On the Rayleigh-Taylor instability for incompressible, inviscid magnetohydrodynamic flows

Ran Duan[a,b], Fei Jiang[a,*], Song Jiang[a]

[a]*Institute of Applied Physics and Computational Mathematics, Beijing, 100088, China.*
[b]*School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, China.*

Abstract

We study the Rayleigh-Taylor instabolity for two incompressible, immiscible, inviscid magnetohydrodynamic (MHD) fluids with zero resistivity, evolving with a free interface in the presence of a uniform gravitational field. We first construct the Rayleigh-Taylor steady-state solution with a denser fluid lying above the light one. Then, we turn to an analysis of the equations obtained from linearizing around such a steady state. By solving a system of ordinary differential equations, we construct the normal mode solutions to the linearized problem that grow exponentially in time. A Fourier synthesis of these normal mode solutions allows us to construct solutions that grow arbitrarily quickly in the Sobolev space H^k, thus leading to an ill-posedness result for the linearized problem in the sense of Hadamard. Using these pathological solutions, we can then demonstrate the ill-posedness of the original non-linear problem in some sense.

Keywords: Rayleigh-Taylor instability, MHD, ill-posedness, Hadamard sense.

1. Introduction

Consider two completely plane-parallel layers of immiscible fluid, the heavier on top of the light one and both subject to the earth's gravity. In this case, the equilibrium is unstable to sustain small perturbations or disturbances. An unstable disturbance will grow and lead to a release of potential energy, as the heavier fluid moves down under the (effective) gravitational field, and the lighter one is displaced upwards. This phenomenon was first studied by Rayleigh [7, 8] and then Taylor [10], and therefore, is called Rayleigh-Taylor instability. In the last decades, a lot of works related to this phenomena have been made from both physical and numerical point of view. However, there are only few analytical results published in the recent years. In 2011, Y. Guo and I. Tice established a variational framework for nonlinear instability in [3], where with the help of the method of Fourier synthesis, they constructed solutions that grow arbitrarily quickly in time in the Sobolev space lead to the ill-posedness of the perturbed problem. It should be noted that they also investigated the stabilized effect of viscosity and surface tension to the linear Rayleigh-Taylor instability (see [4]).

The magnetohydrodynamic (MHD) analogue of the Rayleigh-Taylor instability arises when the fluids are electrically conducting and a magnetic field is present, and the growth of the instability will be influenced by the magnetic field due to the generated electromagnetic induction and the Lorentz force. This has been analyzed from the physical point of view in many monographs, see, for example, [1, 11]. Recently, Hwang [5] investigated the MHD Rayleigh-Taylor instability mathematically. He derived the nonlinear instability around different steady states

*Corresponding author: Tel.: +86-15001201710.
Email addresses: `duanran@mail.ccnu.edu.cn` (Ran Duan), `jiangfei0591@163.com` (Fei Jiang), `jiang@iapcm.ac.cn` (Song Jiang)

for both incompressible and compressible ideal MHD flows when the density is continuous. When two incompressible immiscible fluids evolve with a free interface (the density is discontinuous at interface), it was first showed by Kruskal and Schwarzschild [6] that a horizontal magnetic field has no effect on the development of the linear Rayleigh-Taylor instability for the case of whole space. Recently, for the case of finite slab, Wang [12] obtained the critical magnetic number for the linear Rayleigh-Taylor instability. Namely, he gave an instability criterion to the linearized problem. In particular, he also remarked that the linearized problem was unstable for the initially horizonal magnetic field $\bar{B} = (B, 0, 0)$. To our best knowledge, however, the nonlinear Rayleigh-Taylor instability for two uniform MHD flows is still not shown mathematically in the literature.

In this paper, we will study the Rayleigh-Taylor instability for two unform inviscid MHD flows with a free interface when the initial magnetic field $\bar{B}$ is vertical to the direction of gravity. We will prove that the corresponding linearized system is unstable in the sense of Hadamard, and moreover, the original nonlinear problem is ill-posed in some sense. We point out that in [2] Guo and Hwang introduced a variational approach to deal with the Rayleigh-Taylor instability for incompressible Euler fluids, but it is not obvious whether their approach can be directly extended to two uniform MHD incompressible as well as compressible flows since the MHD flow has a more complicated structure due to presence of the magnetic field. In the current paper, a crucial point in our proof lies in the observation that the growth rate $\lambda(\xi)$ goes to infinity in some unbounded domain (cf. Lemma 3.2), and the normal modes with a higher spatial frequency grow faster in time, providing consequently a mechanism for the Rayleigh-Taylor instability.

Next, we formulate our problem in details for further discussion.

1.1. Formulation in Eulerian coordinates

We consider the two-phase free boundary problem for the equations of magnetohydrodynamics within the infinite slab $\Omega = \mathbb{R}^2 \times (-1, 1) \subset \mathbb{R}^3$ and for time $t \geq 0$. The fluids are separated by a moving free interface $\sum(t)$ that extends to infinity in every horizontal direction. The interface divides Ω into two time-dependent, disjoint, open subsets $\Omega_\pm(t)$, so that $\Omega = \Omega_+(t) \cup \Omega_-(t) \cup \sum(t)$ and $\sum(t) = \bar{\Omega}_+(t) \cap \bar{\Omega}_-(t)$. The motion of the fluids is driven by the constant gravitational field along e_3–the x_3 direction, $G = (0, 0, -g)$ with $g > 0$ and the Lorentz force induced by the magnetic fields. The two fluids are described by their velocity, pressure and magnetic field functions, which are given for each $t \geq 0$ by

$$(u_\pm, \bar{p}_\pm, h_\pm)(t, \cdot) : \Omega_\pm(t) \to (\mathbb{R}^3, \mathbb{R}^+, \mathbb{R}^3),$$

respectively. We assume that at a given time $t \geq 0$, these functions have well-defined traces onto $\sum(t)$.

The fluids under consideration are incompressible, inviscid and of zero resistivity. Hence, for $t > 0$ and $x = (x_1, x_2, x_3) \in \Omega_\pm(t)$ the fluids satisfy the following magnetohydrodynamic equations:

$$\begin{cases} \partial_t(\varrho_\pm u_\pm) + \mathrm{div}(\varrho_\pm u_\pm \otimes u_\pm) + \mathrm{div} S_\pm = -g\varrho_\pm e_3, \\ \mathrm{div} u_\pm = 0, \\ \partial_t h_\pm + \mathrm{div}(u_\pm \otimes h_\pm) - \mathrm{div}(h_\pm \otimes u_\pm) = 0, \\ \mathrm{div} h_\pm = 0, \end{cases} \tag{1.1}$$

where the stress tensor, consisting of both fluid and magnetic parts, is given by

$$S_\pm = \bar{p}_\pm I + \frac{|h_\pm|^2}{2} I - h_\pm \otimes h_\pm$$

with I being the 3×3 identity matrix, the positive constants $\varrho_\pm$ denote the densities of the respective fluids.

Now, we prescribe the jump conditions that the normal component of the velocity is continuous across the free interface. Since we do not take into account the surface tension, it is standard to assume that the normal stress is continuous across the free interface (cf. [1, 13]). Therefore, we impose the jump conditions at the free interface

$$[u \cdot \nu]|_{\sum(t)} = 0,$$
$$[S\nu]|_{\sum(t)} = 0, \tag{1.2}$$

where ν denotes the normal vector to the free surface $\sum(t)$, and $f|_{\sum(t)}$ the trace of a quantity f on $\sum(t)$, the interfacial jump is defined by

$$[f]|_{\sum(t)} := f_+|_{\sum(t)} - f_-|_{\sum(t)}.$$

We also enforce the condition that the normal component of the fluid velocity vanishes at the fixed boundaries, that is,

$$u_+(t, x', -1) \cdot e_3 = u_-(t, x', 1) \cdot e_3 = 0, \quad \text{for all } t \geq 0,\ x' := (x_1, x_2) \in \mathbb{R}^2.$$

The motion of the free interface is coupled to the evolution equations for the fluids (1.1) by requiring that the surface be advected with the fluids. This means that the velocity of the surface is given by $(u \cdot \nu)\nu$. Since the normal component of the velocity is continuous across the surface, there is no ambiguity in writing $u \cdot \nu$. The tangential components of $u_\pm$ need not be continuous across $\sum(t)$, and indeed there may be jumps in these. This allows for the possibility of slipping: the upper and lower fluids moving in different directions tangent to $\sum(t)$. Since only the normal component of the velocity vanishes at the fixed upper and lower boundaries, $\{x_3 = 1\}$ and $\{x_3 = -1\}$, the fluids may also slip along the fixed boundaries.

To complete the statement of the problem, we have to specify initial conditions. We give the initial interface $\sum(0) = \sum_0$, which yields the open sets $\Omega_\pm(0)$ on which we specify the initial data for the velocity and magnetic field

$$(u_\pm, h_\pm)(0, \cdot) : \Omega_\pm(0) \to (\mathbb{R}^3, \mathbb{R}^3).$$

To simply the equations, introducing the indicator functions $\chi_{\Omega_\pm}$ and denoting

$$\varrho = \varrho_+\chi_{\Omega_+} + \varrho_-\chi_{\Omega_-}, \quad u = u_+\chi_{\Omega_+} + u_-\chi_{\Omega_-},$$
$$h = h_+\chi_{\Omega_+} + h_-\chi_{\Omega_-}, \quad \bar{p} = \bar{p}_+\chi_{\Omega_+} + \bar{p}_-\chi_{\Omega_-}.$$

we define the modified pressure by

$$p = \bar{p} + \frac{|h|^2}{2} + g\varrho x_3.$$

Thus, the equations (1.1) can be rewritten as

$$\begin{cases} \varrho\partial_t u + \varrho u \cdot \nabla u + \nabla p = h \cdot \nabla h, \\ \partial_t h + u \cdot \nabla h - h \cdot \nabla u = 0, \\ \mathrm{div} u = \mathrm{div} h = 0, \end{cases}$$

and the jump condition (1.2) becomes, setting $[\varrho] = \varrho_+ - \varrho_-$,

$$[p\,\nu]|_{\sum(t)} = g[\varrho]x_3\nu + [h \otimes h\nu]|_{\sum(t)}.$$

1.2. Formulation in Lagrangian coordinates

Time-evolution of the free interface $\sum(t)$ and the subsequent change of the domains $\Omega_\pm(t)$ in Eulerian coordinates will lead to mathematical difficulties. To circumvent the difficulties, as usual, we use the Lagrangian coordinates to make the interface and the domains fixed in time. To this end we define the fixed Lagrangian domains $\Omega_+ = \mathbb{R}^2 \times (0,1)$ and $\Omega_- = \mathbb{R}^2 \times (-1,0)$. We assume that there exist invertible mappings

$$\eta_\pm^0 : \Omega_\pm \to \Omega_\pm(0),$$

such that

$$\sum\nolimits_0 = \eta_+^0(\{x_3 = 0\}), \quad \{x_3 = 1\} = \eta_+^0(\{x_3 = 1\}), \quad \{x_3 = -1\} = \eta_-^0(\{x_3 = -1\}).$$

The first condition means that $\sum_0$ is parameterized by the either of the mappings $\eta_\pm^0$ restricted to $\{x_3 = 0\}$, and the latter two conditions mean that $\eta_\pm^0$ map the fixed upper and lower boundaries into themselves. Define the flow maps $\eta_\pm$ as the solutions to

$$\begin{cases} \partial_t \eta_\pm(t,x) = u_\pm(t, \eta_\pm(t,x)), \\ \eta_\pm(0,x) = \eta_\pm^0(x). \end{cases}$$

Without yielding confusion, we denote the Eulerian coordinates by (t,y) with $y = \eta(t,x)$ and the fixed Lagrangian coordinates by $(t,x) \in \mathbb{R}^+ \times \Omega$, this implies that $\Omega_\pm(t) = \eta_\pm(t, \Omega_\pm)$ and that $\sum(t) = \eta_+(t, \{x_3 = 0\})$, i.e., that the Eulerian domains of upper and lower fluids are the image of $\Omega_\pm$ under the mapping $\eta_\pm$ and that the free interface is parameterized by $\eta_+(t,\cdot)$ restricted to $\mathbb{R}^2 \times \{0\}$. In order to switch back and forth from Lagrangian to Eulerian coordinates, we assume that $\eta_\pm(t,\cdot)$ is invertible. Since the upper and lower fluids may slip across one another, we have to introduce the slip map $S_\pm : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times (-1,1)$ defined by

$$S_-(t,x') = \eta_-^{-1}(t, \eta_+(t,x',0)), \quad x' \in \mathbb{R}^2 \tag{1.3}$$

and $S_+(t,\cdot) = S_-^{-1}(t,\cdot)$. The slip map S_- gives the particle in the lower fluid that is in contact with the particle of the upper fluid at $x = (x_1, x_2, 0)$ on the contact surface at time t.

