ON AN INVERSE PROBLEM FOR SCALAR CONSERVATION LAWS

HELGE HOLDEN, FABIO SIMONE PRIULI, AND NILS HENRIK RISEBRO

ABSTRACT. We study in what sense one can determine the function $k = k(x)$ in the scalar hyperbolic conservation law $u_t + (k(x)f(u))_x = 0$ by observing the solution $u(t, \cdot)$ of the Cauchy problem with initial data $u|_{t=0} = u_o$.

1. INTRODUCTION

In this paper, we deal with the inverse problem for scalar conservation laws. More precisely, we consider a scalar conservation law of the form

$$\partial_t u + \partial_x\big(k(x)f(u)\big) = 0\,, \tag{1.1}$$

with $(t,x) \in [0,\infty) \times \mathbb{R}$, $u(t,x) \in \mathbb{R}$, $k\colon \mathbb{R} \to (0,\infty)$ and $f\colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ flux functions whose smoothness will be prescribed later. It is well known that if k is a constant function and f is locally Lipschitz continuous, then for every initial data $u_o \in \mathbf{L}^\infty(\mathbb{R}) \cap \mathbf{L}^1(\mathbb{R})$ there exists a unique entropy solution $u(t, \cdot) \in \mathbf{C}^0([0,\infty), \mathbf{L}^1(\mathbb{R}))$, see [2, 5, 10]. In recent years, motivated by problems arising in traffic flow models [14, 15, 19] and in multiphase flow models in porous media [1, 6, 7, 9], the equation (1.1) has been widely studied also in the case where k is a discontinuous and piecewise constant function. In this latter case, assuming, e.g., that the flux function f is strictly concave and defined in a compact interval $[u_1, u_2]$ with $f(u_1) = f(u_2) = 0$, it has been proved in [15] that a unique entropy solution exists for every initial data in $\mathbf{BV}(\mathbb{R})$.

The goal of this paper is to find a reconstruction procedure which allows us to approximate the unknown coefficient k in (1.1) starting from the observation of the solutions $u_{\mathrm{obs}}(t,x)$ corresponding to Cauchy problems with suitably chosen initial data. This is a so-called *coefficient inverse problem*, because an observer has complete access to both initial data and solutions of the problem, but only partial informations on the structure of the equation itself.

This kind of inverse problem has many applications, depending on the underlying physical phenomena described by (1.1). For instance, we can consider models of traffic flow on highways (see [15, 16, 17]). Here the unknown $u(t,x)$ denotes the density of cars at time t in the position x, the product $k(x)f(u(t,x))$ represents the flux of cars which cross each position x at a time t per unit of time, and the function $k(x)$ describes specific characteristics of the road in the position x. The inverse problem, in this case, corresponds to the problem of determining the unknown properties k and f of the considered road by only monitoring the resulting density of cars u_{obs} along the road.

Also, we want a procedure that can handle problems where parts of the spatial domain are not directly observable and, hence, where data from the observable regions has to be used to reconstruct the characteristics of the physical system

Date: September 17, 2013.
2010 *Mathematics Subject Classification.* Primary: 35L03; Secondary: 35R30.
Key words and phrases. scalar hyperbolic conservation laws, inverse problems.
Supported in part by the Research Council of Norway.

also in the unobservable regions. To fix the ideas, think of a highway where a tunnel is present in an interval $[a, b]$, or where the traffic data is monitored by using sensors which cannot cover the whole road. In this situation, in addition to reconstructing k and f in the observable region $\mathbb{R} \setminus [a, b]$, we would like to determine the flux function $k(x)$ in $[a, b]$, relying on the observed data $u_{\text{obs}}|_{\mathbb{R}\setminus[a,b]}$, to detect the possible obstructions, due to car accidents or other events, and to locate their precise position inside the region $[a, b]$.

Despite the ample spectrum of applications, to our knowledge only few attempts of addressing inverse problems for conservation laws (1.1) have been made.

