1 can be estimated as:

$$I_1 \leq O(1)\int_0^{\infty} |U_0(x)|^2 dx. \tag{5.29}$$

As for I_2, since $\frac{1}{|g_+(\epsilon\xi) - g_-(\epsilon\xi)|}$ is uniformly bounded, one has

$$I_2 \leq O(1)\int_0^{\infty} dx \int_{-\infty}^{\infty} e^{2Re\mu_-(\epsilon\xi)\frac{x}{\epsilon}} [|v_R|^2 + O(1)|u_R|^2] d\beta.$$

Then by (5.9) (5.10) (5.18) and (5.26), one obtains:

$$\begin{aligned} g_+(\epsilon\xi)u_R - v_R &= \tilde{w}_{IBVP}, \\ -g_-(\xi)u_R - v_R &= \tilde{w}_{IBVP2}. \end{aligned}$$

Thus $u_R = O(1)\tilde{w}_{IBVP}(\epsilon\xi) + O(1)\tilde{w}_{IBVP2}(\xi)$, $v_R = O(1)\tilde{w}_{IBVP}(\epsilon\xi) + O(1)\tilde{w}_{IBVP2}(\xi)$. Finally by Lemma 7 and Corollary 9,

$$I_2 \leq -O(1)\frac{\epsilon}{2Re\mu_-(\epsilon\xi)} \int_{-\infty}^{\infty} |U_0(x)|^2 dx. \tag{5.30}$$

Then by (4.17) (4.18), one sees that $\int_0^{\infty} dx \int_{-\infty}^{\infty} |\tilde{U}^{\epsilon}(x,\xi)|^2 d\beta$ is uniformly bounded. In the same way, one can prove

$$\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |\tilde{U}^{\epsilon}(x,\xi)|^2 d\beta \leq O(1)\int_{-\infty}^{\infty} |U_0(x)|^2 dx. \tag{5.31}$$

Till now we have proved the stiff well-posedness of the original system.

5.3 The Asymptotic convergence

Next we will prove the asymptotic convergence. The first step is also using the Laplace transform to represent the exact solution. We will consider the case $\lambda < 0$ first. Consider the domain decomposition system (3.2)–(3.1). When $x > 0$, the solution is $u^0(x,t) = u_0(x - \lambda t)$. After the Laplace transform, one gets:

$$\tilde{u}^0(x,\xi) = -\frac{1}{\lambda}\int_x^\infty u_0(y)e^{-\frac{\xi}{\lambda}(x-y)}dy, \tag{5.32}$$

$$\tilde{v}^0 = \lambda \tilde{u}^0. \tag{5.33}$$

For $x < 0$, the solution to (3.2) can be represented as

$$\begin{aligned}\tilde{\overline{U}}(x,\xi) &= e^{\mu_+(\xi)x}\Phi_+(\xi)(\tilde{D}(\xi) + \int_0^x e^{-\mu_+(\xi)y}A^{-1}U_0(y)dy)\\ &\quad + e^{\mu_-(\xi)x}\Phi_-(\xi)(\tilde{D}(\xi) + \int_0^x e^{-\mu_-(\xi)y}A^{-1}U_0(y)dy).\end{aligned} \tag{5.34}$$

Here $\tilde{D}(\xi) = \begin{pmatrix} \tilde{D}_u(\xi) \\ \tilde{D}_v(\xi) \end{pmatrix}$ is determined by:

$$(-g_-(\xi)\ 1)\begin{pmatrix} \tilde{D}_u(\xi) \\ \tilde{D}_v(\xi) \end{pmatrix} + \int_0^{-\infty} e^{-\mu_-(\xi)y}(-g_-(\xi)\ 1)\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}(y)dy = 0, \tag{5.35}$$

$$\frac{1}{g_+(\xi) - g_-(\xi)}[(\tilde{D}_u(\xi)g_+(\xi) - \tilde{D}_v(\xi))g_-(\xi) + (\tilde{D}_v(\xi) - \tilde{D}_u(\xi)g_-(\xi))g_+(\xi)] = -\int_0^\infty u_0(y)e^{\frac{\xi}{\lambda}y}dy,$$

where the second equation is simplified as

$$\tilde{D}_v(\xi) = -\int_0^\infty u_0(y)e^{\frac{\xi}{\lambda}y}dy. \tag{5.36}$$

Now one needs to compare (5.32) with (5.8), and (5.34) with (5.7). Still, in order to avoid the difficulties caused by Laplace transform, we turn to the help of the initial value problem (5.14) with its reduced system:

$$\begin{cases} u_t^0 + \lambda u_x^0 = 0, & (5.37a) \\ u^0(x,0) = u_0(x). & (5.37b) \end{cases}$$

Here we assume $u_0(x)$ is supported on $[0,\infty)$.