Setting $\eta = \chi_+\eta_+ + \chi_-\eta_-$ with $\chi_\pm = \chi_{\Omega_\pm}$, we define the Lagrangian unknowns

$$(v,b,q)(t,x) = (u,h,p)(t,\eta(t,x)), \ (t,x) \in \mathbb{R}^+ \times \Omega.$$

Defining the matrix $A := (A_{ij})_{3\times3}$ via $A^T = (D\eta)^{-1} := (\partial_j \eta_i)_{3\times3}^{-1}$, and the identity matrix $I = (I_{ij})_{3\times3}$, then in Lagrangian coordinates the evolution equations for η, v, b and q read as, writing $\partial_j = \partial/\partial_{x_j}$,

$$\begin{cases} \partial_t \eta_i = v_i \\ \varrho \partial_t v_i + A_{ik}\partial_k q = b_j A_{jk} \partial_k b_i, \\ A_{jk}\partial_k v_j = 0, \\ \partial_t b_i = b_j A_{jk} \partial_k v_i, \\ A_{jk}\partial_k b_j = 0. \end{cases} \tag{1.4}$$

Here we have used the Einstein convention of summing over repeated indices.

Since the boundary jump conditions in Eulerian coordinates are phrased in terms of jumps across the surface, the slip map must be employed in Lagrangian coordinates. The jump conditions in Lagrangian coordinates are

$$\Big(v_+(t,x',0) - v_-(t, S_-(t,x'))\Big) \cdot n(t,x',0) = 0, \tag{1.5}$$

$$\begin{aligned}\Big(q_+(t,x',0)-q_-(t,S_-(t,x'))\Big)n(t,x',0)&=g[\varrho]\eta_+^3(t,x',0)n(t,x',0)\\&\quad+((h_+\otimes h_+)(t,x',0)-(h_-\otimes h_-)(t,S_-(t,x')))n(t,x',0),\end{aligned}\tag{1.6}$$

where we have written $n:=(n_1,n_2,n_3)=\nu(\eta)$, i.e.,

$$n=\frac{\partial_1\eta_+\times\partial_2\eta_+}{|\partial_1\eta_+\times\partial_2\eta_+|}$$

for the normal vector to the surface $\sum(t)=\eta_+(t,\{x_3=0\})$, and η_+^3 is the third-component of η_+. Note that we could also phrase the jump conditions in terms of the slip map S_+ and define the surface and its normal vector in terms of η_-. Finally, we require

$$v_-(t,x',-1)\cdot e_3=v_+(t,x',1)\cdot e_3=0.\tag{1.7}$$

Note that since $\partial_t\eta=v$,

$$e_3\cdot\eta_+(t,x',1)=e_3\cdot\eta_+^0(x',1)+\int_0^t e_3\cdot v_+(t,x',1)\mathrm{d}s=1,$$

which implies that $\eta_+(t,x',1)\in\{x_3=1\}$ for all $t\geq 0$, i.e. that the part of the upper fluid in contact with the fixed boundary $\{x_3=1\}$ never flows down from the boundary. It may, however, slip along the fixed boundary since we do not require $v_+(t,x_1,x_2,1)\cdot e_i=0$ for $i=1,2$. A similar result holds for η_- on the lower fixed boundary $\{x_3=-1\}$.

For convenience in the subsequent analysis, we will use the natation

$$[\![f]\!]:=f_+|_{x_3=0}-f_-|_{x_3=0}$$

for the jump of a quantity f across the set $\{x_3=0\}$.

1.3. Reduction of the problem

In this subsection we reformulate the free boundary problem (1.4)–(1.6) and (1.7). Our goal is to eliminate b by expressing it in terms of η, and this can be achieved in the same manner as in [12]. For the reader's convenience, we give the derivation here.

Applying A_{il} to $(1.4)_4$, we have

$$A_{il}\partial_t b_i=b_jA_{jk}\partial_k v_iA_{il}=b_jA_{jk}\partial_t(\partial_k\eta_i)A_{il}=-b_jA_{jk}\partial_k\eta_i\partial_tA_{il}=-b_i\partial_tA_{il},$$

which implies that $\partial_t(A_{jl}b_j)=0$, and hence,

$$A_{jl}b_j=A_{jl}^0b_j^0,\tag{1.8}$$

$$b_i=\partial_l\eta_iA_{jl}^0b_j^0,\tag{1.9}$$

Hereafter, the superscript 0 means the initial value.

With the help of (1.9), we first evaluate the divergence of b, i.e., $(1.4)_5$. Using the geometric identities

$$J=J^0\text{ and }\partial_k(JA_{ik})=0,$$

where $J=|D\eta|$, and applying $A_{ik}\partial_k$ to (1.9), we see that

$$A_{ik}\partial_kb_i=\frac{J}{J^0}A_{ik}\partial_k(\partial_l\eta_iA_{jl}^0b_j^0)=\frac{1}{J^0}\partial_k(JA_{ik}\partial_l\eta_iA_{jl}^0b_j^0)=\frac{1}{J^0}\partial_k(J^0A_{jk}^0b_j^0)=A_{jk}^0\partial_kb_j^0.\tag{1.10}$$

Hence, if we assume the compatibility conditions on the initial data

$$A^0_{jk}\partial_k b^0_j = 0, \tag{1.11}$$

then from (1.10), we have

$$A_{jk}\partial_k b_j = 0.$$

Moreover, for simplicity, we assume that

$$A^0_{ml}b^0_m = \bar{B}_l \text{ with } \bar{B} = \{\bar{B}_1, \bar{B}_2, \bar{B}_3\} \text{ being a constant vector.} \tag{1.12}$$

We should point out here that the class of the pairs of the data (η^0,b^0) that satisfy the constraints (1.11), (1.12) is quite large. For example, if we choose η^0 =Id and b^0 =constant, then by virtue of (1.8) and (1.10), any pair of data (η, b) which is transported by the flow will satisfy (1.11), (1.12).

Now, in view of (1.8), (1.10) and (1.12), we can represent the Lorentz force term as

$$b_jA_{jk}\partial_k b_i = \partial_l\eta_j A^0_{ml}b^0_m A_{jk}\partial_k(\partial_r\eta_i A^0_{sr}b^0_s) = A^0_{mk}b^0_m\partial_k(\partial_r\eta_i A^0_{sr}b^0_s) = \bar{B}_l\bar{B}_m\partial^2_{lm}\eta_i.$$

Hence, the equations (1.4) become a Navier-Stokes system with the force term induced by the flow map η:

$$\begin{cases} \partial_t\eta_i = v_i \\ \varrho\partial_t v_i + A_{ik}\partial_k q - \bar{B}_l\bar{B}_m\partial^2_{lm}\eta_i = 0, \\ A_{jk}\partial_k v_j = 0, \end{cases} \tag{1.13}$$

where the magnetic number $\bar{B}$ can be regarded as a vector parameter. Accordingly, the jump condition (1.6) becomes

$$\begin{aligned} &\big(q_+(t,x',0) - q_-(t,S_-(t,x'))\big)n_i(t,x',0) = g[\varrho]\eta^3_+(t,x',0)n_i(t,x',0) \\ &\quad + \bar{B}_l\bar{B}_m\big((\partial_l\eta^i_+\partial_m\eta^j_+)(t,x',0) - (\partial_l\eta^i_-\partial_m\eta^j_-)(t,S_-(t,x'))\big)n_j(t,x',0). \end{aligned} \tag{1.14}$$

Finally, we require the other jump condition (1.5) and the boundary condition (1.7).

1.4. Linearization around the steady state

The system (1.13), (1.14), (1.5) and (1.7) admits the steady solution with $v=0$, $\eta=\mathrm{Id}$, q =constant with the interface given by $\eta(\{x_3=0\}) = \{x_3=0\}$ and hence $n=e_3$, $A=I$, and $S_- = \mathrm{Id}_{\{x_3=0\}}$. Here Id denotes the identity map. Now we linearize the equations (1.13) around such a steady-state solution, the resulting linearized equations are

$$\begin{cases} \partial_t\eta = v, \\ \varrho\partial_t v + \nabla q - \bar{B}_l\bar{B}_m\partial^2_{lm}\eta = 0, \\ \mathrm{div}v = 0. \end{cases} \tag{1.15}$$

The corresponding linearized jump conditions read as

$$[\![v\cdot e_3]\!] = 0,\ [\![q]\!]e_3 = g[\varrho]\eta_3 e_3 + \bar{B}_3\bar{B}_l[\![\partial_l\eta]\!] + \bar{B}_l[\![\partial_l\eta_3]\!]\bar{B}, \tag{1.16}$$

while the boundary conditions are

$$v_-(t,x',-1)\cdot e_3 = v_+(t,x',1)\cdot e_3 = 0. \tag{1.17}$$

As aforementioned, the aim of this paper is to study the Rayleigh-Taylor instability of electrically conducting fluids in the presence of a magnetic field. Hence, we assume that the upper fluid is heavier than the lower fluid, i.e.,

$$\varrho_+ > \varrho_- \Leftrightarrow [\varrho] > 0.$$

We end this section by giving the outline of this paper. In Section 2 we state our results concerning the linearized equations (1.15) and nonlinear equations (1.13), see Theorems 2.1, 2.2. In Section 3 we construct the growing solutions to the linearized equations, while in Section 4 we analyze the linear problem, and prove the uniqueness and Theorem 2.1. In Section 5, we prove the ill-posedness of the nonlinear problem, i.e. Theorem 2.2.

