Existence of Global Bounded Weak Entropy Solutions to Extended Keyfitz-Kranzer System Modelling Polymer Flooding

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Abstract

In this paper, the Cauchy problem for extended Keyfitz-Kranzer system of conservation laws modelling polymer flooding is studied, and a short proof of global existence of weak entropy solutions is obtained by using a new technique from the compensated compactness theorem coupled with the total variation estimate on one family of Riemann invariant. This work extends in some sense the previous works, [Barkve, SIAM J. on Appl. Math., 49(1989), 784-798] and [Johansen and Winther, SIAM J. Math. Anal., 19 (1988), 541-566], which provided the global existence of weak solutions for Riemann problem; and [Lu, J. Funct. Anal., 261(2011), 2797-2815], which gave the global weak solution of the Cauchy problem for nonstrictly hyperbolic system of Keyfitz-Kranzer or Aw-Rascle type.

1 Introduction

In this paper, we study the Cauchy problem for extended Keyfitz-Kranzer system of conservation laws modelling polymer flooding

$$\begin{cases} S_t + f(S,T)_x = 0, \\ (ST + \beta(T))_t + (Tf(S,T) + \alpha(T))_x = 0, \end{cases} \tag{1.1}$$

with bounded measurable initial data

$$(S(x,0), T(x,0)) = (S_0(x), T_0(x)), \quad S_0(x) \geq 0. \tag{1.2}$$

When $\alpha(T) = aT, \beta(T) = bT$, where a, b are constants, system (1.1) represents a simple model for nonisothermal two-phase flow in a porous medium [Ba, Fa], where S and T are fluid saturation and temperature, respectively. The function $f = f(S,T)$ is the fractional flow function. The Riemann problem was resolved and the entropy conditions for the system were discussed in [Ba] under suitable conditions on f. In [JB], the analytical solutions of Riemann problem were studied for the following equivalent system

$$\begin{cases} S_t + f(S,T)_x = 0, \\ C_t + ((1 - f(S,T))\frac{C}{1-S})_x = 0, \end{cases} \tag{1.3}$$

where $C = T(S-1)$.

When $\beta(T) = 0$ and $\alpha(T) = 0$, system (1.1) is the famous Keyfitz-Kranzer [KK] or Aw-Rascle model [AR], which is of interest because it arises in such areas as elasticity theory, magnetohydrodynamics [JPP], traffic flow [AR] and enhanced oil recovery [KK]. Its Riemann problem was first resolved in [KK, AR] and the existence of a weak solution of the Cauchy problem with initial data of bounded variation was obtained in [IT]. The existence of a unique and stable solution was proved in [TW] for initial data that are constant outside an interval and with T sufficiently smooth and by Klingenberg and Risebro with no smoothness assumptions [KR]. The Cauchy problem with general L^∞ initial data can be found in [LW, KRe, FR, Ch, Pa, Lu2, Lu3, JPP, GP, GO] and the references cited therein. Roughly speaking, the author obtained in [Lu2] the global existence of a weak solution when $f(S,T) = ST - SP(S)$ with the weakest condition on $P(S)$

$$\lim_{S \to 0} SP(S) = 0, \; SP''(S) + 2P'(S) > 0 \text{ for } S > 0, \tag{1.4}$$

which covers the prototype function $P(S) = \frac{1}{\gamma}S^\gamma + A$ with $\gamma > -1$.

When $\alpha(T) = 0$ and $\beta(T) \neq 0$, system (1.1) arises in connection with enhanced oil recovery, where S is the saturation of the mixture of water and polymer, and called the aqueous phase; T is the concentration of polymer in the aqueous phase; the function f describes the fractional flow of the aqueous phase, which is assumed to be immiscible with oil; the function $\beta(T)$ models adsorption of the polymer on rock. The Riemann problem was resolved in [JW] in the domain $(S,T) \in I \times I$, where $I = [0.1]$ under the following conditions (A) and (B):

(A). The real-valued function $f = f(S,T)$ is a smooth function for $(S,T) \in I \times I$ with the properties

(i) $\quad f(0,T) = f(1,T) = 1;$ $\quad (ii)$ $\quad f_S(S,T) > 0$ for $0 < S < 1, 0 \leq T \leq 1;$

(iii) $\quad f_T(S,T) > 0$ for $0 < S < 1, 0 \leq T \leq 1;$

(iv) $\quad$ For each $T \in I, f(\cdot,T)$ has a unique point of inflection, $S^I = S^I(T) \in (0,1)$, such that $f_{SS}(S,T) > 0$ for $0 < S < S^I$ and $f_{SS}(S,T) < 0$ for $S^I < S < 1$.

(B). The function $\beta(T)$ is a smooth function of $T \in I$ such that

(i) $\quad \beta(0) = 0;$ $\quad (ii)$ $\quad \beta'(T) > 0$ for $0 < T < 1;$ $\quad (iii)$ $\quad \beta''(T) < 0$ for $0 < T < 1;$

For general functions $\beta(T)$ and $\alpha(T)$, as far as we know, there is no any existence result about the Cauchy problem of system (1.1).

The main object of this paper is to present the global weak solution of the Cauchy problem (1.1) and (1.2), where the initial data $(S_0(x), T_0(x)), S_0(x) \geq 0$, are bounded and $T_0(x)$ is of bounded total variation. Our technique is the vanishing viscosity method coupled with the compensated compactness theory developed by Tartar [Ta] and Murat [Mu]. The classical vanishing viscosity method is to add the viscosity terms to the right-hand side of system (1.1) and consider the Cauchy problem for the related parabolic system

$$\begin{cases} S_t + (S\phi(S,T))_x = \varepsilon S_{xx}, \\ \\ (ST + \beta(T))_t + (ST\phi(S,T) + \alpha(T))_x = \varepsilon(ST + \beta(T))_{xx}, \end{cases} \tag{1.5}$$

with initial data

$$(S^\varepsilon(x,0), T^\varepsilon(x,0)) = (S_0^\varepsilon(x), T_0^\varepsilon(x)), \tag{1.6}$$

where $T_0^\varepsilon(x) = T_0(x) * G^\varepsilon, S_0^\varepsilon(x) = (\varepsilon + (1-\varepsilon)S_0(x)) * G^\varepsilon$ are the smooth approximations of $T_0(x), S_0(x)$, G^ε is a mollifier, and $\phi(S,T) = \frac{f(S,T)}{S}$. However, if we consider (S,m), where $m = ST$ as two independent variables in the parabolic system (1.5), then many terms in (1.5) are singular near the line $S = 0$ since $T = \frac{m}{S}$.