Lemma 11. *Let U_{IVP}^ϵ and U_{IVP}^0 be the solution of (5.14) and (5.37) respectively, then*

$$\int_0^\infty dx \int_{-\infty}^\infty |\tilde{U}_{IVP}^\epsilon - \tilde{U}_{IVP}^0|^2 d\beta \le O(1)\epsilon^2 \|U_0\|_{H^2}^2 + O(1)\epsilon\|v_0 - \lambda u_0\|_{L^2[0,\infty)}^2. \tag{5.38}$$

Proof. The proof is based on the Fourier transform, and one can refer to [31] for details. □

Go back to the difference of (5.32) and (5.12), since the solution to (5.37) is (5.32), and part of (5.12) is (5.15), one has

$$\begin{aligned}
&\int_0^\infty dx \int_{-\infty}^\infty |\tilde{u}^\epsilon - \tilde{\bar{u}}|^2 d\beta \leq \int_0^\infty dx \int_{-\infty}^\infty |\tilde{u}^\epsilon_{IVP} - \tilde{u}^0_{IVP}|^2 d\beta \\
&+ \int_0^\infty dx \int_{-\infty}^\infty \left| \frac{1}{g_+(\epsilon\xi) - g_-(\epsilon\xi)} \right|^2 (1 + |g_+(\epsilon\xi)|^2) |e^{\mu_-(\epsilon\xi)\frac{x}{\epsilon}} (\tilde{v}_R(\xi) - g_-(\epsilon\xi)\tilde{u}_R(\xi))|^2 d\beta \\
= \;& \mathbb{I}_1 + \mathbb{I}_2,
\end{aligned}$$

$$\begin{aligned}
\mathbb{I}_1 &\leq O(1)\epsilon^2 \|U_0\|^2_{H^2} + O(1)\epsilon \|v_0 - \lambda u_0\|^2_{L^2[0,\infty)}, && (5.39)\\
\mathbb{I}_2 &\leq O(1) \int_0^\infty e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}} dx \int_{-\infty}^\infty |\tilde{v}(\xi)_R - g_-(\epsilon\xi)\tilde{u}_R(\xi)|^2 d\beta \\
&\leq O(1)\epsilon \|U_0\|^2_{L^2}. && (5.40)
\end{aligned}$$

The calculation of the last inequality is the same as (5.30). Notice here that the term that contains $e^{2\mathrm{Re}\mu_-(\epsilon\xi)\frac{x}{\epsilon}}$ is due to the interface layer, as we expected here that the initial data can induce an interface layer at the interface in this case.

Now compare (5.34) with (5.7). The difference comes from the difference in coefficients, thus

$$\begin{aligned}
&\int_{-\infty}^0 dx \int_{-\infty}^\infty |\tilde{\bar{U}} - \tilde{U}^\epsilon|^2 d\beta \\
\leq \;& O(1) \int_{-\infty}^0 dx \int_{-\infty}^\infty |e^{\mu_+(\xi)x}|^2 (1 + |g_-(\xi)|^2) [g_+(\xi)(\tilde{D}_u - \tilde{u}_L) - (\tilde{D}_v - \tilde{v}_L)]^2 d\beta \\
&+ \int_{-\infty}^0 dx \int_{-\infty}^\infty |e^{\mu_-(\xi)x}|^2 (1 + |g_+(\xi)|^2) [-g_-(\xi)(\tilde{D}_u - \tilde{u}_L) + (\tilde{D}_v - \tilde{v}_L)]^2 d\beta.
\end{aligned}$$

By boundary conditions (5.10) and (5.35), the second term vanishes, so

$$\int_{-\infty}^0 dx \int_{-\infty}^\infty |\tilde{\bar{U}} - \tilde{U}^\epsilon|^2 d\beta \leq O(1) \int_{-\infty}^0 dx \int_{-\infty}^\infty e^{2Re\mu_+(\xi)x} (|\tilde{D}_u - \tilde{u}_L|^2 + |\tilde{D}_v - \tilde{v}_L|^2) d\beta.$$

Next compare (5.9)–(5.11) with (5.35) (5.36), one gets

$$
\begin{aligned}
&\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} |\tilde{\overline{U}} - \tilde{U}^{\epsilon}|^2 d\beta = O(1) \int_{-\infty}^{\infty} (|\tilde{D}_u - \tilde{u}_L|^2 + |\tilde{D}_v - \tilde{v}_L|^2) d\beta \\
= \;& O(1) \int_{-\infty}^{\infty} |\tilde{D}_v - \tilde{v}_L|^2 d\beta \\
= \;& O(1) \int_{-\infty}^{\infty} \left| - \int_0^{\infty} e^{\frac{\xi}{\lambda} y} u_0(y) dy - \frac{-\tilde{w}_{IBVP} + \frac{g_+(\epsilon\xi)}{g_-(\xi)} \tilde{w}_{IBVP2}}{1 - \frac{g_+(\epsilon\xi)}{g_-(\xi)}} \right|^2 d\beta \\
\le \;& O(1) \int_{-\infty}^{\infty} \left| \int_0^{\infty} (u_0(y) - v_0(y) g_+(\epsilon\xi)) e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} dy - \int_0^{\infty} u_0(y) e^{\frac{\epsilon}{\lambda} y} dy \right|^2 d\beta \\
& + O(1) \int_{-\infty}^{\infty} \left| g_+(\epsilon\xi) \int_0^{-\infty} (u_0(y) - v_0(y) g_-(\xi)) e^{-\mu_-(\xi) y} dy \right|^2 d\beta \\
& + O(1) \int_{-\infty}^{\infty} \left| g_+(\epsilon\xi) \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda} y} dy \right|^2 d\beta \\
= \;& \mathbb{J}_1 + \mathbb{J}_2 + \mathbb{J}_3.
\end{aligned}
$$

We begin with the simplest part $\mathbb{J}_3$ first. Since when $\lambda < 0$, $g_+(\epsilon\xi) = O(1)\epsilon\xi$, thus

$$
\mathbb{J}_3 \le O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \xi \int_0^{\infty} u_0(y) e^{\frac{\xi}{\lambda} y} dy \right|^2 d\beta.
$$