2. Main results

Before stating the main results, we introduce the notation that will be used throughout the paper. For a function $f \in L^2(\Omega)$, we define the horizontal Fourier transform via

$$\hat{f}(\xi, x_3) = \int_{\mathbb{R}^2} f(x', x_3) e^{-ix'\cdot\xi} \mathrm{d}x', \tag{2.1}$$

where $x', \xi \in \mathbb{R}^2$ and $x' \cdot \xi = x_1\xi_1 + x_2\xi_2$. By the Fubini and Parseval theorems, we have that

$$\int_\Omega |f(x)|^2 \mathrm{d}x = \frac{1}{4\pi^2} \int_\Omega \left|\hat{f}(\xi, x_3)\right|^2 \mathrm{d}\xi \mathrm{d}x_3. \tag{2.2}$$

We now define a function space suitable for our analysis of two disjoint fluids. For a function f defined on Ω we write f_+ for the restriction to $\Omega_+ = \mathbb{R}^2 \times (0, 1)$ and f_- for the restriction to $\Omega_- = \mathbb{R}^2 \times (-1, 0)$. For $s \in \mathbb{R}$, we define the piecewise Sobolev space of order s by

$$H^s(\Omega) = \{f \mid f_+ \in H^s(\Omega_+), f_- \in H^s(\Omega_-)\} \tag{2.3}$$

endowed with the norm $\|f\|^2_{H^s} = \|f\|^2_{H^s(\Omega_+)} + \|f\|^2_{H^s(\Omega_-)}$. For $k \in \mathbb{N}$ we can take the norms to be given by

$$\begin{aligned}
\|f\|^2_{H^k(\Omega_\pm)} &:= \sum_{j=0}^{k} \int_{\mathbb{R}^2 \times I_\pm} (1 + |\xi|^2)^{k-j} \left|\partial^j_{x_3} \hat{f}_\pm(\xi, x_3)\right|^2 \mathrm{d}\xi \mathrm{d}x_3 \\
&= \sum_{j=0}^{k} \int_{\mathbb{R}^2} (1 + |\xi|^2)^{k-j} \left\|\partial^j_{x_3} \hat{f}_\pm(\xi, \cdot)\right\|^2_{L^2(I_\pm)} \mathrm{d}\xi
\end{aligned}$$

for $I_- = (-1, 0)$ and $I_+ = (0, 1)$. The main difference between the piecewise Sobolev space $H^s(\Omega)$ and the usual Sobolev space lies in that we do not require functions in the piecewise Sobolev space to have weak derivatives across the set $\{x_3 = 0\}$.

Now, we are in a position to state our first result, i.e. the result of ill-posedness for the linearized problem (1.15).

Theorem 2.1. *Assume $\bar{B} = (B, 0, 0)$ is a constant vector. Then, the linear problem (1.15) with the corresponding jump and boundary conditions is ill-posed in the sense of Hadamard in $H^k(\Omega)$ for every k. More precisely, for any k, $j \in \mathbb{N}$ with $j \geq k$ and for any $T_0 > 0$ and $\alpha > 0$ there exists a sequence of solutions $\{(\eta_n, v_n, q_n)\}^\infty_{n=1}$ to (1.15), satisfying the corresponding jump and boundary conditions, so that*

$$\|\eta_n(0)\|_{H^j} + \|v_n(0)\|_{H^j} + \|q_n(0)\|_{H^j} \leq \frac{1}{n}, \tag{2.4}$$

but

$$\|v_n(t)\|_{H^k} \geq \|\eta_n(t)\|_{H^k} \geq \alpha \ for\ all\ t \geq T_0. \tag{2.5}$$

Theorem 2.1 shows discontinuous dependence on the initial data. In fact, we show that there is a sequence of solutions with initial data tending to 0 in $H^k(\Omega)$, but the solutions grow arbitrarily large in $H^k(\Omega)$. The proof of Theorem 2.1 is inspired by [3] and its basic idea is the following. First, we notice that the resulting linear equations have coefficient functions that depend only on the vertical variable, $x_3 \in (-1, 1)$. This allows us to seek "normal mode" solutions by taking the horizontal Fourier transform of the equations and assuming the solution grows exponentially in time by the factor $e^{\lambda(\xi)t}$, where $\xi \in \mathbb{R}^2$ is the horizontal spatial frequency and $\lambda(\xi) > 0$. This reduces the equations to a second order linear ODE with $\lambda(\xi)$ for each ξ (see (3.9)). Then, solving the ODE, we show in Lemma 3.2 that $\lambda(\xi) \to \infty$ in some unbounded domain, the normal modes with a higher spatial frequency grow faster in time, providing a mechanism for the Rayleigh-Taylor instability. Indeed, we can form a Fourier synthesis of the normal mode solutions constructed for each spatial frequency ξ to construct solutions of the linearized incompressible equations that grow arbitrarily quickly in time, when measured in $H^k(\Omega)$ for any $k \geq 0$. This is the content of Section 3. At last, in Section 4, we show the uniqueness result for linear problem (see Theorem 4.1). In spite of the uniqueness, the linear problem is ill-posed in the sense of Hadamard in $H^k(\Omega)$ for any k because solutions do not depend continuously on the initial data.

With the linear ill-posedness established, we can obtain the ill-posedness of the fully non-linear problem in some sense. Recalling that the steady state solution to (1.13) is given by $v = 0$, $\eta = \eta^{-1} = \mathrm{Id}$, q =constant with $A = I$ and $S_- = S_+ = \mathrm{Id}_{\{x_3=0\}}$, we now rewrite the non-linear equations (1.15) in a perturbation formulation around the steady state. Let

$$\eta = \mathrm{Id} + \tilde{\eta}, \quad \eta^{-1} = \mathrm{Id} - \zeta, \quad q = \text{constant} + \sigma, \quad v = 0 + v, \quad A = I - G,$$

where

$$G^T = \sum_{n=1}^{\infty} (-1)^{n-1} (D\tilde{\eta})^n.$$

Then the evolution equations (1.13) with $\bar{B} = (B, 0, 0)$ can be rewritten for $\tilde{\eta}$, v, σ as

$$\begin{cases} \partial_t \tilde{\eta} = v, \\ \mathrm{div} v - \mathrm{tr}(G\nabla v) = 0, \\ \varrho \partial_t v + (I - G)\nabla\sigma - |B|^2 \partial_{11}^2 \tilde{\eta} = 0, \end{cases} \tag{2.6}$$

where $\mathrm{tr}(\cdot)$ denotes the matrix trace. We require the compatibility between ζ and $\tilde{\eta}$ given by

$$\zeta = \tilde{\eta} \circ (\mathrm{Id} - \zeta).$$

The jump conditions across the interface are

$$(v_+(t, x', 0) - v_-(t, S_-(t, x'))) \cdot n(t, x', 0) = 0 \tag{2.7}$$

$$\begin{aligned} &(\sigma_+(t, x', 0) - \sigma_-(t, S_-(t, x')))n(t, x', 0) \\ &\quad = g[\varrho]\tilde{\eta}_+^3(t, x', 0)n(t, x', 0) + |B|^2(((e_1 + \partial_1\tilde{\eta}_+)(e_1 + \partial_1\tilde{\eta}_+^j))(t, x', 0) \\ &\qquad - ((e_1 + \partial_1\tilde{\eta}_-)(e_1 + \partial_1\tilde{\eta}_-^j))(t, S_-(t, x')))n_j(t, x', 0), \end{aligned} \tag{2.8}$$

where the slip map (1.3) is rewritten as

$$S_- = (\mathrm{Id}_{\mathbb{R}^2} - \zeta_-) \circ (\mathrm{Id}_{\mathbb{R}^2} + \tilde{\eta}_+) = \mathrm{Id}_{\mathbb{R}^2} + \tilde{\eta}_+ - \zeta_- \circ (\mathrm{Id}_{\mathbb{R}^2} + \tilde{\eta}_+).$$

Finally, we require the boundary condition

$$v_-(t, x', -1) \cdot e_3 = v_+(t, x', 1) \cdot e_3 = 0. \tag{2.9}$$

We collectively refer to the evolution, jump, and boundary equations (2.6)–(2.9) as "the perturbed problem".

To shorten notation, for $k \geq 0$ we define

$$\|(\tilde{\eta}, v, \sigma, \partial_t\sigma)(t)\|_{H^k} = \|\tilde{\eta}(t)\|_{H^k} + \|v(t)\|_{H^k} + \|\sigma(t)\|_{H^k} + \|\partial_t\sigma(t)\|_{H^k}.$$

Definition 2.1. *We say that the perturbed problem has property $EE(k)$ for some $k \geq 4$ if there exist δ, t_0, $c > 0$ and a function $F : [0, \delta) \to \mathbb{R}^+$ satisfying $F(z) \leq cz$ for $z \in [0, \delta)$, so that the following holds. For any $\tilde{\eta}_0$, v_0, σ_0 satisfying*

$$\|(\tilde{\eta}_0, v_0, \sigma_0)\|_{H^k} < \delta,$$

there exist $(\tilde{\eta}, v, \sigma) \in L^\infty(0, t_0; H^4(\Omega))$, so that

1. *$(\tilde{\eta}, v, \sigma)(0) = (\tilde{\eta}_0, v_0, \sigma_0)$,*
2. *$\eta(t) = \mathrm{Id} + \tilde{\eta}(t)$ is invertible and $\eta^{-1}(t) = \mathrm{Id} - \zeta(t)$ for $0 \leq t < t_0$,*
3. *$\tilde{\eta}$, v, σ solve the perturbed problem on $(0, t_0) \times \Omega$, and*
4. *we have the estimate*

$$\sup_{0 \leq t < t_0} \|(\tilde{\eta}, v, \sigma, \partial_t\sigma)(t)\|_{H^4} \leq F(\|(\tilde{\eta}_0, v_0, \sigma_0)\|_{H^4}). \tag{2.10}$$

Similar to [3], we can show that the property $EE(k)$ cannot hold for any $k \geq 4$, i.e. the following Theorem 2.2, which will be proved in Section 5. In the proof we utilize the Lipschitz structure of F to show that the property $EE(k)$ would give rise to certain estimates of solutions to the linearized equations (1.15) that cannot hold in general because of Theorem 2.1.

Theorem 2.2. *Assume $\bar{B} = (B, 0, 0)$ is a constant vector, the perturbed problem does not have property $EE(k)$ for any $k \geq 4$.*

Remark 2.1. Here the magnetic field $\bar{B} = (B, 0, 0)$ is incorporated into the reformulated system (1.13) instead of the original system (1.4). Notice that by the systems (1.4) and (1.13), together with the assumption of (1.12), we immediately have the ill-posedness of the original system (1.4) in the sense of (2.10).

Remark 2.2. Theorems 2.1, 2.2 also hold for the general horizontal magnetic field $\bar{B} = (B_1, B_2, 0)$. In fact, rotating the o-xy coordinates properly so that $\bar{B} = (B, 0, 0)$ under the rotated coordinates o-$\tilde{x}\tilde{y}$, where $B = \sqrt{B_1^2 + B_1^2}$, we have the same case as in Theorem 2.2, since our system is symmetric on horizontal plane, the above rotation will not break the system structure.

Remark 2.3. Theorems 2.1 and 2.2 show that a horizontal magnetic field can not prevent the linear and nonlinear Rayleigh-Taylor instability in the sense described in Theorems 2.1 and 2.2. However, in the construction of the normal mode solution to the linearized system, the horizontal magnetic field does have a stabilizing effect on the growth rate $\lambda(\xi)$ (cf.(3.11)). In particular, the horizontal magnetic field can succeed in stabilizing a potentially unstable arrangement for some spatial frequency ξ (for example, $|\xi| = |\xi_1|$ is sufficiently large). Of course, it is easy to see that if the magnetic field is neglected, i.e. $\bar{B} = (0, 0, 0)$, then the growth rate reduces to the one for the corresponding equations of incompressible inviscid fluids.

3. Construction of a growing solution to the linearized equations

3.1. Growing mode ansatz

We wish to construct a solution to the linearized equations (1.15) that has a growing H^k-norm for any k. We will construct such solutions via Fourier synthesis by first constructing a growing mode for fixed spatial frequency.