Comparing the previous results of (1.1) introduced above, mainly we meet the following three difficulties when we study the Cauchy problem for system (1.1).

Difficulty I. How to obtain the positive, lower bound of the viscosity solutions S^ε for the Cauchy problem (1.5) with suitable initial data?

As we introduced in [Lu2], there are three basic methods to obtain the bound $S^\varepsilon \geq c(\varepsilon, t) > 0$ for some suitable function $c(\varepsilon, t)$ (See [Di, Pe, Lu4] for the details). However, a basic boundedness condition of $\phi(S,T)$ is necessary in all these papers. In this paper, we always assume that $f(S,T)$ is smooth for $S > 0$ and satisfies

$$\begin{cases} |f(S,T)| = O(S^l), \quad |f_S(S,T)| = O(S^{l-1}), \\ |f_{SS}(S,T)| = O(S^{l-2}), \quad |f_T(S,T)| = O(S), \\ |f_{ST}(S,T)| = O(1), \quad |f_{STT}(S,T)| = O(1) \end{cases} \tag{1.7}$$

near the line $S = 0$, where $l > 0$ and $O(1)$ means a bounded function, which covers the examples $f(S,T) = ST - SP(S), P(S) = \frac{1}{\gamma} S^\gamma + A$ with $\gamma > -1$ studied in [Lu2]. If f is a smooth function on $S \geq 0$, clearly it satisfies all conditions in (1.7).

When $f(S,T)$ satisfies (1.7) and $0 < l < 1, \phi(S,T) = \frac{f(S,T)}{S}$ is singular near the line $S = 0$, all methods given in [Di, Pe, Lu4] to obtain the bound $S^\varepsilon \geq c(\varepsilon, t) > 0$ are invalid for system (1.5). To overcome this difficulty, instead of the classical viscosity approximation, we use again the flux approximation introduced in [Lu2] and consider the following parabolic system

$$\begin{cases} S_t + (\frac{(S-\delta)}{S} f(S,T))_x = \varepsilon S_{xx}, \\ (ST + \beta(T))_t + (\frac{(S-\delta)}{S} T f(S,T) + \alpha(T))_x = \varepsilon (ST + \beta(T))_{xx}, \end{cases} \tag{1.8}$$

with initial data

$$(S^{\varepsilon,\delta}(x,0), T^{\varepsilon,\delta}(x,0)) = (S_0^{\varepsilon,\delta}(x), T_0^{\varepsilon,\delta}(x)), \tag{1.9}$$

where $T_0^{\varepsilon,\delta}(x) = T_0(x) * G^\varepsilon, S_0^{\varepsilon,\delta}(x) = (\varepsilon + \delta + (1 - (\varepsilon + \delta)) S_0(x)) * G^\varepsilon$ are the smooth approximations of $T_0(x), S_0(x)$, G^ε is a mollifier and ε, δ are positive, small perturbation constants. Since $S_0^{\varepsilon,\delta}(x) \geq \varepsilon + \delta$, applying the maximum principle to the first equation in (1.8), we first have $S^{\varepsilon,\delta}(x,t) \geq \delta$. Since $\phi(S,T)$ is bounded in $S \geq \delta$ (the bound could depend on δ), we may obtain the lower bound

$$S^{\varepsilon,\delta} \geq c(t, \varepsilon, \delta) > \delta, \tag{1.10}$$

where $c(t,\varepsilon,\delta)$ could tend to δ as the time t tends to infinity or ε tends to zero (See $Theorem 1.0.2$ in [Lu1] for the details).

Difficulty II. How to obtain the uniformly, upper bound $S^{\varepsilon,\delta} \leq M$?

For Keyfitz-Kranzer system studied in [Lu2] (i.e. $\alpha(T)=\beta(T)=0$), the upper bound $S^{\varepsilon,\delta} \leq M$ is obtained by using the invariant region theorem [CCS] since it is easy to calculate the Riemann invariants. However, for system (1.1) with general functions $\alpha(T), \beta(T)$, the explicit Riemann invariants are only available when both $\alpha(T)$ and $\beta(T)$ are linear functions. If $\alpha(T)=aT$ and $\beta(T)=bT$, where a,b are constants, one Riemann invariant is T and another is $\frac{f+a}{S+b}$ ([Ba]).

In this paper, we will study the upper bound $S^{\varepsilon,\delta} \leq M$ in two different cases. First, f satisfies the condition $f(M,T)=d$ as given in Condition (A) above, where d is a constant, and M a upper bound of the initial data $S_0(x)$; Second, both $\alpha(T)$ and $\beta(T)$ are linear functions.

For general nonlinear functions $f, \alpha(T)$ and $\beta(T)$, we will study the upper estimate $S^{\varepsilon,\delta} \leq M$ in a coming paper.

Difficulty III. How to prove the pointwise convergence of $T^{\varepsilon,\delta}$ as ε,δ go to zero?

When the initial data $T_0(x)$ is of bounded total variation, it is easy to prove that $T_x^{\varepsilon,\delta}(\cdot,t)$ is uniformly bounded in $L^1(R)$ by using a technique by Serre [Se]. If we could also prove that $T_t^{\varepsilon,\delta}(x,\cdot)$ is uniformly bounded in $L^1(R^+)$, then the pointwise convergence of $T^{\varepsilon,\delta}$ would follow immediately. However, to obtain the uniform bound of $T_t^{\varepsilon,\delta}(x,\cdot)$ in $L^1(R^+)$ is very difficult or even impossible.