If the compatibility condition on $u_0(y)$ is assumed such that $u_0(0) = 0$, $\int_0^{\infty} \xi u_0(y) e^{\frac{\xi}{\lambda} y} dy$ can be considered as the Laplace transform to $u_0'(y)$, so

$$
\mathbb{J}_3 \le O(1)\epsilon^2 \int_{-\infty}^{\infty} |\mathcal{L}(u_0'(y))(\xi)|^2 d\beta \le O(1)\epsilon^2 \int_0^{\infty} |u_0'(y)|^2 dy. \tag{5.41}
$$

Next we look at $\mathbb{J}_2$. Similar to $\mathbb{J}_3$, one will first get

$$
\mathbb{J}_2 \le O(1)\epsilon^2 \int_{-\infty}^{\infty} \left| \xi \int_0^{-\infty} (u_0(y) - v_0(y) g_-(\xi)) e^{-\mu_+(\xi) y} dy \right|^2 d\beta.
$$

Recall (5.26) and integration by parts, one gets

$$
\tilde{w}_{IBVP2} = -\frac{1}{\mu_-(\xi)} \int_0^{\infty} e^{-\mu_-(\xi) y} (-u_0' + g_-(\xi) v_0') dy, \tag{5.42}
$$

where the compatibility condition $u_0(0) = 0$ and $v_0(0) = 0$ are used. Since $-\mu_-(\xi) = \mu_+(\xi) - 2\lambda$, one has

$$
(\mu_+(\xi) - 2\lambda) \tilde{w}_{IBVP2} = \int_0^{\infty} e^{-\mu_-(\xi) y} (-u_0' + g_-(\xi) v_0') dy.
$$

Notice when $\lambda < 0$, $\mu_+(\xi) = -\frac{\xi}{g_-(\xi)}$, thus

$$
\xi \tilde{w}_{IBVP2} = 2\lambda g_-(\xi) \tilde{w}_{IBVP2} + g_-(\xi) \int_0^{\infty} (-u_0'(y) + g_-(\xi) v_0'(y)) dy.
$$

Therefore, the following estimate holds:

$$\int_{-\infty}^{\infty}|\xi\tilde{w}_{IBVP2}|^2 d\beta \leq O(1)\int_{-\infty}^{\infty}\left|\int_0^{-\infty}e^{-\mu_-(\xi)y}(u_0'-g_-(\xi)v_0')(y)dy\right|^2 d\beta + O(1)\int_{-\infty}^{\infty}|\tilde{w}_{IBVP2}(\xi)|^2 d\beta. \tag{5.43}$$

The integral with respect to y on the right hand side is similar to $\tilde{w}_{IBVP2}$ in (5.26), except to change u_0 and v_0 to u_0' and v_0'. So one has

$$\int_{-\infty}^{\infty}|\xi\tilde{w}_{IBVP2}|^2 d\beta \leq O(1)\int_0^{\infty}|U_0'(x)|^2 dx + O(1)\int_0^{\infty}|U_0(x)|^2 dx. \tag{5.44}$$

Therefore,

$$\mathbb{J}_2 \leq O(1)\epsilon^2\int_0^{\infty}|U_0'(x)|^2 dx. \tag{5.45}$$

Now we turn to $\mathbb{J}_1$. First using $g_+(\epsilon\xi)\sim O(1)\epsilon\xi$ gives

$$\mathbb{J}_1 \leq \int_{-\infty}^{\infty}\epsilon^2\left|\xi\int_0^{\infty}v_0 e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}dy\right|^2 d\beta + \int_{-\infty}^{\infty}\left|\int_0^{\infty}u_0(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y})(y)dy\right|^2 d\beta. \tag{5.46}$$

Notice in (5.18) if one exchanges u_0 and v_0 and let $u_0\equiv 0$, then use (5.18) (5.43), and similar to (5.44), one will get:

$$\int_{-\infty}^{\infty}\left|\xi\int_0^{\infty}v_0 e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}}dy\right|^2 d\beta \leq O(1)\int_0^{\infty}|v_0'(x)|^2 dx. \tag{5.47}$$

On the other hand,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left|\int_0^{\infty}u_0(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}} - e^{\frac{\xi}{\lambda}y})(y)dy\right|^2 d\beta\\
=\;&\int_{-\infty}^{\infty}\left|\int_0^{\infty}u_0(y)e^{\frac{\xi}{\lambda}y}(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}-\frac{\xi}{\lambda}y} - 1)dy\right|^2 d\beta.
\end{aligned}$$

Since for $\lambda<0$, similar to (4.28), one has $-\frac{\mu_+(\epsilon\xi)}{\epsilon}-\frac{\xi}{\lambda}=a(\epsilon\xi,\lambda)\epsilon\xi^2$, here $a(\epsilon\xi,\lambda)$ is a O(1) function. Denote the upper bound of a as c, i.e., $|a|<c$. Therefore,