To begin, we make a growing mode ansatz, i.e., let us assume that

$$v(t,x) = w(x)e^{\lambda t},\ q(t,x) = \tilde{q}(x)e^{\lambda t},\ \eta(t,x) = \tilde{\eta}(x)e^{\lambda t}, \quad \text{for some } \lambda > 0.$$

Substituting this ansatz into (1.15), eliminating $\tilde{\eta}$ by using the first equation, we arrive at the time-invariant system for $w = (w_1, w_2, w_3)$ and $\tilde{q}$:

$$\begin{cases} \lambda \varrho w + \nabla \tilde{q} - \lambda^{-1} \bar{B}_l \bar{B}_m \partial^2_{lm} w = 0, \\ \operatorname{div} w = 0, \end{cases} \tag{3.1}$$

with the corresponding jump conditions

$$[\![w_3]\!] = 0,\ [\![\tilde{q}]\!] e_3 = \lambda^{-1} g[\varrho] w_3 e_3 + \lambda^{-1} \bar{B}_3 \bar{B}_l [\![\partial_l w]\!] + \lambda^{-1} \bar{B}_l [\![\partial_l w_3]\!] \bar{B}$$

and boundary conditions

$$w_3(t, x', -1) = w_3(t, x', 1) = 0.$$

3.2. Horizontal Fourier transformation

We take the horizontal Fourier transform of w_1, w_2, w_3 in (3.1), which we denote with either $\hat{\cdot}$ or $\mathcal{F}$, and fix a spatial frequency $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. Define the new unknowns $\varphi(x_3) = i\hat{w}_1(\xi, x_3)$, $\theta(x_3) = i\hat{w}_2(\xi, x_3)$, $\psi(x_3) = \hat{w}_3(\xi, x_3)$ and $\pi(x_3) = \hat{\tilde{q}}_3(\xi, x_3)$, such that

$$\mathcal{F}(\mathrm{div} w) = \xi_1 \varphi + \xi_2 \theta + \psi',$$

where $' = d/dx_3$. Since we only consider the case $\bar{B} = (B, 0, 0)$, then for φ, θ, ψ and $\lambda = \lambda(\xi)$ we arrive at the following system of ODEs.

$$\begin{cases} \lambda^2 \varrho \varphi - \lambda \xi_1 \pi + |B|^2 \xi_1^2 \varphi = 0, \\ \lambda^2 \varrho \theta - \lambda \xi_2 \pi + |B|^2 \xi_1^2 \theta = 0, \\ \lambda^2 \varrho \psi + \lambda \pi' + |B|^2 \xi_1^2 \psi = 0, \\ \xi_1 \varphi + \xi_2 \theta + \psi' = 0, \end{cases} \tag{3.2}$$

along with the jump conditions

$$[\![\psi]\!] = 0,\ [\![i|B|^2 \xi_1 \psi]\!] = 0,\ [\![\lambda \pi]\!] = g[\varrho]\psi \tag{3.3}$$

and boundary conditions

$$\psi(-1) = \psi(1) = 0. \tag{3.4}$$

Eliminating π from the third equation in (3.2), we obtain the following ODE for ψ

$$-\lambda^2 \rho (|\xi|^2 \psi - \psi'') = |B|^2 (|\xi|^2 \xi_1^2 \psi - \xi_1^2 \psi'') \tag{3.5}$$

along with the jump conditions

$$[\![\psi]\!] = 0, \tag{3.6}$$

$$\lambda^2 [\![\rho \psi']\!] + B^2 \xi_1^2 [\![\psi']\!] + g[\rho]|\xi|^2 \psi = 0 \tag{3.7}$$

and boundary conditions

$$\psi(-1) = \psi(1) = 0. \tag{3.8}$$

Since we want $\lambda \neq 0$ for a growing mode solution, we can deduce from (3.5)

$$\psi'' = |\xi|^2 \psi \text{ in } (-1, 1)/\{0\}. \tag{3.9}$$

3.3. Construction of a solution to the ODEs

In this section, we will give a solution to (3.2) with $|\xi| > 0$. Throughout this section we assume that $|\xi| > 0$, and we will construct a non-trivial solution with $\lambda = \lambda(\xi) > 0$.

Lemma 3.1. *For any $|\xi| > 0$, if*

$$\lambda^2 = \frac{g[\varrho]|\xi|(e^{2|\xi|} - 1) - 2|B|^2\xi_1^2(e^{2|\xi|} + 1)}{(\varrho_+ + \varrho_-)(e^{2|\xi|} + 1)} > 0,$$

then the function

$$\psi = \begin{cases} c_1(e^{-|\xi|x_3} - e^{|\xi|(2+x_3)}), & \text{if } x_3 \in [-1, 0], \\ c_1(e^{|\xi|x_3} - e^{|\xi|(2-x_3)}), & \text{if } x_3 \in (0, 1], \end{cases} \tag{3.10}$$

is a nontrivial solution to the problem (3.6)–(3.9) for a given constant $c_1 \neq 0$.

Proof. Given $|\xi| > 0$, we know that the solution of (3.9) has the following form:

$$\psi = \begin{cases} c_2 e^{|\xi|x_3} + c_1 e^{-|\xi|x_3}, & \text{if } x_3 \in [-1, 0], \\ c_4 e^{|\xi|x_3} + c_3 e^{-|\xi|x_3}, & \text{if } x_3 \in (0, 1]. \end{cases}$$

Using the condition (3.8), we get

$$\psi = \begin{cases} c_1(e^{-|\xi|x_3} - e^{|\xi|(2+x_3)}), & \text{if } x_3 \in [-1, 0], \\ c_3(e^{|\xi|x_3} - e^{|\xi|(2-x_3)}), & \text{if } x_3 \in (0, 1]. \end{cases}$$

Applying the jump condition (3.6), we obtain $c_1 = c_3$. Finally, by (3.7), when

$$\lambda^2 = \frac{g[\varrho]|\xi|(e^{2|\xi|} - 1) - 2|B|^2\xi_1^2(e^{2|\xi|} + 1)}{(\varrho_+ + \varrho_-)(e^{2|\xi|} + 1)} > 0 \tag{3.11}$$

we can check that

$$\psi = \begin{cases} c_1(e^{-|\xi|x_3} - e^{|\xi|(2+x_3)}), & \text{if } x_3 \in [-1, 0], \\ c_1(e^{|\xi|x_3} - e^{|\xi|(2-x_3)}), & \text{if } x_3 \in (0, 1]. \end{cases}$$

is indeed a solution to (3.6)–(3.9) for any given constant c_1. The desired result immediately follows. □

We see that $\lambda > 0$ in (3.11) is equivalent to

$$\frac{g[\varrho]|\xi|(e^{2|\xi|} - 1)}{e^{2|\xi|} + 1} > 2|B|^2\xi_1^2. \tag{3.12}$$

Hence, (3.12) is the necessary and sufficient condition that guarantees the existence of a nontrivial solutio to (3.6)–(3.9) with $\lambda > 0$. Moreover, when ξ_1 is fixed, the expression (3.11) provides a upper and lower bound for λ for large ξ_2, showing that $\lambda(\xi_1, \xi_2) \to \infty$ as $\xi_2 \to \infty$. In particular, we have the following estimate for λ.

Lemma 3.2. *Let R_1, ξ_1 satisfy*

$$\frac{e^{2R_1} - 1}{e^{2R_1} + 1} \geq \frac{1}{\sqrt{2}}, \text{ and } |\xi_1| \leq \frac{g[\varrho]}{4|B|^2} < R_1. \tag{3.13}$$

Then

$$c_4(R_1) < c_2\sqrt{|\xi_2|} < \lambda < c_3\sqrt{|\xi|} \text{ for any } |\xi| > R_1 > 0, \tag{3.14}$$

where

$$c_2 = \sqrt{\frac{g[\varrho]}{2(\varrho_+ + \varrho_-)}}, \; c_3 = \frac{c_2}{\sqrt{2}}, \; c_4(R_1) = c_2\left[R_1^2 - \left(\frac{g[\varrho]}{4|B|^2}\right)^2\right]^{\frac{1}{4}}.$$

Proof. Let $|\xi| > R_1 > 0$. Making use of (3.11) and (3.13), we can estimate for λ^2 as follows.

$$
\begin{aligned}
\frac{g[\varrho]|\xi|}{(\varrho_+ + \varrho_-)} >& \lambda^2 \\
>& \frac{1}{\varrho_+ + \varrho_-}\left(\frac{g[\varrho]\sqrt{\xi_1^2+\xi_2^2}}{\sqrt{2}} - 2|B|^2\xi_1^2\right) \\
\geq& \frac{1}{\varrho_+ + \varrho_-}\left(\frac{g[\varrho](|\xi_1|+|\xi_2|)}{2} - 2|B|^2\xi_1^2\right) \\
\geq& \frac{g[\varrho]|\xi_2|}{2(\varrho_+ + \varrho_-)} > \frac{g[\varrho]}{2(\varrho_+ + \varrho_-)}\sqrt{R_1^2 - \left(\frac{g[\varrho]}{4|B|^2}\right)^2},
\end{aligned}
$$

which immediately yields (3.14). $\square$

A solution to (3.6)–(3.9) gives rise to a solution of the system (3.2)–(3.4) for the growing mode velocity w, as well.

Lemma 3.3. *Under the assumption of (3.12), there exists a solution $\psi = \psi(\xi, x_3)$, $\pi = \pi(\xi, x_3)$, $\varphi = \varphi(\xi, x_3)$, $\theta = \theta(\xi, x_3)$, and $\lambda(\xi) > 0$ to (3.2)–(3.4). This solution is smooth when restricted to $(-1, 0)$ or $(0, 1)$. Moreover, $\|\psi\|_{H^1(-1,1)} < 3$.*

Proof. By Lemma 3.1, we first construct a solution $\psi = \psi(\xi, x_3)$ which is smooth when restricted to $(-1, 0)$ or $(0, 1)$. Furthermore, we take the value

$$c_1 = \frac{1}{\sqrt{|\xi|(4|\xi|e^{2|\xi|} + e^{4|\xi|} - 1)}}$$

in (3.10), we can check that

$$\|\psi'\|_{L^2(-1,1)} = 1,\ \|\psi\|_{L^2(-1,1)} = \sqrt{\frac{e^{4|\xi|} - 4|\xi|e^{2|\xi|} - 1}{|\xi|^2(e^{4|\xi|} + 4|\xi|e^{2|\xi|} - 1)}} < 2, \tag{3.15}$$

which yield $\|\psi\|_{H^1(-1,1)} < 3$. Thus, by solving (3.2), we get

$$\pi(\xi, x_3) = -\frac{(\lambda^2\varrho + |B|^2\xi_1^2)\psi'}{\lambda|\xi|^2},\ \varphi(\xi, x_3) = -\frac{\psi'\xi_1}{|\xi|^2},\ \theta(\xi, x_3) = -\frac{\psi'\xi_2}{|\xi|^2}. \tag{3.16}$$

From (3.10) and (3.16), we see that $\pi = \pi(\xi, x_3)$, $\varphi = \varphi(\xi, x_3)$ and $\theta = \theta(\xi, x_3)$ are smooth when restricted to $(-1, 0)$ or $(0, 1)$. Furthermore, they satisfy the jump conditions (3.3). $\square$

Remark 3.1. By the expressions (3.10) and (3.16), we observe that

(1) λ, ψ and π are even on ξ_1 or ξ_2, when the another variable is fixed;
(2) φ is odd on ξ_1, but even on ξ_2, when the another variable is fixed;
(3) θ is even on ξ_1, but odd on ξ_2, when the another variable is fixed.

The next result provides an estimate for H^k-norm of the solutions $(\varphi, \theta, \psi, \pi)$ with $|\xi|$ varying, which will be useful in the next section when such solutions are integrated in a Fourier synthesis. To emphasize the dependence on ξ, we will write these solutions as

$$(\varphi(\xi) = \varphi(\xi, x_3), \theta(\xi) = \theta(\xi, x_3), \psi(\xi) = \psi(\xi, x_3), \pi(\xi) = \pi(\xi, x_3)).$$

Denoting

$$\mathbb{D} := \left\{\xi = (\xi_1, \xi_2)\ \middle|\ |\xi_1| < \frac{g[\varrho]}{4|B|^2},\ |\xi| > R_1\right\},$$

where $1 \leq R_1$ satisfies (3.13), we see that (3.12) holds for any $\xi \in \mathbb{D}$.