In this paper, a very short proof of the pointwise convergence of $T^{\varepsilon,\delta}$ is obtained by using a new idea to apply for the div-curl lemma to some special pairs of functions $(c, F(T))$, where c is a constant and $F(T)$ is a suitable function of T. We can prove the H^{-1}_{loc} compactness of $c_t + F(T^{\varepsilon,\delta})_x$ since the $L^1(R)$ estimate of $T_x^{\varepsilon,\delta}(\cdot,t)$.

The main results of this paper are listed in Theorems 1-3.

First, we have the following compactness framework theorem

<u>Theorem</u> 1 *(I). Suppose $(S_0(x),T_0(x))$ are bounded, $S_0(x)\geq 0$ and $T_0(x)$ is of bounded total variation; $\alpha(T),\beta(T)$ are suitable smooth functions and $f(S,T)$ satisfies the condition (1.7); $\beta'(T)\geq 0$, meas $\{T : \beta''(T)=0\}=0$ or $\beta(T)=bT, b>0$; meas $\{S : f_{SS}(S,T)=0\}=0$ for any fixed T. Assume $S^{\varepsilon,\delta}(x,t)$ has a uniformly bounded estimate $S^{\varepsilon,\delta}(x,t)\leq M$, where M is a constant independent* *of ε and δ, then the global smooth solution of the Cauchy problem (1.8) and (1.9) exists, and there exists a subsequence (still labelled $(S^{\varepsilon,\delta}(x,t), T^{\varepsilon,\delta}(x,t))$) such that*

$$(S^{\varepsilon,\delta}(x,t), T^{\varepsilon,\delta}(x,t)) \to (S(x,t), T(x,t)) \tag{1.11}$$

as ε, δ go to zero, a.e. on any bounded and open set $\Omega \subset R \times R^+$, and the limit (S,T) is a weak entropy solution of the Cauchy problem (1.1) and (1.2), namely satisfies the Lax entropy condition.

(II). If $\beta(T) = 0$ and all other conditions in (I) are satisfied, then there exists a subsequence (still labelled $(S^{\varepsilon,\delta}(x,t), T^{\varepsilon,\delta}(x,t))$) such that

$$T^{\varepsilon,\delta}(x,t) \to T(x,t) \tag{1.12}$$

pointwisely on any support set $\{(x,t) : S(x,t) > 0\}$ and

$$S^{\varepsilon,\delta}(x,t) \to S(x,t) \tag{1.13}$$

as ε, δ go to zero, a.e. on any bounded and open set $\Omega \subset R \times R^+$. Particularly, if $\alpha(T) = 0$, the limit (S,T) is a weak entropy solution of the Cauchy problem (1.1) and (1.2).

Remark 1. As a direct corollary of (II) in Theorem 1, and the L^∞ estimates obtained in [Lu2] when the nonlinear function f satisfies (1.4), we have a simple and short proof of Theorems 2,3 in [Lu2] for the Keyfitz-Kranzer or Aw-Rascle model.

Second, we have the following theorems about the upper bound estimate $S^{\varepsilon,\delta} \leq M$.

Theorem 2 *If all conditions in (I) of Theorem 1 are true, f is a suitable smooth function, and satisfies $f(0,T) = 0, f(M,T) = d$, where d is a constant and M is a upper bound of $S_0(x)$, then the a priori L^∞ estimate $S^{\varepsilon,\delta}(x,t) \leq M$ is true, and so the Cauchy problem (1.1) and (1.2) has a weak entropy solution.*

Remark 2. The conditions in Theorems 1 and 2 are much weaker than (A) and (B) above for obtaining the existence of Riemann solution in [Ba, JW].

Theorem 3 *If $\alpha(T) = aT$ and $\beta = bT, b > 0$, $f(0,T) = 0$ and $f(S,T) = F(S,C)$ is a convex, smooth function with respect to the variables S and C, where $C = (S+b)T$, moreover, if $\lim_{S \to +\infty} \frac{f(S,T)}{S} = +\infty$, then the a priori L^∞ estimate $S^{\varepsilon,\delta}(x,t) \leq M$ is true, and so the Cauchy problem (1.1) and (1.2) has a weak entropy solution.*

Remark 3. It is worthwhile to point out that the results in Theorem 1 can be easily extended to the following nonstrictly hyperbolic conservation system of $n+1$ equations

$$\begin{cases} \rho_t + f(\rho, u_1, u_2, ...u_n)_x = 0 \\ (\rho u_i + \beta_i(u_i))_t + (u_i f(\rho, u_1, u_2, ...u_n) + \alpha_i(u_i))_x = 0, \quad i = 1, 2, ...n \end{cases} \tag{1.14}$$

with bounded measurable initial data

$$(\rho(x,0), u_i(x,0)) = (\rho_0(x), u_{i0}(x)), \quad 0 \le \rho_0(x) \le c_0, \quad -c_i^1 \le u_{i0}(x) \le c_i^2. \tag{1.15}$$

Let

$$\begin{cases} |f(\rho, u_1, u_2, ...u_n)| = O(\rho^l), \quad |f_\rho| = O(\rho^{l-1}), \\ |f_{\rho\rho}| = O(\rho^{l-2}), \quad |f_{u_i}| = O(\rho), \\ |f_{\rho u_i}| = O(1), \quad |f_{\rho u_i u_j}| = O(1) \end{cases} \tag{1.16}$$

near the line $\rho = 0$, for $i, j = 1.2...n$, where $l > 0$ and $O(1)$ means a bounded function.