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left|\int_0^{\infty}u_0(y)e^{\frac{\xi}{\lambda}y}(e^{-\mu_+(\epsilon\xi)\frac{y}{\epsilon}-\frac{\xi}{\lambda}y} - 1)dy\right|^2 d\beta\\
=\;&\int_{-\infty}^{\infty}\left|\int_0^{\infty}u_0(y)e^{\frac{\xi}{\lambda}y}(e^{a\epsilon\xi^2 y} - 1)dy\right|^2 d\beta\\
=\;&\int_{-\infty}^{\infty}\left|\int_0^{\infty}u_0(y)e^{\frac{\xi}{\lambda}y}\sum_{k=1}^{\infty}(a\epsilon\xi^2 y)^k\frac{1}{k!}dy\right|^2 d\beta\\
\leq\;& 2\int_{-\infty}^{\infty}\sum_{k=1}^{\infty}(\frac{1}{k!})^2|a|^{2k}\epsilon^{2k}|\xi^2|^{2k}\left|\int_0^{\infty}u_0(y)e^{\frac{\xi}{y}}y^k dy\right|^2 d\beta\\
=\;& 2\sum_{k=1}^{\infty}(\frac{1}{k!})^2\int_{-\infty}^{\infty}|a\epsilon|^{2k}\left|\int_0^{\infty}\xi^{2k}y^k u_0(y)e^{\frac{\xi}{\lambda}y}dy\right|^2 d\beta.
\end{aligned}$$

Let ϵ be small enough such that $|\epsilon c| < 1$, and observe that $\int_0^\infty \xi^{2k} u_0(y) e^{\frac{\xi}{\lambda} y} dy$ can be considered the Laplace transform of the $2k$-th derivative of $u_0(y)$, so if one further assumes $u_0 \in C^\infty$, and u_0 has compact support, then use the same trick as we prove (5.47), one will get:

$$
\begin{aligned}
& \sum_{k=1}^{\infty} (\frac{1}{k!})^2 \int_{-\infty}^{\infty} |a\epsilon|^{2k} \left| \int_0^\infty \xi^{2k} y^k u_0(y) e^{\frac{\xi}{\lambda} y} dy \right|^2 d\beta \\
\leq \quad & O(1) \sum_{k=1}^{\infty} |\epsilon c|^{2k} \parallel u_0(y)^{(2k)} \parallel_{L^2}^2 \leq O(1)\epsilon^2 \, \|u_0(y)\|_{H^2}^2 \, (1 + o(\epsilon)).
\end{aligned}
$$

Here the term $o(\epsilon)$ will depend on the norm $\|(u_0)^{(2k)}\|_{L^2} \quad (k \geq 2)$, but our assumption on u_0 will guarantee that this term is a higher order term w.r.t ϵ. Therefore, one arrives at the estimate for $\mathbb{J}_3$:

$$
\mathbb{J}_3 \leq O(1)\epsilon^2 \|u_0(y)\|_{H^2}^2 (1 + o(\epsilon)) + O(1)\epsilon^2 \|v_0\|_{H^1}^2. \tag{5.48}
$$

In summary,

$$
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |\tilde{U}^\epsilon - \tilde{U}|^2 d\beta \leq O(1)\epsilon \|v_0 - \lambda u_0\|_{L^2}^2 + O(1)\epsilon \|U_0\|_{L^2[0,\infty)}^2 + O(1)\epsilon^2 \|U_0\|_{H^2}^2 (1 + o(\epsilon)). \tag{5.49}
$$

The case with $\lambda > 0$ is rather similar, but there is no interface layer at $x = 0$, so one will find the term that contains $\|U_0\|_{L^2}^2$ will have a convergence rate $O(1)\epsilon^2$ instead of $O(1)\epsilon$.

6 Domain-decomposition based numerical schemes and numerical experiments

We use Δt and Δx to represent the time step and mesh size respectively, u_j^n denotes the value at time $n\Delta t$ and space $j\Delta x$. Let $M = T/\Delta t$, and $N = 2L/\Delta x$. We use the upwind scheme to the Riemann invariants $u \pm v$ to solve the left part and use the Godunov scheme to solve the equilibrium equation.

6.1 The numerical scheme

Case I: $f'(u) < 0$

- **Step 1.** For $j = N/2 + 1, ...N$, $n = 0, 1, ...M$, solve

$$
\frac{\widetilde{u}_j^{n+1} - \widetilde{u}_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0, \tag{6.1}
$$

$$
\widetilde{u}_j^0 = u_0(x_j), \quad \widetilde{v}_j^0 = v_0(x_j), \tag{6.2}
$$

$$
\widetilde{u}_N^n = b_R(t^n), \tag{6.3}
$$

where $F^n_{j+\frac{1}{2}} = f(R(0, \widetilde{u}^n_j, \widetilde{u}^n_{j+1}))$, $F^n_{j-\frac{1}{2}} = f(R(0, \widetilde{u}^n_{j-1}, \widetilde{u}^n_j))$, and $R(0, \zeta, \eta)$, the Riemann solver, is defined as:

$$R(0,\zeta,\eta) = \begin{cases} \zeta, & \text{if } f'(\zeta),\ f'(\eta) \le 0, \\ \eta, & \text{if } f'(\zeta),\ f'(\eta) \ge 0, \\ \zeta, & \text{if } f'(\zeta) > 0 > f'(\eta),\ s > 0, \\ \eta, & \text{if } f'(\zeta) > 0 > f'(\eta),\ s < 0, \\ f'^{-1}(0), & \text{otherwise} \end{cases}$$

where $s = \frac{f(\zeta)-f(\eta)}{\zeta-\eta}$ is the shock speed.