In [12] a special class of inverse problems is solved for scalar conservation laws (1.1) with $k(x) \equiv 1$ and f of class $\mathbf{C}^2$ and uniformly convex. Namely, it is assumed that the initial data for (1.1) is such that the observed solution u_{obs} consists only of a single shock wave, after a large enough time T. In this particular case, f can be expressed as limit of functions explicitly depending on the shock wave and on the initial data. Unfortunately, the requirement that the solution develops a single discontinuity is very strong in the context of conservation laws, making this approach infeasible for general equations of the form (1.1).

In [11], a more general approach is presented to deal with the inverse problem for (1.1) under the assumptions of $k(x) \equiv 1$ and f locally Lipschitz continuous. Namely, the flux function f is uniquely identified by minimizing, over a compact set of Lipschitz continuous fluxes, a suitable cost functional $J(f)$ which measures the distance between the observed solution u_{obs} and the solution corresponding to any choice of the flux. The functional has the following form

$$J(f) := \frac{1}{2} \left\| u_f(T, \cdot\,) - u_{\text{obs}} \right\|_{\mathbf{L}^2}^2 + \frac{\rho}{2} \left| \int_{\mathbb{R}} x(u_f(T, x) - u_{\text{obs}}(x))\, dx \right| \tag{1.2}$$

where u_f is the solution to the conservation law with flux f and $\rho > 0$ is a fixed constant. The first term is the cost used in the well-known output least square method and is sensitive to the shape of the observed function, while the second term is more sensitive to the localization of the observed function on the x-axis. Dealing with a minimization problem for (1.2), differentiability of J with respect to f is important, since both optimality conditions and gradient algorithms rely on it, however, in general the function is nondifferentiable. Yet, minimization is possible if additional assumptions are posed on the number and location of jumps in the observed solution u_{obs}. Unfortunately, one cannot in general expect these additional hypotheses to hold, and in the general setting the problem remains open.

Similar results are obtained in [4] where the flux f is obtained by minimizing the functional

$$J(f) := \frac{1}{2} \| u_f(T, \cdot\,) - u_{\text{obs}} \|_{\mathbf{L}^2}^2 + \frac{\rho}{2} \int_{u_1}^{u_2} |f'(u)|\, du.$$

If the penalization parameter ρ is zero, then the above functional does not have a unique minimizer as can easily be demonstrated by an example where u_{obs} contains shocks. Nevertheless, in [4] efficient algorithms are developed for the numerical calculation of minimizers f even if the observed solution has discontinuities.

In this paper we follow a different approach. We exploit the complete and detailed knowledge of the approximation procedure used to obtain solutions to the Cauchy problem for (1.1), the so-called *front-tracking* algorithm [2, 10], in order to somehow revert the construction and deduce properties of the flux functions k, f starting from the observed solutions. Our analysis is restricted to one space dimension due to the constructive method that we advocate. For applications to traffic flow, this suffices.

This produces an *ad hoc* procedure which allows us to solve the inverse problem, both in the case of homogeneous conservation laws where $k \equiv \text{const}$, and in the case of a piecewise constant function k, as long as we assume that we can observe the solutions corresponding to suitable families of initial data. Namely, for the case $k \equiv 1$, i.e., for the homogeneous conservation law

$$\partial_t u + \partial_x f(u) = 0\,, \tag{1.3}$$

we prove in Theorem 2.1 the following: If f is of class $\mathbf{C}^{1,1}$ with a finite number of inflection points, then we can always find a piecewise linear interpolation f_ν of f, by using a single observation at a fixed time $T > 0$ of a finite number of solutions u_{obs}, corresponding to properly chosen initial data. Such approximate flux f_ν coincides with f in suitable nodes $u_1 < \cdots < u_\nu$, and it is close to f in the sense that the $\mathbf{L}^1$ distance between u_{obs} and the solution of the conservation law with flux f_ν converges to 0 as $\nu \to \infty$.

To deal with the general case of a piecewise constant function $k(x)$, we notice that by applying Theorem 2.1 in a region where k is known to be constant, say $k(x) \equiv k_o$, it is always possible to reconstruct as accurately as we want the function f. Hence, we can assume $f(u)$ to be a known function and to focus our study on the coefficient $k(x)$.