We consider the flux-viscosity approximation

$$\begin{cases} \rho_t + (\frac{\rho-\delta}{\rho} f(\rho, u_1, u_2, ...u_n))_x = \varepsilon \rho_{xx} \\ (\rho u_i + \beta_i(u_i))_t + (\frac{\rho-\delta}{\rho} u_i f(\rho, u_1, u_2, ...u_n) + \alpha_i(u_i))_x = \varepsilon(\rho u_i + \beta_i(u_i))_{xx} \end{cases} \tag{1.17}$$

with suitable smooth initial data like (1.9). Then we have

Theorem 4 *(I). Suppose $(\rho_0(x), u_{i0}(x))$ are bounded, $\rho_0(x) \ge 0$ and $u_{i0}(x)$ are of bounded total variation; $\alpha_i(u_i), \beta_i(u_i)$ are suitable smooth functions and f satisfies the condition (1.16); $\beta_i'(u_i) \ge 0$, meas $\{u_i : \beta_i''(u_i) = 0\} = 0$ or $\beta_i(u_i) = b_i u_i, b_i > 0$ are constants; meas $\{\rho : f_{\rho\rho}(\rho, u_1, u_2, ...u_n) = 0\} = 0$ for any fixed $(u_1, u_2, ...u_n)$. Assume $\rho^{\varepsilon,\delta}(x,t)$ has a uniformly bounded estimate $\rho^{\varepsilon,\delta}(x,t) \le M$, where M is a constant independent of ε, then the global smooth solution of the Cauchy problem (1.17) with suitable initial data exists, and there exists a subsequence (still labelled $(\rho^{\varepsilon,\delta}(x,t), u_i^{\varepsilon,\delta}(x,t))$) such that*

$$(\rho^{\varepsilon,\delta}(x,t), u_i^{\varepsilon,\delta}(x,t)) \to (\rho(x,t), u_i(x,t)) \tag{1.18}$$

as ε, δ go to zero, a.e. on any bounded and open set $\Omega \subset R \times R^+$, and the limit (ρ, u_i) is a weak entropy solution of the Cauchy problem (1.14) and (1.15).

(II). If $\beta_i(u_i) = 0$ and all other conditions in (I) are satisfied, then there exists a subsequence (still labelled $(S^{\varepsilon,\delta}(x,t), T^{\varepsilon,\delta}(x,t))$) such that

$$u_i^{\varepsilon,\delta}(x,t) \to u_i(x,t) \tag{1.19}$$

pointwisely on any support set $\{(x,t) : \rho(x,t) > 0\}$ and

$$u_i^{\varepsilon,\delta}(x,t) \to u_i(x,t) \tag{1.20}$$

as ε, δ go to zero, a.e. on any bounded and open set $\Omega \subset R \times R^+$. Particularly, if $\alpha_i(u_i) = 0$, the limit (ρ, u_i) is a weak entropy solution of the Cauchy problem (1.14) and (1.15).

We will prove Theorems 1-3 in the next two sections.

2 Proof of Theorem 1

The proof of Theorem 1 is divided into several lemmas. From (1.9), it is easy to see that $|T_0^{\varepsilon,\delta}(x)| \le M$ and $0 < \varepsilon + \delta \le S_0^{\varepsilon,\delta}(x) \le M$, where M is a suitable constant, which is independient of ε, δ.

Lemma 5 *For any fixed $\varepsilon > 0, \delta > 0$, if we assume that the viscosity solution of the Cauchy problem (1.8) and (1.9) has the a-priori upper estimate $S^{\varepsilon,\delta}(x,t) \le M$, then its global smooth solution $(S^{\varepsilon,\delta}(x,t), T^{\varepsilon,\delta}(x,t))$ exists, and has the following estimates*

$$0 < \delta < c(t,\varepsilon,\delta) \le S^{\varepsilon,\delta}(x,t) \le M, \quad |T^{\varepsilon}(x,t)| \le M, \tag{2.1}$$

where $c(t,\varepsilon,\delta)$ could tend to δ as the time t tends to infinity or ε tends to zero. Furthermore if $T_0(x)$ is of bounded total variation, then

$$\int_{-\infty}^{\infty} |T_x^{\varepsilon,\delta}|(x,t)dx \le \int_{-\infty}^{\infty} |T_x^{\varepsilon,\delta}|(x,0)dx \le M. \tag{2.2}$$

Proof of Lemma 5. Let $C = ST + \beta(T)$. Since $\beta'(T) \ge 0$, then for any fixed $S \in (0, M]$, there exists a smooth, inverse function $T = g(S,C)$ and system (1.8) can be rewritten as

$$\begin{cases} S_t + (\frac{(S-\delta)}{S} f(S, g(S,C)))_x = \varepsilon S_{xx}, \\ \\ C_t + (\frac{(S-\delta)}{S} g(S,C) f(S, g(S,C)) + \alpha(g(S,C)))_x = \varepsilon C_{xx}, \end{cases} \tag{2.3}$$

where the flux functions $\frac{(S-\delta)}{S}f(S,g(S,C))$, $\frac{(S-\delta)}{S}g(S,C)f(S,g(S,C))+\alpha(g(S,C))$ are smooth in the domain $S>0$.

Substituting the first equation in (1.8) into the second, we may rewrite the second equation in (1.8) as

$$T_t+\frac{\frac{(S-\delta)}{S}f+\alpha'(T)}{S+\beta'(T)}T_x=\varepsilon T_{xx}+\varepsilon\frac{2S_x+\beta''(T)T_x}{S+\beta'(T)}T_x. \tag{2.4}$$

Then we have the estimates $|T^\varepsilon(x,t)|\le M$ by applying the maximum principle to (2.4). Since $S_0^{\varepsilon,\delta}(x)\ge\varepsilon+\delta$, applying the maximum principle to the first equation in (1.8), we may first have $S^{\varepsilon,\delta}(x,t)\ge\delta$. If we assume the a-priori upper estimate $S^\varepsilon(x,t)\le M$, then $\phi(S,T)=\frac{f(S,T)}{S}$ is bounded in $\delta\le S\le M$ (the bound could depend on δ), and we may obtain the lower bound (1.10) (See Theorem 1.0.2 in [Lu1] for the details).

Thus the existence of the viscosity solution for the Cauchy problem (1.8)-(1.9) (or (2.3)-(1.9)) can be proved by the standard theory of semilinear parabolic systems, namely the local existence and the bounded estimate given in (2.1). Using (2.4), the estimate (2.2) can be proved by applying for a technique from [Se] or [Lu2, Lu3]. So, the proof of Lemma 5 is completed.