- **Step 2.** For $j = 0, 1, ...N/2$, $n = 0, 1, ...M$, let the Riemann invariants $\overline{P}^n_j = \overline{u}^n_j + \overline{v}^n_j, \overline{Q}^n_j = \overline{u}^n_j - \overline{v}^n_j$, and solve

$$\frac{\overline{P}^{n+1}_j - \overline{P}^n_j}{\Delta t} + \frac{\overline{P}^n_j - \overline{P}^n_{j-1}}{\Delta x} = -\frac{1}{c_1}(\overline{v}^n_j - f(\overline{u}^n_j)), \tag{6.4}$$
$$\frac{\overline{Q}^{n+1}_j - \overline{Q}^n_j}{\Delta t} - \frac{\overline{Q}^n_{j+1} - \overline{Q}^n_j}{\Delta x} = \frac{1}{c_1}(\overline{v}^n_j - f(\overline{u}^n_j)), \tag{6.5}$$
$$\overline{P}^0_j = u_0(x_j) + v_0(x_j), \quad \overline{Q}^0_j = u_0(x_j) - v_0(x_j), \tag{6.6}$$
$$\overline{u}^{n+1}_0 = b_L(t^{n+1}), \quad \overline{v}^{n+1}_{\frac{N}{2}} = \widetilde{v}^{n+1}_{\frac{N}{2}}; \tag{6.7}$$

where $\widetilde{v}^{n+1}_{\frac{N}{2}}$ was obtained from Step 1.

Case II: $f'(u) > 0$

- **Step 1.** For $j = 0, 1, ...N/2$, $n = 0, 1, ...M$, let the Riemann invariants $\overline{P}^n_j = \overline{u}^n_j + \overline{v}^n_j$, $\overline{Q}^n_j = \overline{u}^n_j - \overline{v}^n_j$, then solve

$$\frac{\overline{P}^{n+1}_j - \overline{P}^n_j}{\Delta t} + \frac{\overline{P}^n_j - \overline{P}^n_{j-1}}{\Delta x} = -\frac{1}{\epsilon}(\overline{v}^n_j - f(\overline{u}^n_j)), \tag{6.8}$$
$$\frac{\overline{Q}^{n+1}_j - \overline{Q}^n_j}{\Delta t} - \frac{\overline{Q}^n_{j+1} - \overline{Q}^n_j}{\Delta x} = \frac{1}{\epsilon}(\overline{v}^n_j - f(\overline{u}^n_j)) \tag{6.9}$$
$$\overline{P}^0_j = u_0(x_j) + v_0(x_j), \quad \overline{Q}^0_j = u_0(x_j) - v_0(x_j), \tag{6.10}$$
$$\overline{u}^{n+1}_0 = b_1(t^{n+1}), \tag{6.11}$$
$$\overline{P}^{n+1}_{\frac{N}{2}} = \overline{u}^{n+1}_{\frac{N}{2}} + f(\overline{u}^{n+1}_{\frac{N}{2}}); \tag{6.12}$$

- **Step 2.** For $j = N/2+1, ...N$, $n = 0, 1, ...M$, solve

$$\frac{\widetilde{u}_j^{n+1} - \widetilde{u}_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}^n - F_{j-\frac{1}{2}}^n}{\Delta x} = 0, \tag{6.13}$$

$$\widetilde{u}_j^0 = u_0(x_j), \quad \widetilde{v}_j^0 = v_0(x_j), \tag{6.14}$$

$$\widetilde{u}_{\frac{N}{2}}^{n+1} = \overline{u}_{\frac{N}{2}}^{n+1}, \tag{6.15}$$

  where $F_{j+\frac{1}{2}}^n$ and $F_{j-\frac{1}{2}}^n$ are defined as in Case I. To solve for $\overline{u}_{\frac{N}{2}}^{n+1}$, since (6.8) is an explicit scheme for $\overline{P}^{n+1}$, we first use it to get $\overline{P}_{\frac{N}{2}}^{n+1}$, and then use Newton iteration for (6.12) to get $\overline{u}_{\frac{N}{2}}^{n+1}$.

Finally, the numerical solution is obtained by piecing together solutions in the two domains:

$$\begin{cases} u_j^n = \overline{u}_j^n, \quad v_j^n = \overline{v}_j^n \;, \; j = 0, ...\dfrac{N}{2}, n = 0, ...M, & (6.16a) \\ u_j^n = \widetilde{u}_j^n, \quad v_j^n = f(\widetilde{u}_j^n)\,, \quad j = \dfrac{N}{2}+1, ...N, n = 0, ...M. & (6.16b) \end{cases}$$

6.2 Numerical examples

The first two examples are given to validate our domain decomposition system numerically. Therefore we focus on the behavior of L^1 error with a changing ϵ (we only change ϵ for $x > 0$, for $x < 0$, let $\epsilon = 1$). Here we use $\Delta x = 10^{-3}$, $\Delta t = 2.5 \times 10^{-4}$, and run the algorithm to T=0.2. We change ϵ from 0.05 to 0.0025, then calculate the error

$$U_{L^1} = \max_{0 \le n \le M} \sum_{j=0}^{N} |(u^\epsilon)_j^n - u_j^n|\Delta x, \quad V_{L^1} = \max_{0 \le n \le M} \sum_{j=0}^{N} |(v^\epsilon)_j^n - v_j^n|\Delta x.$$

Here $(u^\epsilon)_j^n$ and $(v^\epsilon)_j^n$ are obtained by directly solving the original system (1.1)–(1.5) (hereafter called the relaxation method).