Lemma 3.4. *Let $\xi \in \mathbb{D}$, $\varphi(\xi)$, $\theta(\xi)$, $\psi(\xi)$ and $\pi(\xi)$ be constructed as in Lemma 3.3, then for any $k \geq 0$, there exits a positive constant A depending on ϱ, $|B|$, R_1 and g, such that*

$$\|\psi(\xi)\|_{H^k(-1,0)} + \|\psi(\xi)\|_{H^k(0,1)} \leq 3\sum_{j=0}^{k} |\xi|^{j-\delta(j)}, \tag{3.17}$$

$$\|\varphi(\xi)\|_{H^k(-1,0)} + \|\varphi(\xi)\|_{H^k(0,1)} + \|\theta(\xi)\|_{H^k(-1,0)} + \|\theta(\xi)\|_{H^k(0,1)} \leq 2\sum_{j=0}^{k} |\xi|^{j}, \tag{3.18}$$

$$\|\pi(\xi)\|_{H^k(-1,0)} + \|\pi(\xi)\|_{H^k(0,1)} \leq A\sum_{j=0}^{k} |\xi|^{j}, \tag{3.19}$$

where $\delta(j) = 0$ if $j = 0$ and $\delta(j) = 1$ if $j \neq 0$. Moreover

$$\sqrt{\|\varphi\|^2_{L^2(-1,1)} + \|\theta\|^2_{L^2(-1,1)} + \|\psi\|^2_{L^2(-1,1)}} \geq 1. \tag{3.20}$$

Proof. By (3.15) and (3.10), we see that $\psi(\xi)$ is even on x_3,

$$\psi''(\xi) = |\xi|^2\psi(\xi), \ |\xi\psi(\xi)|^2 \leq |\psi'(\xi)|^2 \tag{3.21}$$

and

$$\|\psi(\xi)\|_{L^2(-1,1)} < 2, \quad |\psi'(\xi)\|_{L^2(-1,1)} = 1. \tag{3.22}$$

Thus

$$\|\psi''(\xi)\|_{L^2(-1,1)} = |\xi|\|\xi\psi(\xi)\|_{L^2(-1,1)} \leq |\xi|\|\psi'(\xi)\|_{L^2(-1,1)} < |\xi|. \tag{3.23}$$

Employing (3.21)–(3.23), we get

$$\|\psi^{(k+1)}(\xi)\|_{L^2(-1,1)} \leq |\xi|^k \text{ for any } k \geq 0, \tag{3.24}$$

which immediately implies (3.17).

Combing (3.16) with (3.24), we see that $\varphi(\xi)$ and $\theta(\xi)$ are odd on x_3, and

$$\|\varphi^{(k)}(\xi)\|_{L^2(-1,1)} + \|\theta^{(k)}(\xi)\|_{L^2(-1,1)} \leq 2|\xi|^k \quad \text{for any } k \geq 0$$

which yields (3.18) with $|\xi| > 1$.

Recalling the expression of π, we find by (3.14), (3.24) with $|\xi| > 1$ that

$$\begin{aligned}\|\pi^{(k)}(\xi)\|_{L^2(-1,1)} &\leq \left(\frac{\lambda\varrho}{|\xi|^2} + \frac{|B|^2}{\lambda}\right)\|\psi^{(k+1)}(\xi)\|_{L^2(-1,1)} \\ &\leq \left\{c_2\varrho + \frac{2|B|^3}{c_2\left[(4R_1|B|)^2 - (g[\varrho])^2\right]^{1/4}}\right\}|\xi|^k := A|\xi|^k,\end{aligned}$$

which gives (3.19). Finally, using (3.16) and (3.22), we obtain (3.20). □

3.4. Fourier synthesis

In this section we will use the Fourier synthesis to build growing solutions to (1.15) out of the solutions constructed in the previous section (Lemma 3.3) for fixed spatial frequency $\xi \in \mathbb{R}^2$. The solutions will be constructed to grow in the piecewise Sobolev space of order k, H^k, defined by (2.3).

Theorem 3.1. *Let $1 \leq R_1 \leq R_2 < R_3 < \infty$ with R_1 satisfying (3.13). Let $f \in C_0^\infty(\mathbb{R}^2)$ be a real-valued function, so that $f(\xi) = f(|\xi|)$ and supp$(f) \subset B(0,R_3)/B(0,R_2)$. For $\xi \in \mathbb{R}^2$, define*

$$w(\xi, x_3) = -i\varphi(\xi, x_3)e_1 - i\theta(\xi, x_3)e_2 + \psi(\xi, x_3)e_3,$$

where

$$(\varphi, \theta, \psi, \pi)(\xi, x_3) \begin{cases} \text{are the solutions provided by Lemma 3.3,} & \text{if } \xi \in \mathbb{D}, \\ \text{take } 0, & \text{if } \xi \notin \mathbb{D}. \end{cases} \tag{3.25}$$

Denote

$$\eta(t,x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} f(\xi)w(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}\mathrm{d}\xi, \tag{3.26}$$

$$v(t,x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \lambda(\xi)f(\xi)w(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}\mathrm{d}\xi, \tag{3.27}$$

$$q(t,x) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \lambda(\xi)f(\xi)\pi(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}\mathrm{d}\xi. \tag{3.28}$$

Then, η, v, q are real-valued solutions to the linearized equation (1.15) along with the corresponding jump and boundary conditions. For every $k \in \mathbb{N}$, we have the estimate

$$\|\eta(0)\|_{H^k} + \|v(0)\|_{H^k} + \|q(0)\|_{H^k} \leq \tilde{c}_k \left(\int_{\mathbb{R}^2} (1+|\xi|^2)^{k+1}|f(|\xi|)|^2\mathrm{d}\xi\right)^{1/2} < \infty \tag{3.29}$$

for a constant $\tilde{c}_k > 0$ depending on the parameters ϱ, $|B|$, R_1 and g. Moreover, for every $t > 0$ we have $\eta(t), v(t), q(t) \in H^k(\Omega_\pm)$, and

$$e^{tc_4(R_2)}\|\eta(0)\|_{H^k} \leq \|\eta(t)\|_{H^k} \leq e^{tc_3\sqrt{R_3}}\|\eta(0)\|_{H^k}, \tag{3.30}$$

$$e^{tc_4(R_2)}\|v(0)\|_{H^k} \leq \|v(t)\|_{H^k} \leq e^{tc_3\sqrt{R_3}}\|v(0)\|_{H^k}, \tag{3.31}$$

$$e^{tc_4(R_2)}\|q(0)\|_{H^k} \leq \|q(t)\|_{H^k} \leq e^{tc_3\sqrt{R_3}}\|q(0)\|_{H^k}, \tag{3.32}$$

where

$$c_3 := \sqrt{\frac{g[\varrho]}{(\varrho_+ + \varrho_-)}}, \quad c_4(R_2) := \frac{g[\varrho]}{2(\varrho_+ + \varrho_-)}\left[R_2^2 - \left(\frac{g[\varrho]}{4|B|^2}\right)^2\right]^{1/4}$$

Proof. For each fixed $\xi \in \mathbb{R}^2$,

$$\begin{aligned} \eta(t,x) &= f(\xi)w(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi}, \\ v(t,x) &= \lambda(\xi)f(\xi)w(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi} \\ q(t,x) &= \lambda(\xi)f(\xi)\pi(\xi,x_3)e^{\lambda(\xi)t}e^{ix'\xi} \end{aligned}$$

give a solution to (1.15). Since supp$(f) \subset B(0,R_3)/B(0,R_2)$, Lemma 3.4 implies that

$$\sup_{\xi \in \text{supp}(f)} \|\partial_{x_3}^k w(\xi,\cdot)\|_{L^\infty} < \infty \text{ for all } k \in \mathbb{N}.$$

Also, $\lambda(\xi) \leq c_3\sqrt{|\xi|}$. These bounds show that the Fourier synthesis of these solutions given by (3.26)-(3.28) is also a solution of (1.15). Because f is real-valued and radial, $\mathbb{D}$ is a symmetrical domain, combined with Remark 3.1 and (3.25), we can easily verify that the Fourier synthesis is real-valued.

The bound (3.29) follows by applying Lemma 3.4 with arbitrary $k \geq 0$ and utilizing the fact that f is compactly supported. At last, note $R_2 \geq R_1$, we can use (3.14) and (3.26)-(3.28) to infer the bounds (3.30)-(3.32). □

Remark 3.2. It holds that

$$\eta_3(0, x_1, x_2, 0) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(\xi)\psi(\xi, 0)e^{ix'\cdot\xi}\mathrm{d}\xi$$

is the vertical component of the initial linearized flow map at the interface between the two fluids. Since $\psi(\xi, 0) \neq 0$ for any choice of $\xi \in \mathbb{D}$, a nonzero f in general gives rise to a nonzero $\eta_3(0, x_1, x_2, 0)$.

4. Ill-posedness for the linear problem

4.1. Estimates for band-limited solutions

We assume that η, v, q are the real-valued solutions to (1.15) along with the corresponding jump and boundary conditions. Furthermore, suppose that the solutions are band-limited at radius $R > 0$, that is,

$$\bigcup_{x_3\in(-1,1)} \mathrm{supp}(|\hat{\eta}(\cdot, x_3)| + |\hat{v}(\cdot, x_3)| + |\hat{q}(\cdot, x_3)|) \subset B(0, R),$$

where $\hat{v}$ denotes the horizontal Fourier transform defined by (2.1). We will derive estimates for band-limited solutions in terms of R.

Differentiating the second equation in (1.15) with respect to t and eliminating the η term by using the first equation, we obtain

$$\begin{cases} \varrho\partial_{tt}^2 v + \nabla\partial_t q - |B|^2\partial_{11}^2 v = 0 \\ \mathrm{div} v = \mathrm{div}\partial_t v = 0 \end{cases} \tag{4.1}$$

for $\bar{B} = (B, 0, 0)$, along with the jump and boundary conditions

$$[\![v_3]\!] = [\![\partial_t v_3]\!] = 0, \ [\![\partial_t q]\!]e_3 = g[\varrho]v_3 e_3 + B[\![\partial_1 v_3]\!]\bar{B}, \tag{4.2}$$

$$\partial_t v_3(t, x', -1) = \partial_t v_3(t, x', 1) = 0. \tag{4.3}$$

The band limited assumption implies that $\mathrm{supp}(\hat{v}(\cdot, x_3)) \subset B(0, R)$ for all $x_3 \in (-1, 1)$. The initial datum for $\partial_t v(0)$ is given in terms of the initial data $q(0)$ and $\eta(0)$ via the second linear equation, i.e.

$$\varrho\partial_t v(0) = -\nabla q(0) + |B|^2\partial_{11}^2\eta(0).$$

Our first result gives an evolution equation for an energy associated to v.

Lemma 4.1. *For solutions to (4.1)–(4.3) it holds that*

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \varrho|\partial_t v|^2 + |B|^2|\partial_1 v|^2\mathrm{d}x - \int_{\mathbb{R}^2} g[\varrho]|v_3(x', 0)|^2\mathrm{d}x'\right) = 0. \tag{4.4}$$

Proof. Multiply the equation $(4.1)_1$ by $\partial_t v$ and integrate over Ω. After integrating by parts respectively in Ω_+ and Ω_-, and using the jump and boundary conditions (4.2), (4.3), and $(4.1)_2$, we obtain (4.4).

The next result allows us to estimate the energy in terms of R.