Since $T_x^{\varepsilon,\delta}$ are uniformly bounded both in $L^1_{loc}(R\times R^+)$ and $W^{-1,\infty}_{loc}(R\times R^+)$ from the estimates (2.1)-(2.2), we have from Murat's lemma that

Lemma 6

$$c_t+h(T^{\varepsilon,\delta})_x \quad \textit{are compact in} \quad H^{-1}_{loc}(R\times R^+) \tag{2.5}$$

for any constant c and smooth function h.

Furthermore we have

Lemma 7

$$S_t^{\varepsilon,\delta}+f^\delta(S^{\varepsilon,\delta},T^{\varepsilon,\delta})_x \quad \textit{are compact in} \quad H^{-1}_{loc}(R\times R^+), \tag{2.6}$$

where $f^\delta(S,T)=\frac{(S-\delta)}{S}f(S,T)$.

Proof of Lemma 7. We multiply $g'(S)$ to the first equation in (2.3) to obtain (for simplicity, we omit the superscript ε)

$$\begin{aligned}&g(S)_t+(\textstyle\int_0^S g'(\tau)f_S^\delta(\tau,T)d\tau)_x+(g'(S)f_T^\delta-\int_0^S g'(\tau)f_{ST}^\delta(\tau,T)d\tau)T_x\\&=\varepsilon g(S)_{xx}-\varepsilon g''(S)S_x^2.\end{aligned} \tag{2.7}$$

Since T_x is bounded in $L^1_{loc}(R \times R^+)$ and the condition (1.7), then the term in (2.7)

$$(g'(S)f^\delta_T - \int_0^S g'(\tau)f^\delta_{ST}(\tau,T)d\tau)T_x$$

is bounded in $L^1_{loc}(R \times R^+)$.

Choosing a strictly convex function g and multiplying a suitable nonnegative test function to (2.7), we have that εS_x^2 is bounded in $L^1_{loc}(R \times R^+)$. So the right-hand side of the first equation in (2.3) is compact in $H^{-1}_{loc}(R \times R^+)$. Lemma 7 is proved.

Lemma 8

$$(S^{\varepsilon,\delta}h(T^{\varepsilon,\delta}) + \int_0^{T^{\varepsilon,\delta}} h'(\tau)\beta'(\tau)d\tau)_t + (h(T^{\varepsilon,\delta})f^\delta(S^{\varepsilon,\delta},T^{\varepsilon,\delta}) + \int_0^{T^{\varepsilon,\delta}} h'(\tau)\alpha'(\tau)d\tau)_x \tag{2.8}$$

are compact in $H^{-1}_{loc}(R \times R^+)$*, where* h *is an arbitrary smooth function of* T*.*

Proof of Lemma 8. We multiply $S + \beta'(T)$ to (2.4) to obtain

$$ST_t + \beta(T)_t + f^\delta T_x + \alpha(T)_x = \varepsilon(S + \beta'(T))T_{xx} + \varepsilon(2S_x + \beta''(T)T_x)T_x. \tag{2.9}$$

We multiply (2.9) by $h'(T)$, the first equation in (2.3) by $h(T)$ and then add the result to obtain

$$\begin{aligned}&(Sh(T) + \textstyle\int_0^T h'(\tau)\beta'(\tau)d\tau)_t + (h(T)f^\delta(S,T) + \textstyle\int_0^T h'(\tau)\alpha'(\tau)d\tau)_x \\ &= \varepsilon(Sh(T))_{xx} - \varepsilon(Sh''(T) + h''(T)\beta'(T))(T_x)^2 + \varepsilon(h'(T)\beta'(T)T_x)_x.\end{aligned} \tag{2.10}$$

If we multiply a suitable nonnegative test function to (2.10) and choose a strictly convex function $h(T)$, we may first prove from (2.10) that

$$\varepsilon S^{\varepsilon,\delta}(T_x^{\varepsilon,\delta})^2, \quad \varepsilon\beta'(T^{\varepsilon,\delta})(T_x^{\varepsilon,\delta})^2 \quad \text{are bounded in } L^1_{loc}(R \times R^+), \tag{2.11}$$

which imply that, for any smooth function h, the first and third terms on the right-hand side of (2.10) are compact in $H^{-1}_{loc}(R \times R^+)$ and the second term is uniformly bounded in $L^1_{loc}(R \times R^+)$. Thus the right-hand side of (2.10) is compact in $W^{-1,q}_{loc}(R \times R^+), 1 < q < 2$ by Sobolev's embedding theorem. Furthermore since the left-hand side of (2.10) is bounded in $W^{-1,\infty}(R \times R^+)$, we may use Murat's lemma to obtain the proof of Lemma 8.

<u>Lemma</u> 9 *If meas* $\{T : \beta''(T) = 0\} = 0$ *or* $\beta(T) = bT, b > 0$, *then there exists a subsequence (still labelled* $T^{\varepsilon,\delta}(x,t)$ *) such that*

$$T^{\varepsilon,\delta}(x,t) \to T(x,t) \tag{2.12}$$

a.e. on any bounded and open set $\Omega \subset R \times R^+$.

Proof of Lemma 9. We first apply for the div-curl lemma in the compensated compactness theory [Ta] to the following special pairs of functions

$$(c, h(T^{\varepsilon,\delta})), \quad (S^{\varepsilon,\delta}, f^\delta(S^{\varepsilon,\delta}, T^{\varepsilon,\delta})) \tag{2.13}$$

to obatin

$$\overline{S^{\varepsilon,\delta}} \cdot \overline{h(T^{\varepsilon,\delta})} = \overline{S^{\varepsilon,\delta} h(T^{\varepsilon,\delta})}, \tag{2.14}$$

where $\overline{f(\theta^{\varepsilon,\delta})}$ denotes the weak-star limit of $f(\theta^{\varepsilon,\delta})$.

We choose $h(T) = T$ in (2.8), and have from (2.6) that

$$(S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k) + \beta(T^{\varepsilon,\delta}) - \beta(k))_t + ((T^{\varepsilon,\delta} - k)f^\delta(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) + \alpha(T^{\varepsilon,\delta}))_x \tag{2.15}$$

are compact in $H^{-1}_{loc}(R \times R^+)$ for any constant k.