Example 1. Let $f(u^\epsilon) = \frac{1}{4}(e^{-u^\epsilon} - 1)$ in (1.1), with initial condition $u^\epsilon(x, 0) = sin(\pi x)^3$, and boundary condition $u(-1, t) = u(1, t) = 0$. In this case, $f'(u) < 0$, so there will be an interface layer at the interface $x = 0$. Figure fig:error1 gives the $\log(error)$ versus $\log(\epsilon)$ and one can see that the convergence rate is $O(\epsilon)$.

Example 2. Now we consider the case $f'(u) > 0$. Let $f(u^\epsilon) = \frac{1}{4}(e^{u^\epsilon} - 1)$, initial condition $u^\epsilon(x, 0) = \sin(\pi x)^3$, and boundary condition $u(-1, t) = u(1, t) = 0$. Still one sees that the convergence rate is $O(\epsilon)$, as shown in Figure fig:error2.

Next we will compare our domain decomposition method with the relaxation method. Thus in the following examples we will use underresolved mesh, and let $\epsilon = 0.002$ be fixed for $x > 0$. We use the relaxation method with resolved mesh to serve as the analytical solution to (1.1)–(1.5).

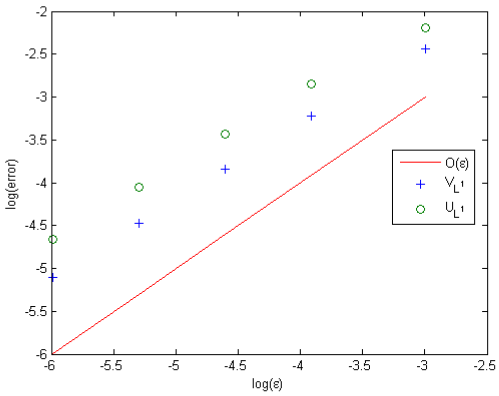


Figure 1: convergence rate for Example 1

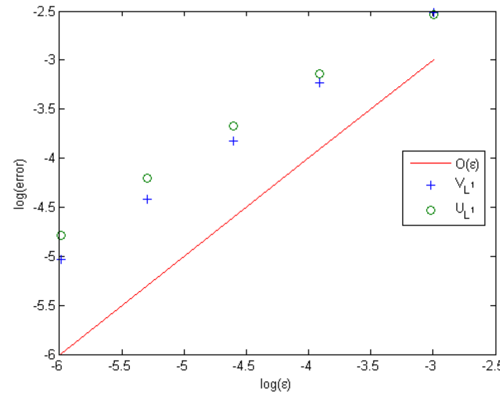


Figure 2: convergence rate for Example 2

Example 3. The set up is the same as Example 1, and here fix ϵ to be 0.002 for $x > 0$. The solutions are plotted at T=0.5. In this case, there is an interface layer in u at$x = 0$, as one can see from Figures 3 and 4. In comparison, one can see that the relaxation method with a relatively large mesh size gives poor results at the interface which results in larger numerical errors away from the interface. The error becomes smaller if the mesh size is reduced (yet still underresolved). On the other hand, the domain decomposition method gives good approximation even when the mesh size is large ($\Delta x >> \epsilon$).

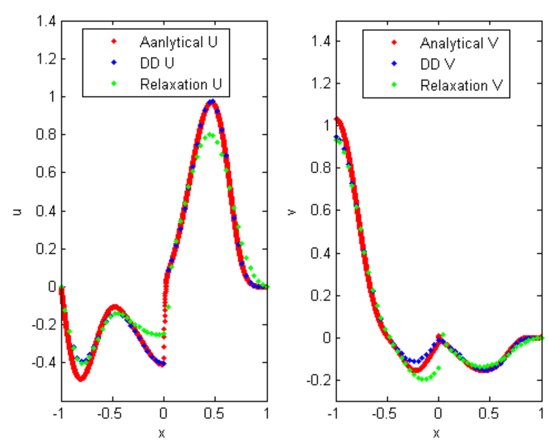


Figure 3: Example 3, $\Delta x = 0.04$, $\Delta t = 0.02$.

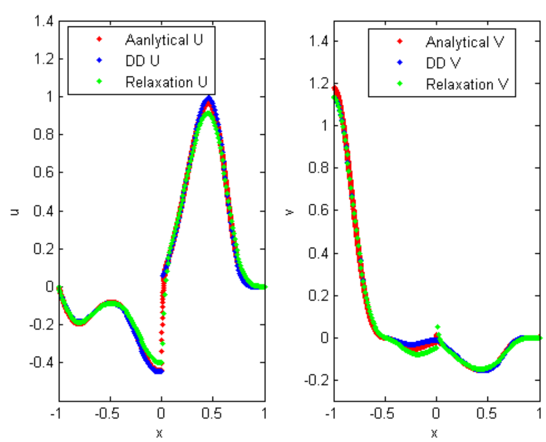
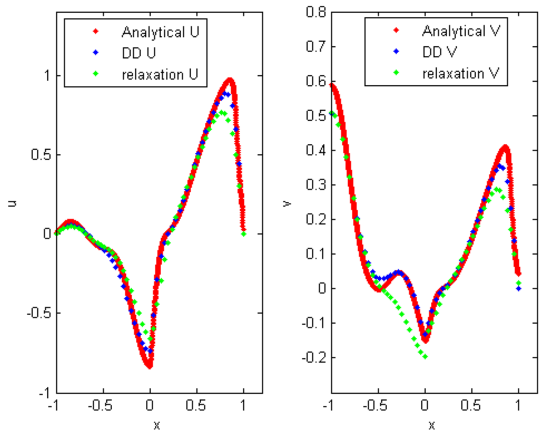


Figure 4: Example 3, $\Delta x = 0.01$, $\Delta t = 0.005$.