Lemma 4.2. *Let $v \in H^1(\Omega)$ be band-limited at radius $R > 0$, and satisfy $\mathrm{div}\, v = 0$ and the boundary conditions $v_3(t, x', -1) = v_3(t, x', 1) = 0$. Then,*

$$\int_{\mathbb{R}^2} g[\varrho]|v_3(x', 0)|^2\mathrm{d}x' \leq \frac{(R^2 + 1)g[\varrho]}{2} \int_\Omega |v|^2\mathrm{d}x. \tag{4.5}$$

Proof. Since

$$\partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 = 0, \tag{4.6}$$

applying the horizontal Fourier transform (2.1) to (4.6), writing $\varphi(x_3) = i\hat{v}_1(\xi_1, \xi_2, x_3)$, $\theta(x_3) = i\hat{v}_2(\xi_1, \xi_2, x_3)$, $\psi(x_3) = \hat{v}_3(\xi_1, \xi_2, x_3)$, we find that

$$\xi_1\varphi + \xi_2\theta + \psi' = 0. \tag{4.7}$$

From (2.2) and (4.7) we get

$$\begin{aligned}
\int_{\mathbb{R}^2} g[\varrho]|v_3(x', 0)|^2 \mathrm{d}x' \leq & \frac{g[\varrho]}{4\pi^2} \int_{\mathbb{R}^2} |\hat{v}_3|^2 \mathrm{d}\xi = \frac{g[\varrho]}{4\pi^2} \int_{\mathbb{R}^2} |\psi(0)|^2 \mathrm{d}\xi \\
\leq & \frac{g[\varrho]}{8\pi^2} \left[\int_{\mathbb{R}^2} \int_{-1}^{1} (|\psi|^2 + |\psi'|^2) \mathrm{d}x_3 \mathrm{d}\xi \right] \\
\leq & \frac{g[\varrho]}{8\pi^2} \left[\int_{-1}^{1} \int_{\mathbb{R}^2} (|\psi|^2 + |\xi_1|^2|\varphi|^2 + |\xi_2|^2|\theta|^2) \mathrm{d}\xi \mathrm{d}x_3 \right] \\
\leq & \frac{(R^2+1)g[\varrho]}{2} \left[\int_{-1}^{1} \int_{\mathbb{R}^2} (|v_1|^2 + |v_2|^2 + |v_3|^2) \mathrm{d}x' \mathrm{d}x_3 \right] \\
= & \frac{(R^2+1)g[\varrho]}{2} \int_{\Omega} |v|^2 \mathrm{d}x,
\end{aligned}$$

which gives (4.5). □

Now, we may derive growth estimates in terms of the initial data and R.

Lemma 4.3. *Let v be a solution to (1.15) along with the corresponding jump and boundary conditions (1.16), (1.17), which is also band-limited at radius $R > 0$. Then*

$$\|v(t)\|^2_{L^2(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Omega)} \leq c_5 e^{((R^2+1)g[\varrho]/\varrho_- + 1)t} (\|v(0)\|_{H^1(\Omega)} + \|\partial_t v(0)\|_{L^2(\Omega)}),$$

where the constant c_5 depends on ϱ, B, g and R.

Proof. Integrating the result of Lemma 4.1 in time from 0 to t, we deduce that

$$\int_\Omega \varrho |\partial_t v|^2(t, x) \mathrm{d}x \leq C + \int_{\mathbb{R}^2} g[\varrho]|v_3(t, x', 0)|^2 \mathrm{d}x', \tag{4.8}$$

where

$$C = \int_\Omega \varrho|\partial_t v(0, x)|^2 + |B|^2|\partial_1 v(0, x)|^2 \mathrm{d}x.$$

We may then apply Lemma 4.2 to get the inequality

$$\int_\Omega \varrho|\partial_t v|^2(t) \mathrm{d}x \leq C + \frac{(R^2+1)g[\varrho]}{2} \int_\Omega |v|^2(t) \mathrm{d}x,$$

which yields

$$\frac{1}{2}\|\partial_t v(t)\|^2_{L^2(\Omega)} \leq \frac{C}{\varrho_-} + \frac{(R^2+1)g[\varrho]}{2\varrho_-}\|v(t)\|^2_{L^2(\Omega)}. \tag{4.9}$$

Combing the Cauchy inequality with (4.9), we infer that

$$\begin{aligned}
\partial_t \|v(t)\|^2_{L^2(\Omega)} &= 2\langle \partial_t v(t), v(t)\rangle \leq (\|v(t)\|^2_{L^2(\Omega)} + \|\partial_t v(t)\|^2_{L^2(\Omega)}) \\
&\leq \frac{2C}{\varrho_-} + \left\{ \frac{(R^2+1)g[\varrho]}{\varrho_-} + 1 \right\} \|v(t)\|^2_{L^2(\Omega)} \\
&:= c_6 + c_7\|v(t)\|^2_{L^2(\Omega)},
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the L^2 inner product. $c_6 = 2C/\varrho_-$ and $c_7 = (R^2+1)g[\varrho]/\varrho_- + 1$. An application of the Gronwall inequality gives

$$\|v(t)\|^2_{L^2(\Omega)} \leq e^{c_7 t}\left(\|v(0)\|^2_{L^2(\Omega)} + c_6/c_7\right), \qquad \text{for all } t \geq 0. \tag{4.10}$$

To derive the corresponding bound for $\|\partial_t v(t)\|^2_{L^2(\Omega)}$ we return to (4.10) and plug in (4.9) to see that

$$\|\partial_t v(t)\|^2_{L^2(\Omega)} \leq c_6 + (c_7-1)e^{c_7 t}(\|v(0)\|^2_{L^2(\Omega)} + c_6/c_7). \tag{4.11}$$

Note that the constant C in (4.8) is bounded by

$$\begin{aligned} C \leq & |B|^2\|v(0)\|^2_{H^1(\Omega)} + \varrho_+\|\partial_t v(0)\|^2_{L^2(\Omega)} \\ \leq & c_8(\|v(0)\|^2_{H^1(\Omega)} + \|\partial_t v(0)\|^2_{L^2(\Omega)}), \end{aligned} \tag{4.12}$$

where $c_8 = \max\{|B|^2, \varrho_+\}$. Making use of (4.10)–(4.12), we conclude

$$\begin{aligned} & \|\partial_t v(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)} \\ & \leq c_6 + (c_7-1)e^{c_7 t}(\|v(0)\|^2_{L^2(\Omega)} + c_6/c_7) + c_8(\|v(0)\|^2_{H^1(\Omega)} + \|\partial_t v(0)\|^2_{L^2(\Omega)}) \\ & \leq c_5 e^{c_7 t}(\|v(0)\|_{H^1(\Omega)} + \|\partial_t v(0)\|_{L^2(\Omega)}). \end{aligned}$$

Thus, the desired conclusion follows. □

4.2. Uniqueness

Similar to [3], once we get Lemma 4.3, through constructing the horizontal spatial frequency projection operator, we can obtain the uniqueness. Here we give the proof for reader's convenience.

Let $\Phi \in C_0^\infty(\mathbb{R}^2)$ satisfy $0 \leq \Phi \leq 1$, $\mathrm{supp}(\Phi) \subset B(0,1)$ and $\Phi(x) = 1$ for $x \in B(0,1/2)$. For $R > 0$, let Φ_R be the function defined by $\Phi_R(x) = \Phi(x/R)$. We define the projection operator P_R via

$$P_R f = \mathcal{F}^{-1}(\Phi_R \mathcal{F} f), \qquad f \in L^2(\Omega),$$

where $\mathcal{F} = \hat{}$ denotes the horizontal Fourier transform in x' defined by (2.1). It is easy to see that P_R satisfies the following.

(1) $P_R f$ is band-limited at radius R.
(2) P_R is a bounded linear operator on $H^k(\Omega)$ for all $k \geq 0$.
(3) P_R commutes with partial differentiation and multiplication by functions depending only on x_3.
(4) $P_R f = 0$ for all $R > 0$ if and only if $f = 0$.

Now, we begin with the proof of the uniqueness on η and v.

Theorem 4.1. *Assume that (η_1, v_1, q_1) and (η_2, v_2, q_2) are two solutions to (1.15). Then $\eta_1 = \eta_2$, $v_1 = v_2$, and $\nabla(q_1 - q_2) = 0$.*

Proof. It suffices to show that solutions to (1.15) with 0 initial data remain 0 for $t > 0$. Suppose that η, v are solutions with vanishing initial data. For arbitrary but fixed $R > 0$, define $\eta_R = P_R\eta$, $v_R = P_R v$, $q_R = P_R q$. The properties of P_R show that η_R, v_R, q_R are also solutions to (1.15) but that they are band-limited at radius R. Turning to the second order formulation, we find that v_R is a solution to (4.1) with initial data $v_R(0) = \partial_t v_R(0) = 0$. We may then apply Lemma 4.3 to deduce that

$$\|v_R(t)\|_{L^2(\Omega)} = \|\partial_t v_R(t)\|_{L^2(\Omega)} = 0 \text{ for all } t \geq 0,$$

which implies that $\eta_R(t)$ and $v_R(t)$ all vanish for $t \geq 0$. Thus, $\eta(t)$ and $v(t)$ also vanish for $t \geq 0$ since R is arbitrary. Therefore, $\nabla q = 0$. □

The solutions to the linear problem (1.15) constructed in Theorem 3.1 are sufficiently pathological to give rise to a result showing the discontinuous dependence of the solutions on initial data. Thus, in spite of the previous uniqueness result, the linear problem is still ill-posed in the sense of Hadamard, i.e. Theorem 2.1. Next, we prove Theorem 2.1.

4.3. Proof of Theorem 2.1

Fix $j \geq k \geq 0$, $\alpha > 0$, $T_0 > 0$ and let $\tilde{c}_j$, R_1 be the constants from Theorem 3.1. For each $n \in \mathbb{N}$, let $R(n)$ be sufficiently large so that $R(n) > R_1$ and

$$\frac{\exp(2T_0 c_4(n))}{(1+(R(n)+1)^2)^{j-k+1}} \geq \alpha^2 n^2 \tilde{c}_j^2,$$

where

$$c_4(n) = \frac{g[\varrho]}{2(\varrho_+ + \varrho_-)}\left[R^2(n) - \left(\frac{g[\varrho]}{4|B|^2}\right)^2\right]^{1/4} \geq 1.$$

Choose $f_n \in C_0^\infty(\mathbb{R}^2)$, such that $\mathrm{supp}(f_n) \subset B(0, R(n)+1)\backslash B(0, R(n))$, f_n is real-valued and radial, and

$$\int_{\mathbb{R}^2} (1+|\xi|^2)^{j+1} |f_n(|\xi|)|^2 \mathrm{d}\xi = \frac{1}{\tilde{c}_j^2 n^2}. \tag{4.13}$$

Now, we can apply Theorem 3.1 with f_n, $R_2 = R(n)$, and $R_3 = R(n)+1$ to find that $\eta_n, v_n, q_n \in H^j(\Omega)$ $(t \geq 0)$ solve the problem (1.15)–(1.17). It follows from (3.29) and (4.13) that (2.4) holds for all n.