Second, we apply for the div-curl lemma to the following pairs of functions

$$(c, \beta(T^{\varepsilon,\delta}) - \beta(k)), \quad (S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k) + \beta(T^{\varepsilon,\delta}) - \beta(k), (T^{\varepsilon,\delta} - k)f^\delta(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) + \alpha(T^{\varepsilon,\delta})) \tag{2.16}$$

to obatin

$$\begin{aligned} &\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k)(\beta(T^{\varepsilon,\delta}) - \beta(k))} + \overline{(\beta(T^{\varepsilon,\delta}) - \beta(k))^2} \\ &= \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k)} \cdot \overline{\beta(T^{\varepsilon,\delta}) - \beta(k)} + (\overline{\beta(T^{\varepsilon,\delta})} - \beta(k))^2. \end{aligned} \tag{2.17}$$

Finally, letting $h(T) = \beta(T) - \beta(k)$ in (2.10), we apply for the div-curl lemma to the following pairs of functions $(c, T^{\varepsilon,\delta} - k)$ and

$$(S^{\varepsilon,\delta}(\beta(T^{\varepsilon,\delta}) - \beta(k)) + \int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau, h(T^{\varepsilon,\delta})f(S^{\varepsilon,\delta}, T^\varepsilon) + \int_0^{T^\varepsilon} h'(\tau)\alpha'(\tau)d\tau) \tag{2.18}$$

to obatin

$$\begin{aligned} &\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - k)(\beta(T^{\varepsilon,\delta}) - \beta(k))} + \overline{(T^{\varepsilon,\delta} - k)\int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau} \\ &= (\overline{T^{\varepsilon,\delta}} - k) \cdot \overline{S^{\varepsilon,\delta}(\beta(T^{\varepsilon,\delta}) - \beta(k))} + (\overline{T^{\varepsilon,\delta}} - k) \cdot \overline{\int_k^{T^{\varepsilon,\delta}} (\beta'(\tau))^2 d\tau}. \end{aligned} \tag{2.19}$$

Since we have from (2.14) that

$$\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta}-k)} = \overline{S^{\varepsilon,\delta}} \cdot \overline{(T^{\varepsilon,\delta}-k)}, \quad \overline{S^{\varepsilon,\delta}(\beta(T^{\varepsilon,\delta})-\beta(k))} = \overline{S^{\varepsilon,\delta}} \cdot \overline{(\beta(T^{\varepsilon,\delta})-\beta(k))}, \tag{2.20}$$

then (2.17) and (2.19) reduce that

$$\begin{aligned} &\overline{(T^{\varepsilon,\delta}-k)\int_k^{T^{\varepsilon,\delta}}(\beta'(\tau))^2 d\tau} - \overline{(\beta(T^{\varepsilon,\delta})-\beta(k))^2} \\ &= (\overline{T^{\varepsilon,\delta}}-k)\cdot \overline{\int_k^{T^{\varepsilon,\delta}}(\beta'(\tau))^2 d\tau} - (\overline{\beta(T^{\varepsilon,\delta})}-\beta(k))^2. \end{aligned} \tag{2.21}$$

Let $\overline{T^{\varepsilon,\delta}} = T$. By simple calculations (see Theorem 3.1.1 in [Lu1] for the details), we have

$$\overline{(T^{\varepsilon,\delta}-T)\int_T^{T^{\varepsilon,\delta}}(\beta'(\tau))^2 d\tau - (\beta(T^{\varepsilon,\delta})-\beta(T))^2} + (\overline{\beta(T^{\varepsilon,\delta})}-\beta(T))^2 = 0, \tag{2.22}$$

which implies the pointwise compactness of T^{ε} if meas $\{T : \beta''(T) = 0\} = 0$.

If $\beta(T)$ is a linear function, $\beta(T) = bT, b > 0$, then letting $h(T) = T$ in (2.8) and applying for the div-curl lemma to the following special pairs of functions

$$(c, T^{\varepsilon,\delta}), \quad (S^{\varepsilon,\delta}T^{\varepsilon,\delta} + bT^{\varepsilon,\delta}, T^{\varepsilon,\delta}f^{\delta}(S^{\varepsilon,\delta}, T^{\varepsilon,\delta}) + \alpha(T^{\varepsilon,\delta})) \tag{2.23}$$

to obatin

$$\overline{(S^{\varepsilon,\delta}T^{\varepsilon,\delta} + bT^{\varepsilon,\delta})} \cdot \overline{T^{\varepsilon,\delta}} = \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta})^2} + b\overline{(T^{\varepsilon,\delta})^2}, \tag{2.24}$$

Let $h(T) = T$ in (2.14). We have from (2.14) and (2.24) that

$$\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta}-T)^2} = \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta})^2} - 2\overline{S^{\varepsilon,\delta}T^{\varepsilon,\delta}}T + ST^2 = bT^2 - b\overline{(T^{\varepsilon,\delta})^2} \tag{2.25}$$

Since the left-hand side of (2.25) is nonnegative, and the right-hand side is non-positive, we know that both sides of (2.25) must be zero. From

$$T^2 - \overline{(T^{\varepsilon,\delta})^2} = 0, \tag{2.26}$$

we get the pointwise convergence of $T^{\varepsilon,\delta}$. The proof of Lemma 9 is completed.

Lemma 10 *If $\beta(T) = 0$, then there exists a subsequence (still labelled $T^{\varepsilon,\delta}(x,t)$) such that*

$$T^{\varepsilon,\delta}(x,t) \to T(x,t) \tag{2.27}$$

pointwisely on any support set $\{(x,t) : S(x,t) > 0\}$, where $S(x,t)$ is the weak limit of $S^{\varepsilon,\delta}(x,t)$.