Example 4. The set up is the same as Example 2. The results at $T = 0.6$ are plotted in Figure 5 and Figure 6. Similar to Example 3, one can find that the relaxation method behaves much better with the decreasing of the mesh size, while the domain decomposition method gives good approximation even with the large mesh size compared to ϵ.

Example 5. Let $f(u^\epsilon)$ be the same as in Example 2, but consider the Riemann initial data:

$$u^\epsilon(x, 0) = \begin{cases} -1, & \text{if } -1 \le x \le -0.2, \\ 1, & \text{if } -0.2 < x \le 1. \end{cases}$$



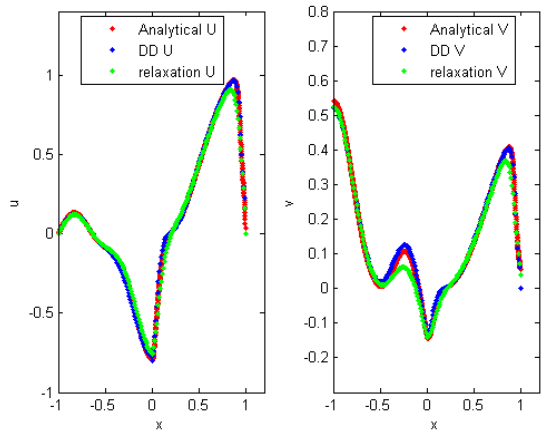


Figure 5: Example 4, $\Delta x = 0.04$, $\Delta t = 0.02$. Figure 6: Example 4, $\Delta x = 0.01$, $\Delta t = 0.005$.

In this case a contact discontinuity formed at the left hand side will propagate across the interface to the right. Let $\Delta x = 0.02$, $\Delta t = 0.01$. From Figure 7, one will see that, before the contact discontinuity passes through the interface, there is not much difference between the relaxation method and the domain decomposition method, but after that the domain decomposition method has an obvious advantage in producing more accurate results. The results are given at different times to show the dynamics of the solution.

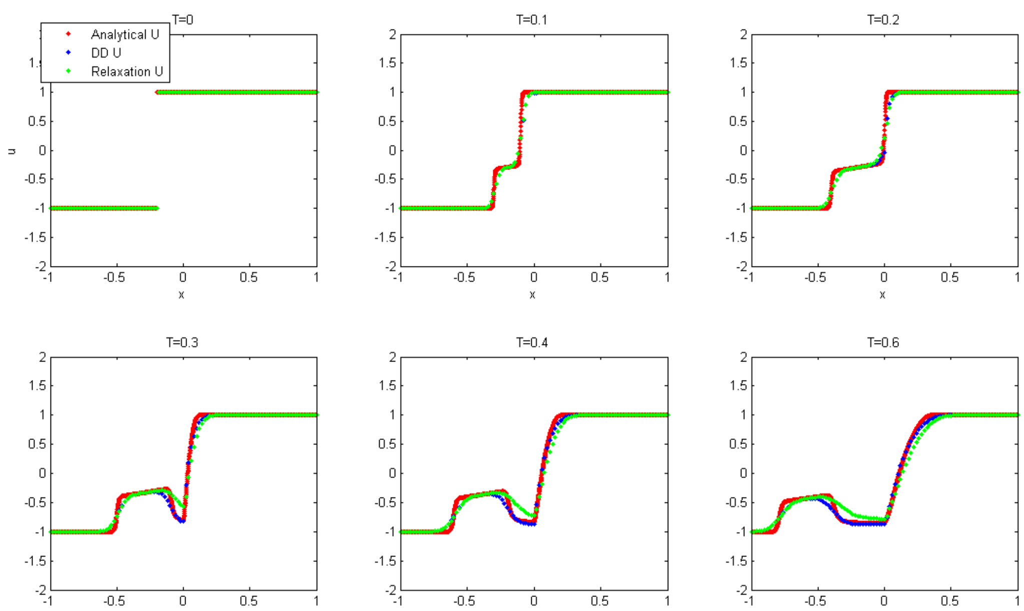


Figure 7: Example 5, a contact discontinuity passing through the interface.

Example 6. Let $f(u^\epsilon)$ be the same as in Example 1, and consider the following Riemann initial data:

$$u^\epsilon(x,0) = \begin{cases} 1, & \text{if } -1 \le x \le 0.2, \\ -1, & \text{if } 0.2 < x \le 1. \end{cases}$$

Here we use $\Delta x = 0.02$ and $\Delta t = 0.01$. In this case, a shock forms at the right region and propagates to the left region. From Figure 8, one can see that, when the shock crosses the interface, the domain decomposition method gives spurious solution at the interface. This is because our interface layer analysis assumes that the solution is smooth, yet here the interaction between the interface layer and shock complicates the problem, thus our domain decomposition system may not be valid here.