After a straightforward calculation, we see that

$$\begin{aligned}
\|\eta_n(T_0)\|_{H^k}^2 &\geq \int_{\mathbb{R}^2} (1+|\xi|^2)^k e^{2T_0\lambda(\xi)} |f_n(\xi)|^2 \|w(\xi, x_3)\|_{L^2(-1,1)}^2 \mathrm{d}\xi, \\
&\geq \frac{\exp(2T_0 c_4(n))}{\left(1+(R(n)+1)^2\right)^{j-k+1}} \int_{\mathbb{R}^2} (1+|\xi|^2)^{j+1} |f_n(\xi)|^2 \|w(\xi, x_3)\|_{L^2(-1,1)}^2 \mathrm{d}\xi \\
&\geq \alpha^2 n^2 \tilde{c}_j^2 \int_{\mathbb{R}^2} (1+|\xi|^2)^{j+1} |f_n(\xi)|^2 \mathrm{d}\xi \\
&= \alpha^2,
\end{aligned}$$

where the second bound follows from the fact that $\mathrm{supp}(f_n) \subset B(0, R(n)+1)$ and $\lambda(\xi) \geq c_4(n)$, and the third one from the choice of $R(n)$ and the lower boundedness (3.20).

Since $\lambda(\xi) \geq c_4(n) \geq 1$ on the support of f_n, we conclude

$$\|v_n(t)\|_{H^k}^2 \geq \|\eta_n(t)\|_{H^k}^2 \geq \|\eta_n(T_0)\|_{H^k}^2 \text{ for all } t \geq T_0.$$

This complete the proof of Theorem 2.1

5. Proof of Theorem 2.2

The proof is similar to [3] under necessary modifications. We argue by contradiction. Suppose that the perturbed problem has the property $EE(k)$ for some $k \geq 4$. Let δ, t_0, $c > 0$ and $F : [0, \delta] \to \mathbb{R}^+$ be the constants and function provided by the property $EE(k)$. Fix $n \in \mathbb{N}$, such that $n > c$. Applying Theorem 2.1 with this n, $T_0 = t_0/2$, $k \geq 4$ and $\alpha = 1$, we find that $\bar{\eta}$, $\bar{v}$, $\bar{\sigma}$ solve (1.15) with $\bar{B} = (B, 0, 0)$, satisfying

$$\|(\bar{\eta}, \bar{v}, \bar{\sigma})(0)\|_{H^k} < \frac{1}{n},$$

but

$$\|\bar{v}(t)\|_{H^4} \geq \|\bar{\eta}(t)\|_{H^4} \geq 1 \text{ for } t \geq t_0/2. \tag{5.1}$$

For $\varepsilon > 0$ we then define $\bar{\eta}_0^\varepsilon = \varepsilon\bar{\eta}(0)$, $\bar{v}_0^\varepsilon = \varepsilon\bar{v}(0)$, and $\bar{\sigma}_0^\varepsilon = \varepsilon\bar{\sigma}(0)$. Then, for $\varepsilon < \delta n$ we have $\|(\bar{\eta}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{\sigma}_0^\varepsilon)\|_{H^k} < \delta$. So, according to $EE(k)$, there exist $\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon \in L^\infty(0, t_0; H^4(\Omega))$ that solve the perturbed problem (2.6)-(2.9) with the initial data satisfying $\|(\bar{\eta}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{\sigma}_0^\varepsilon)\|_{H^k} < \delta$, and satisfy the inequality

$$\sup_{0 \leq t < t_0} \|(\tilde{\eta}^\varepsilon, v^\varepsilon, \sigma^\varepsilon, \partial_t\sigma^\varepsilon)(t)\|_{H^4} \leq F(\|(\bar{\eta}_0^\varepsilon, \bar{v}_0^\varepsilon, \bar{\sigma}_0^\varepsilon)\|_{H^4}) \leq c\varepsilon\|(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon)(0)\|_{H^4} < \varepsilon. \tag{5.2}$$

Now, defining the rescaled functions $\bar{\eta}^\varepsilon = \tilde{\eta}^\varepsilon/\varepsilon$, $\bar{v}^\varepsilon = v^\varepsilon/\varepsilon$, $\bar{\sigma}^\varepsilon = \sigma^\varepsilon/\varepsilon$, and rescaling (5.2), one gets

$$\sup_{0 \leq t < t_0} \|(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon, \partial_t\bar{\sigma}^\varepsilon)(t)\|_{H^4} \leq 1. \tag{5.3}$$

Note that by construction $(\bar{\eta}^\varepsilon, \bar{v}^\varepsilon, \bar{\sigma}^\varepsilon)(0) = (\bar{\eta}, \bar{v}, \bar{\sigma})(0)$. Our next goal is to show that the rescaled functions converge as $\varepsilon \to 0$ to the solution $(\bar{\eta}, \bar{v}, \bar{\sigma})$ of the linearized equations (1.15) with $\bar{B} = (B, 0, 0)$.

We may further assume that ε is sufficiently small so that

$$\sup_{0 \leq t < t_0} \|\varepsilon D\bar{\eta}^\varepsilon(t)\|_{L^\infty} < \frac{1}{9} \quad \text{and} \quad \varepsilon < 1/(2K_1), \tag{5.4}$$

where $K_1 > 0$ is the best constant in the inequality

$$\|FG\|_{H^2} \leq K_1\|F\|_{H^2}\|G\|_{H^2}$$

for 3×3 matrix-valued functions F, G. Then,

$$\bar{G}^\varepsilon := \big(I - (I + \varepsilon D\bar{\eta}^T)^{-1}\big)/\varepsilon$$

is well-defined by (5.4) and uniformly bounded in $L^\infty(0, t_0; H^2(\Omega))$ since

$$\begin{aligned} \|\bar{G}^\varepsilon\|_{H^2} &= \left\|\sum_{n=1}^\infty (-\varepsilon)^{n-1}(D\bar{\eta}^\varepsilon)^n\right\|_{H^2} \leq \sum_{n=1}^\infty \varepsilon^{n-1}\|(D\bar{\eta})^n\|_{H^2} \\ &\leq \sum_{n=1}^\infty (\varepsilon K_1)^{n-1}\|D\bar{\eta}^\varepsilon\|_{H^2}^n \leq \sum_{n=1}^\infty \frac{1}{2^{n-1}}\|\bar{\eta}^\varepsilon\|_{H^4}^n < \sum_{n=1}^\infty \frac{1}{2^{n-1}} = 2. \end{aligned} \tag{5.5}$$

Since $\mathrm{Id} + \varepsilon\bar{\eta}^\varepsilon$ is invertible, we can define $\bar{\zeta}^\varepsilon$ via $(\mathrm{Id} + \varepsilon\bar{\eta}^\varepsilon)^{-1} = \mathrm{Id} - \varepsilon\bar{\zeta}^\varepsilon$, which implies that $\bar{\zeta}^\varepsilon = \bar{\eta}^\varepsilon \circ (\mathrm{Id} - \varepsilon\bar{\zeta}^\varepsilon)$. The slip map $S_-^\varepsilon : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2 \times \{0\}$ is then given by

$$S_-^\varepsilon = \mathrm{Id}_{\mathbb{R}^2} + \varepsilon\bar{\eta}_+^\varepsilon - \varepsilon\bar{\zeta}_-^\varepsilon \circ (\mathrm{Id}_{\mathbb{R}}^2 + \varepsilon\bar{\eta}_+^\varepsilon).$$

The bounds on $\bar{\eta}^\varepsilon$ and the equation satisfied by $\bar{\zeta}^\varepsilon$ then imply that

$$\sup_{0 \leq t < t_0} \|S_-^\varepsilon - \mathrm{Id}_{\mathbb{R}^2}\|_{L^\infty} \leq 2\varepsilon \sup_{0 \leq t < t_0} \|\bar{\eta}^\varepsilon\|_{L^\infty} \leq 2\varepsilon K_2 \sup_{0 \leq t < t_0} \sup_{0 \leq t < t_0} \|\bar{\eta}^\varepsilon(t)\|_{H^4} < 2\varepsilon K,$$

where $K_2 > 0$ is the embedding constant for the trace map $H^4(\Omega) \hookrightarrow L^\infty(\mathbb{R}^2 \times \{0\})$. This boundedness allows us to define the normalized slip map $\bar{S}_-^\varepsilon := (S_-^\varepsilon - \mathrm{Id}_{\mathbb{R}^2})/\varepsilon$ as a well-defined and uniformly bounded function in $L^\infty(0, t_0; L^\infty(\mathbb{R}^2 \times \{0\})$.

Next, we exploit the boundedness of $\bar{\eta}^\varepsilon$, $\bar{v}^\varepsilon$, $\bar{\sigma}^\varepsilon$ and $\bar{G}^\varepsilon$ to control $\partial_t\bar{\eta}$, $\partial_t\bar{v}^\varepsilon$ and to give some convergence results. The first equation in (2.6) implies that $\partial_t\bar{\eta}^\varepsilon=\bar{v}^\varepsilon$, therefore

$$\sup_{0\le t<t_0}\|\partial_t\bar{\eta}^\varepsilon(t)\|_{H^4}=\sup_{0\le t<t_0}\|\bar{v}^\varepsilon(t)\|_{H^4}\le 1. \tag{5.6}$$

By virtue of (5.3) and (5.5), the second equation in (2.6) yields

$$\lim_{\varepsilon\to 0}\sup_{0\le t<t_0}\|\mathrm{div}\bar{v}^\varepsilon(t)\|_{H^3}=\lim_{\varepsilon\to 0}\sup_{0\le t<t_0}\|\varepsilon\mathrm{tr}(\bar{G}^\varepsilon D\bar{v}^\varepsilon)(t)\|_{H^3}=0. \tag{5.7}$$

Expanding the third equation in (2.6), one sees

$$\varrho\partial_t\bar{v}^\varepsilon+\nabla\bar{\sigma}^\varepsilon-|B|^2\partial_{11}^2\bar{\eta}^\varepsilon-\varepsilon\bar{G}^\varepsilon\nabla\bar{\sigma}^\varepsilon=0, \tag{5.8}$$

whence,

$$\lim_{\varepsilon\to 0}\sup_{0\le t<t_0}\|\varrho\partial_t\bar{v}^\varepsilon+\nabla\bar{\sigma}^\varepsilon-|B|^2\partial_{11}^2\bar{\eta}^\varepsilon\|_{H^3}=0$$

and

$$\sup_{0\le t<t_0}\|\partial_t\bar{v}^\varepsilon\|_{H^2}\le K_3 \qquad \text{for some constant } K_3>0. \tag{5.9}$$

We proceed to show some convergence results for the jump conditions. We first write the normal at the interface as $n^\varepsilon=N^\varepsilon/|N^\varepsilon|$ with

$$\begin{aligned}N^\varepsilon=&(e_1+\varepsilon\partial_1\bar{\eta}_+^\varepsilon)\times(e_2+\varepsilon\partial_2\bar{\eta}_+^\varepsilon)\\ =&e_3+\varepsilon(e_1\times\partial_x\bar{\eta}_+^\varepsilon+\partial_1\bar{\eta}_+^\varepsilon\times e_2)+\varepsilon^2(\partial_1\bar{\eta}_+^\varepsilon\times\partial_2\bar{\eta}_+^\varepsilon):=e_3+\varepsilon\bar{N}^\varepsilon.\end{aligned}$$

As $\varepsilon\to 0$, we have $|N^\varepsilon|>0$, so we may rewrite the jump condition (2.7) as

$$(\bar{v}_+^\varepsilon-\bar{v}_-^\varepsilon\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon))\cdot(e_3+\varepsilon\bar{N}^\varepsilon)=0.$$

Clearly $\sup_{0\le t<t_0}\|\bar{N}^\varepsilon(t)\|_{L^\infty}$ is uniformly bounded and

$$\sup_{0\le t<t_0}\|\bar{v}_-^\varepsilon\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon)-\bar{v}_-^\varepsilon\|_{L^\infty}\le\sup_{0\le t<t_0}\|D\bar{v}^\varepsilon(t)\|_{L^\infty}\sup_{0\le t<t_0}\|\varepsilon\bar{S}^\varepsilon(t)\|_{L^\infty}\to 0 \text{ as } \varepsilon\to 0.$$