Proof of Lemma 10. If $\beta(T) = 0$, we have from (2.25) that

$$\overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta} - T)^2} = \overline{S^{\varepsilon,\delta}(T^{\varepsilon,\delta})^2} - 2\overline{S^{\varepsilon,\delta}T^{\varepsilon,\delta}}T + ST^2 = 0 \tag{2.28}$$

which reduces the conclusion (2.27) in Lemma 10.

<u>**Lemma**</u> **11** *If meas* $\{S : f_{SS}(S,T) = 0\} = 0$ *for any fixed* T*, then there exists a subsequence (still labelled* $S^{\varepsilon}(x,t)$ *) such that*

$$S^{\varepsilon}(x,t) \to S(x,t) \tag{2.29}$$

a.e. on any bounded and open set $\Omega \subset R \times R^+$*.*

Proof of Lemma 11. Letting $F = f^{\delta}$ and multiplying rS^{r-1} to the first equation in (2.3), we have

$$v_t + G(v,T)_x - \int_c^S r\tau^{r-1}F_{ST}(\tau,T)d\tau T_x + rS^{r-1}F_T(S,T)T_x = \varepsilon rS^{r-1}S_{xx}, \tag{2.30}$$

where $r \geq 2$ and c are constants, $v = S^r$ and $G(v,T) = \int_c^S r\tau^{r-1}F_S(\tau,T)d\tau$.

By simple calculations,

$$\begin{cases} G_v = rS^{r-1}F_S(S,T)\frac{ds}{dv} = F_S(S,T), \quad G_T = \int_c^S r\tau^{r-1}F_{ST}(\tau,T)d\tau, \\ G_{vv} = \frac{1}{rS^{r-1}}F_{SS}(S,T), \quad G_{vT} = F_{ST}. \end{cases} \tag{2.31}$$

Multiplying (2.30) by G_v, (2.4) by G_T respectively, then adding the result, we have (for simplicity, we omit the superscript ε, δ)

$$\begin{aligned} &G_t + (\textstyle\int_c^v G_v^2(\tau,T)d\tau)_x - \int_c^v 2G_v(\tau,T)G_{vT}(\tau,T)d\tau T_x + rS^{r-1}F_T(S,T)T_x \\ &+G_T\frac{F+\alpha'(T)}{S+\beta'(T)}T_x = \varepsilon rS^{r-1}G_vS_{xx} + \varepsilon G_TT_{xx} + \varepsilon\frac{2G_T}{S+\beta'(T)}S_xT_x + \varepsilon\frac{\beta''(T)G_T}{S+\beta'(T)}T_x^2. \end{aligned} \tag{2.32}$$

Since the pointwise convergence of $T^{\varepsilon}(x,t)$ proved in Lemmas 9 and 10, to prove Lemma 11, we may use the compensated compactness lemma on the scalar conservation equation with a space-time discontinuous flux [KT, Lu1]. This means that we only need to prove that both $(v-c)_t + (G(v,T) - G(c,T))_x$ and $(G(v,T) - G(c,T))_t + (\int_c^v G_v^2(\tau,T)d\tau)_x$ are compact in $H_{loc}^{-1}(R \times R^+)$.

If we choose $g(S) = S^{l_1}$ in (2.7), where l_1 is an arbitrary positive constant, we may prove that

$$\varepsilon S^{l_1-2}S_x^2 \quad \text{are bounded in} \quad L_{loc}^1(R \times R^+). \tag{2.33}$$

Using (2.33) and the boundedness of $T_x(\cdot,t)$ in $L^1(R)$, we have from (2.30) that

$$(v-c)_t + (G(v,T) - G(c,T))_x \quad \text{are compact in} \quad H^{-1}_{loc}(R\times R^+). \tag{2.34}$$

Since the conditions about f in (1.7), by simple calculations, we have

$$\begin{cases} |F(S,T)| = O(S^l), \quad |F_S(S,T)| = O(S^{l-1}), \\ |F_{SS}(S,T)| = O(S^{l-2}), \quad |F_T(S,T)| = O(S), \\ |F_{ST}(S,T)| = O(1), \quad |F_{STT}(S,T)| = O(1) \end{cases} \tag{2.35}$$

near the line $S=\delta$. Then we can estimate the terms in (2.32) as follows.

First,

$$\int_c^v G_v^2(\tau,T)d\tau \approx \int_c^v \tau^{\frac{2(l-1)}{r}} d\tau \approx v^{\frac{2(l-1)}{r}+1} + d \tag{2.36}$$

near the line $v=0$, where d is a constant. So, $\int_c^v G_v^2(\tau,T)d\tau$ is a continuous function;

Second

$$|(-\int_c^v 2G_v(\tau,T)G_{vT}(\tau,T)d\tau + rS^{r-1}F_T(S,T) + G_T\frac{F+\alpha'(T)}{S+\beta'(T)})T_x| \le M|T_x| \tag{2.37}$$

are bounded in $L^1_{loc}(R\times R^+)$ and so compact in $W^{-1,q}_{loc}(R\times R^+), 1<q<2$;

Third, since the estimates (2.11),(2.31), (2.33) and (2.35), we have that

$$\varepsilon\frac{2G_T}{S+\beta'(T)}S_xT_x + \varepsilon\frac{\beta''(T)G_T}{S+\beta'(T)}T_x^2 \tag{2.38}$$

are bounded in $L^1_{loc}(R\times R^+)$ and so compact in $W^{-1,q}_{loc}(R\times R^+), 1<q<2$, and

$$\begin{aligned} &\varepsilon rS^{r-1}G_vS_{xx} + \varepsilon G_TT_{xx} = \varepsilon rS^{r-1}F_SS_{xx} + \varepsilon G_TT_{xx} \\ &= \varepsilon(rS^{r-1}F_SS_x)_x + \varepsilon(G_TT_x)_x - \varepsilon(rS^{r-1}F_{SS} + r(r-1)S^{r-2}F_S)S_x^2 - \varepsilon G_{TT}T_x^2, \end{aligned} \tag{2.39}$$

where on the right-hand side of (2.39), the last two terms are bounded in $L^1_{loc}(R\times R^+)$ and so compact in $W^{-1,q}_{loc}(R\times R^+), 1<q<2$; the first two terms are compact in $H^{-1}_{loc}(R\times R^+)$.