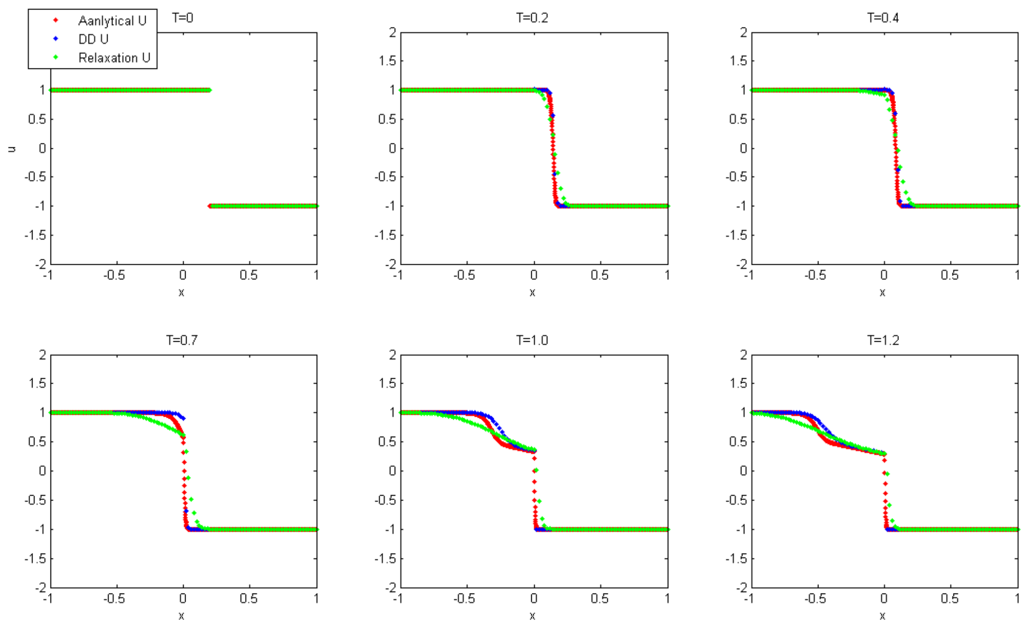


Figure 8: Example 6, a shock from the right region passing through the interface.

Example 7. Let $f(u^\epsilon)$ be the same as in Example 1, and consider the following Riemann initial data:

$$u^\epsilon(x,0) = \begin{cases} -1, & \text{if } -1 \le x \le 0.2, \\ 1, & \text{if } 0.2 < x \le 1. \end{cases}$$

With this initial data, a rarefaction wave forms in the right region, and propagates across the interface to the left. We still let $\Delta x = 0.02$ and $\Delta t = 0.01$, and the solutions are plotted at different times in Figure 9. One can see that, unlike a shock, the domain decomposition method gives a good approximation when the rarefaction wave crosses the interface.

7 Conclusion

In this paper, a domain decomposition method is presented and analyzed on a semilinear hyperbolic system with multiple relaxation times. In the region where the relaxation time

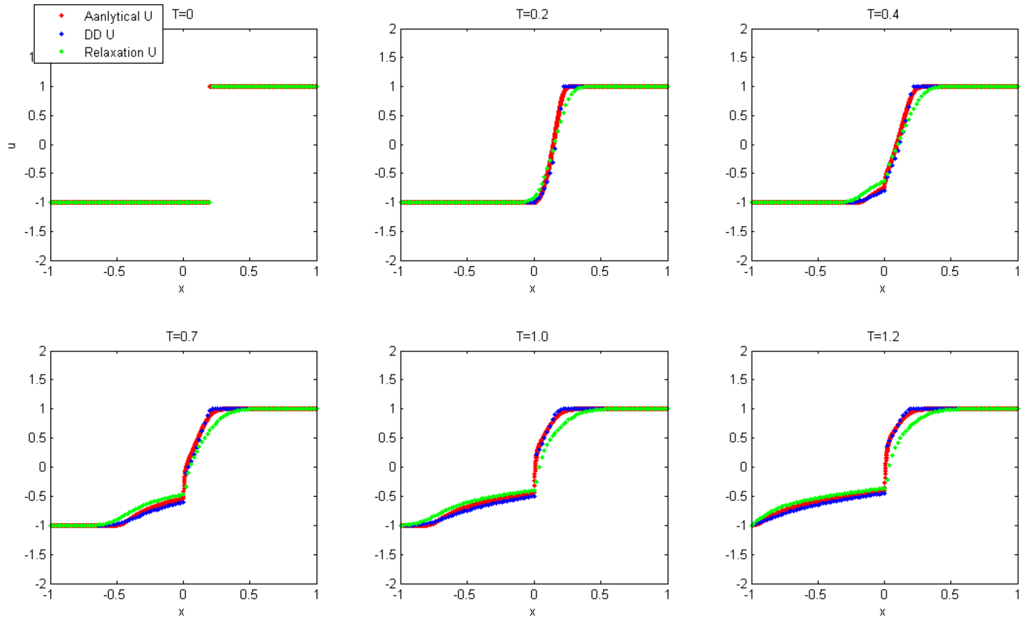


Figure 9: Example 7, rarefaction wave

is small, an asymptotic equilibrium equation is used for computational efficiency which is coupled with the original relaxation system on the other part of the region through an interface condition. A rigorous analysis established the well-posedness and error estimate in terms of the relaxation time on this domain decomposition method, and numerical results are presented to study the performance of this method.

This is a prototype model for the more general coupling of kinetic and hydrodynamic equations which are competitive multiscale computational methods using multi-physics, thus a deep mathematical understanding of this simpler model problem will shed light on the more general physical problems.

There are still remaining problems to be studied. Among them we mention the problem of shock passing through the interface, transonic solutions in the equilibrium domain, nonlinear hyperbolic systems with relaxation, and the error estimate on the numerical schemes based on such a domain decomposition method.

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