Therefore,

$$\sup_{0\le t<t_0}\|e_3\cdot(\bar{v}_+^\varepsilon(t)-\bar{v}_-^\varepsilon(t))\|_{L^\infty}\to 0 \text{ as } \varepsilon\to 0. \tag{5.10}$$

We rewrite the jump condition (2.8) as

$$\begin{aligned}&(\bar{\sigma}_+^\varepsilon-\bar{\sigma}_-^\varepsilon\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon)-g[\varrho]\bar{\eta}_{+3}^\varepsilon)(e_3+\varepsilon\bar{N}^\varepsilon)\\ &=|B|^2\Big\{(e_1+\varepsilon\partial_1\bar{\eta}_+^\varepsilon)((e_1+\varepsilon\partial_1\bar{\eta}_+^\varepsilon)\cdot(e_3+\varepsilon\bar{N}^\varepsilon))/\varepsilon\\ &\quad-((e_1+\varepsilon\partial_1\bar{\eta}_-^\varepsilon)\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon))\big[\big((e_1+\varepsilon\partial_1\bar{\eta}_-^\varepsilon)\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon)\big)\cdot(e_3+\varepsilon\bar{N}^\varepsilon)\big]/\varepsilon\Big\},\end{aligned}$$

and find, after a further rearrangement, that

$$\begin{aligned}&(\bar{\sigma}_+^\varepsilon-\bar{\sigma}_-^\varepsilon\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon)-g[\varrho]\bar{\eta}_{+3}^\varepsilon)(e_3+\varepsilon\bar{N}^\varepsilon)\\ &=|B|^2(\bar{N}_1^\varepsilon-\bar{N}_1^\varepsilon\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon)+\partial_1\bar{\eta}_{+3}^\varepsilon-\partial_1\bar{\eta}_{-3}^\varepsilon\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon))e_1+\varepsilon\bar{F}^\varepsilon,\end{aligned}$$

where

$$\begin{aligned}\bar{F}^\varepsilon=&\partial_1\bar{\eta}_+^\varepsilon\cdot\bar{N}^\varepsilon+\partial_1\bar{\eta}_+^\varepsilon(\bar{N}_1^\varepsilon+\partial_1\bar{\eta}_{+3}^\varepsilon+\varepsilon\partial_1\bar{\eta}_+^\varepsilon\cdot\bar{N}^\varepsilon)\\ &-(\partial_1\bar{\eta}_-^\varepsilon\cdot\bar{N}^\varepsilon+\partial_1\bar{\eta}_-^\varepsilon(\bar{N}_1^\varepsilon+\partial_1\bar{\eta}_{-3}^\varepsilon+\varepsilon\partial_1\bar{\eta}_-^\varepsilon\cdot\bar{N}^\varepsilon))\circ(\mathrm{Id}_{\mathbb{R}^2}+\varepsilon\bar{S}_-^\varepsilon)).\end{aligned}$$

Obviously $\sup_{0\le t<t_0}\|\bar{F}^\varepsilon(t)\|_{L^\infty}$ is uniformly bounded. Similar to (5.10), we obtain

$$\sup_{0\le t<t_0}\|(\bar{\sigma}_+^\varepsilon-\bar{\sigma}_-^\varepsilon-g[\varrho]\bar{\eta}_{+3}^\varepsilon)e_3-|B|^2(\partial_1\bar{\eta}_{+3}^\varepsilon-\partial_1\bar{\eta}_{-3}^\varepsilon)e_1\|_{L^\infty}\to 0 \text{ as } \varepsilon\to 0. \tag{5.11}$$

By (5.3), (5.6) and the sequential weak-*compactness, we see that up to the extraction of a subsequence (which we still denote using only ε),

$$(\bar{\eta}^\varepsilon,\bar{v}^\varepsilon,\bar{\sigma}^\varepsilon,\partial_t\bar{\sigma}^\varepsilon)\to(\bar{\eta}^0,\bar{v}^0,\bar{\sigma}^0,\partial_t\bar{\sigma}^0) \text{ weakly-* in } L^\infty(0,t_0;H^4(\Omega))$$

and

$$\partial_t\bar{v}^\varepsilon\to\partial_t\bar{v}^0 \text{ weakly-* in } L^\infty(0,t_0;H^2(\Omega)). \tag{5.12}$$

From lower semi-continuity one gets

$$\sup_{0\le t<t_0}\|(\bar{\eta}^0,\bar{v}^0,\bar{\sigma}^0,\partial_t\bar{\sigma}^0)(t)\|_{H^4}\le 1. \tag{5.13}$$

On the other hand, by (5.3), (5.6), and (5.9), we deduce

$$\limsup_{\varepsilon\to 0}\sup_{0\le t<t_0}\|(\partial_t\bar{\eta}^\varepsilon,\partial_t\bar{v}^\varepsilon,\partial_t\bar{\sigma}^\varepsilon)\|_{H^2}<\infty.$$

By a result in [9], we then find that the set $\{(\bar{\eta}^\varepsilon,\bar{v}^\varepsilon,\bar{\sigma}^\varepsilon)\}$ is strongly pre-compact in the space $L^\infty(0,t_0;H^{11/4}(\Omega))$, thus

$$(\bar{\eta}^\varepsilon,\bar{v}^\varepsilon,\bar{\sigma}^\varepsilon)\to(\bar{\eta}^0,\bar{v}^0,\bar{\sigma}^0) \text{ strongly in } L^\infty(0,t_0;H^{11/4}(\Omega)). \tag{5.14}$$

This strongly convergence, together with the equation $\partial_t\bar{\eta}^\varepsilon=\bar{v}^\varepsilon$, implies that

$$\begin{aligned}&\partial_t\bar{\eta}^\varepsilon\to\partial_t\bar{\eta}^0 \text{ strongly in } L^\infty(0,t_0;H^{11/4}(\Omega)),\\ &\partial_t\bar{v}^\varepsilon\to\partial_t\bar{v}^0 \text{ strongly in } L^\infty(0,t_0;L^2(\Omega)).\end{aligned} \tag{5.15}$$

Utilizing (5.7), (5.8), (5.12), (5.14) and (5.15), we conclude

$$\begin{cases}\partial_t\bar{\eta}^0=\bar{v}^0,\\ \mathrm{div}\bar{v}^0=0,\\ \varrho\partial_t\bar{v}^0+\nabla\bar{\sigma}^0-|B|^2\partial_{11}^2\bar{\eta}^0=0.\end{cases} \tag{5.16}$$

We may pass to the limit in the initial conditions $(\bar{\eta}^\varepsilon,\bar{v}^\varepsilon,\bar{\sigma}^\varepsilon)(0)$=$(\bar{\eta},\bar{v},\bar{\sigma})(0)$ to find that

$$(\bar{\eta}^0,\bar{v}^0,\bar{\sigma}^0)(0)=(\bar{\eta},\bar{v},\bar{\sigma})(0) \tag{5.17}$$

as well.

We now derive the jump and boundary conditions for the limiting functions. The index 11/4 is sufficiently large to give $L^\infty(0,t_0;L^\infty)$-convergence of $(\bar{\eta}^\varepsilon,\bar{v}^\varepsilon,\bar{\sigma}^\varepsilon)$ when restricted to $\{x_3=0\}$, $\{x_3=-1\}$ and $\{x_3=1\}$, i.e. the interface, the lower and upper boundaries. Combing this with (5.10) and (5.11), we infer that

$$\bar{v}_+^0\cdot e_3=0 \text{ on } \{x_3=1\} \text{ and } \bar{v}_-^0\cdot e_3=0 \text{ on } \{x_3=-1\}, \tag{5.18}$$

$$(\bar{v}_+^0-\bar{v}_-^0)\cdot e_3=0 \text{ on } \{x_3=0\}, \tag{5.19}$$

and

$$(\bar{\sigma}_+^0-\bar{\sigma}_-^0-g[\varrho]\bar{\eta}_{+3}^0)e_3-|B|^2(\partial_1\bar{\eta}_{+3}^0-\partial_1\bar{\eta}_{-3}^0)e_1=0 \text{ on } \{x_3=0\}. \tag{5.20}$$

By virtue of (5.16)–(5.20), $(\bar{\eta}^0, \bar{v}^0, \bar{\sigma}^0)$ is a solution of (1.15)–(1.17) with $\bar{B} = (B, 0, 0)$, satisfying the initial conditions (5.17). Thus, Theorem 4.1 guarantees that

$$(\bar{\eta}^0, \bar{v}^0, \bar{\sigma}^0) = (\bar{\eta}, \bar{v}, \bar{\sigma}) \text{ on } [0, t_0) \times \Omega.$$

Hence, we can combine the inequality (5.13) with (5.1) to get

$$2 \leq \sup_{t_0/2 \leq t < t_0} \|(\bar{\eta}^0, \bar{v}^0, \bar{\sigma}^0)(t)\|_{H^4} \leq \sup_{0 \leq t < t_0} \|(\bar{\eta}^0, \bar{v}^0, \bar{\sigma}^0)(t)\|_{H^4} \leq 1,$$

which is a contraction. Therefore, the perturbed problem does not have the property $EE(k)$ for any $k \geq 4$. This completes the proof of Theorem 2.2

Acknowledgements. The research of Ran Duan was supported by NSFC (Grant No. 11001096) and the Special Fund for Basic Scientific Research of Central Colleges, the research of Fei Jiang by the Fujian Provincial Department of Science and Technology (Grant No. JK2009045), and the research of Song Jiang by the National Basic Research Program under the Grant 2011CB309705 and NSFC (Grant No. 40890154)

References

[1] S. Chandrasekhar, Hydrodynamic and Hydromagnetic Stability, The International Series of Monographs on Physics, Oxford, Clarendon Press, 1961.

[2] Y. Guo, H.J. Hwang, On the dynamical Rayleigh-Taylor instability, Arch. Ration. Mech. Anal. 167 (2003) 235–253.

[3] Y. Guo, I. Tice, Compressible, inviscid Rayleigh-Taylor instability, arXiv:0911.4098v2 [math.AP] 23 Feb 2011, to appear in Indiana Univ. Math. J. (2011).

[4] Y. Guo, I. Tice, Linear Rayleigh-Taylor instability for viscous, compressible fluids, SIAM J. Math. Anal. **42** (2011) 1688–1720.

[5] H.J. Hwang, Variational approach to nonlinear gravity-driven instability in a MHD setting, Quart. Appl. Math. **66** (2008) 303–324.

[6] M. Kruskal, M. Schwarzschild, Some instabilities of a completely ionized plasma, Proc. Roy. Soc. (London) A **233** (1954) 348–360.

[7] L. Rayleigh, Analytic solutions of the Rayleigh equations for linear density profiles, Proc. London. Math. Soc. **14** (1883) 170–177.

[8] L. Rayleigh, Investigation of the character of the equilibrium of an in compressible heavy fluid of variable density, Scientific Paper, **II** (1990) 200–207.

[9] J. Simon, Compact sets in the space $l^p(0, t;\, B)$, Ann. Mat. Pura Appl. **146** (1987) 65–96.

[10] G. I. Taylor, The stability of liquid surface when accelerated in a direction perpendicular to their planes, Proc. Roy Soc. A **201** (1950) 192–196.

[11] J.H. Wang, Two-Dimensional Nonsteady Flows and Shock Waves (in Chinese), Beijing, China, Science Press, (1994).

[12] Y. Wang, Critical Magnetic Number in the MHD Rayleigh-Taylor instability, arXiv:1009.5422v1 [math.AP] 28 Sep 2010 (2010).

[13] J. Wehausen, E. Laitone, Surface waves, Handbuch der Physik Vol. 9, Part 3, Springer-Verlag, Berlin, 1960.