Therefore, we proved from (2.32) that

$$G(v,T)_t + (\int_c^v G_v^2(\tau,T)d\tau)_x \quad \text{are compact in} \quad W^{-1,q}_{loc}(R\times R^+) \tag{2.40}$$

for a constant $q \in (1,2)$. Since they are also bounded in $W^{-1,\infty}(R \times R^+)$, we thus have that

$$G(v,T)_t + (\int_c^v G_v^2(\tau,T)d\tau)_x \quad \text{are compact in} \quad H_{loc}^{-1}(R \times R^+) \tag{2.41}$$

by using the Murat's Theorem [Mu]. Lemma 11 is proved.

Combining (2.12) and (2.29), we complete the proof of Part (I) in Theorem 1 by letting ε, δ go to zero in (1.8). When $\beta(T) = \alpha(T) = 0$, since S appears in all terms in system (1.8), we have the proof of Part (II) in Theorem 1 from (2.27) and (2.29).

3 Proofs of Theorems 2 and 3

In this section, we shall prove Theorems 2 and 3.

Proof of Theorem 2. In the proof of Theorem 1, the technique to add the approximation perturbation δ to the flux function since $\frac{f}{S}$ could be singular near the line $S = 0$. If f is a smooth function as stated in Theorem 2, this technique is not necessary since we may easily obtain the lower, positive bound $S^\varepsilon \geq c(\varepsilon, t) > 0$ for the classical viscosity solutions of system (1.5) ([Di, Pe, Lu4]). So, we just let $\delta = 0$ in (1.8) or consider (1.5) directly. We rewrite the first equation in (1.5) as follows:

$$S_t + f_S(S,T)S_x + f_T(S,T)T_x = \varepsilon S_{xx}. \tag{3.1}$$

Since $f(M,T) = d$ uniformly with respect to T, then $f_T(M,T) = 0$. Since M is a upper bound of the initial data $S_0(x)$, the standard maximum principle reduces the upper bound $S^\varepsilon \leq M$. Theorem 2 is proved.

Proof of Theorem 3. When $\beta(T) = bT$ and $\alpha(T) = aT$, one family of the Riemann invariant of system (1.1) is

$$\begin{cases} z(S,T) = \frac{f(S,T)+a}{S+b} \text{ or} \\ \\ Z(S,C) = \frac{f(S,\frac{C}{S+b})+a}{S+b} = \frac{F(S,C)+a}{S+b}, \end{cases} \tag{3.2}$$

where $C = (S+b)T$.

Since f is a smooth function, we also let $\delta = 0$ in (1.8) or consider (1.5) directly.

Substituting the first equation in (1.5) into the second, we may rewrite the second equation in (1.5) as

$$T_t + \frac{f+a}{S+b}T_x = \frac{\varepsilon}{S+b}C_{xx} - \varepsilon\frac{C}{(S+b)^2}S_{xx}. \tag{3.3}$$

We multiply (3.3) by z_T, the first equation in (1.5) by z_S and then add the result to obtain one equation, whose left-hand side is

$$\begin{aligned} & z_t + z_S(f_S S_x + f_T T_x) + z_T \tfrac{f+a}{S+b}T_x = z_t + z_S f_S S_x + (z_S f_T + z z_T)T_x \\ & = z_t + z_S f_S S_x + z_T f_S T_x = z_t + f_S z_x = Z_t + f_S Z_x; \end{aligned} \tag{3.4}$$

and the right-hand side is

$$\begin{aligned} & \varepsilon z_s S_{xx} + z_T(\tfrac{\varepsilon}{S+b}C_{xx} - \varepsilon\tfrac{C}{(S+b)^2}S_{xx}) \\ & = \varepsilon(Z_S + Z_C\tfrac{C}{S+b})S_{xx} + Z_C(S+b)(\tfrac{\varepsilon}{S+b}C_{xx} - \varepsilon\tfrac{C}{(S+b)^2}S_{xx}) \\ & = \varepsilon(Z_S S_{xx} + Z_C C_{xx}) = \varepsilon Z_{xx} - \varepsilon(Z_{SS}S_x^2 + 2Z_{SC}S_xC_x + Z_{CC}C_x^2). \end{aligned} \tag{3.5}$$

By simple calculations,

$$\begin{cases} Z_S = -\frac{1}{(S+b)^2}F + \frac{F_S}{S+b} - \frac{a}{(S+b)^2}, \quad Z_C = \frac{F_C}{S+b}, \quad Z_{CC} = \frac{F_{CC}}{S+b}, \\ Z_{SC} = -\frac{1}{(S+b)^2}F_C + \frac{F_{SC}}{S+b}, \quad Z_{SS} = \frac{2}{(S+b)^3}F - \frac{2F_S}{(S+b)^2} + \frac{2a}{(S+b)^3} + \frac{F_{SS}}{S+b} \end{cases} \tag{3.6}$$

and

$$Z_x = (-\frac{1}{(S+b)^2}F + \frac{F_S}{S+b} - \frac{a}{(S+b)^2})S_x + \frac{F_C}{S+b}Z_x. \tag{3.7}$$

Thus we have from (3.4)-(3.7) that

$$\begin{aligned} & Z_t + f_S Z_x = \varepsilon Z_{xx} - \tfrac{\varepsilon}{S+b}(F_{SS}S_x^2 + 2F_{SC}S_xC_x + F_{CC}C_x^2) + \tfrac{2\varepsilon}{S+b}S_x Z_x \\ & \leq \varepsilon Z_{xx} + \tfrac{2\varepsilon}{S+b}S_x Z_x \end{aligned} \tag{3.8}$$

since conditions on F given in Theorem 3. Applying for the maximum principle to (3.8), we have $Z(S,C) \leq M$, where M is a upper bound of the initial data $Z(S_0(x), C_0(x))$. Using the conditions in Theorem 3 again, we have the upper estimate $S^\varepsilon \leq M$, so we end the proof of Theorem 3.

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