In Theorem 2.3, assuming that f is defined on an interval $[u_1, u_2]$, is strictly concave and such that $f(u_1) = f(u_2) = 0$ (which is the case, e.g., in the Lighthill–Whitham–Richards traffic flow model [16, 17]), we prove that in order to reconstruct exactly the function $k(x)$ on any compact interval $J \subseteq \mathbb{R}$, it is enough to observe the solution u_{obs} in $[0, T] \times \mathbb{R}$, for a single suitable initial data u_o^J.

Finally, we have studied the case in which the solution can only be observed in $[0, T] \times (\mathbb{R} \setminus I)$, for some unobservable open interval I and for some time T large enough. In this case, the expression of $k(x)$ outside I can be obtained by using Theorem 2.3, but $k(x)$ can also be reconstructed inside I, if we assume that no more than two jumps are present inside the unobservable interval. Namely, in Theorem 2.4 we prove that a suitable choice of the initial data in the region $\{x \in \mathbb{R}\ ;\ x < \inf I\}$ allows us to reconstruct the position and the size of the jumps of $k(x)$ inside I from the observed solution u_{obs}. Moreover, in Theorem 2.7 we prove that the reconstruction is also possible when the initial data cannot be chosen freely but it is given by a constant state $\bar{u}_o$. This is for instance the case when considering a physical system whose inhomogeneity appears at time $t = 0$, due to some external event (like a car accident) which modifies the properties of the flux function in a specific region. In this latter case, we prove that it is still possible to determine positions and sizes of the jumps of $k(x)$ in I, provided that the jump is large enough to influence the dynamics outside I.

We remark that the assumption on the number of jumps in the unobservable region I is rather strong, because we are basically assuming that only a single obstruction can be present. However, this appears to be unavoidable, because if more than two jumps are allowed in I, then the inverse problem is in general ill-posed. Indeed, in Section 3 we present a few examples where relaxing the assumption on k leads to infinitely many piecewise constant functions $k(x)$ on I, all giving the same observed solution in $[0, T] \times (\mathbb{R} \setminus I)$. This means that in many real situations it is impossible, based only on the observations of the solution in $[0, T] \times (\mathbb{R} \setminus I)$, to distinguish between a single large obstruction or many smaller ones. In such a context, one can apply Theorem 2.4 in order to obtain a reconstructed flux function k with a single jump and consider such a single obstruction as an approximation of the real one, whose structure can be very complex.

This paper represents the first steps towards a more complete understanding of the *coefficient inverse problem*. Further study is necessary in order to address the fundamental question of stability. Furthermore, extensions to multi-dimensional cases, will require novel techniques.

2. Main results

We start by studying the inverse problem for (1.3), i.e., in the case of $k(x) \equiv$ const. We recall that a Riemann problem for (1.3) is a Cauchy problem with initial data of the form

$$u_o(x) = \begin{cases} u^-, & x < 0, \\ u^+, & x > 0, \end{cases} \tag{2.1}$$

for given values $u^- \neq u^+$. Here, we assume that the function

(F) $f\colon \mathbb{R} \to \mathbb{R}$ is piecewise C^1 with a finite number of inflection points on any bounded interval $[u_1, u_2]$.

Furthermore, we call a function *observable* if we know its values (almost everywhere). Please note that the use of observable in this paper differs from that in control theory.

Our first result states that if the flux function f is assumed to satisfy **(F)** and if all solutions $u_{\text{obs}}(T, \cdot)$ to Riemann problems (1.3)–(2.1) at some fixed time $T > 0$ are observable, then it is possible to construct on every bounded interval $I \subseteq \mathbb{R}$ a piecewise linear interpolation $\tilde{f}$ of the flux f, which is close to f in the following sense: at a time T, the solution to every Cauchy problem for

$$\partial_t u + \partial_x \tilde{f}(u) = 0\,,$$

is close in $\mathbf{L}^1$ to the solution to (1.3) with the same initial data. More precisely, we prove the following:

Theorem 2.1. *Let $T > 0$, $u_*, u^* \in \mathbb{R}$ such that $u_* < u^*$ and $c \in \mathbb{R}$ be fixed. Assume that f satisfies* **(F)***, that $[u_*, u^*] \subseteq \Omega$, that $f(u_*) = c$ and that the solution to any Riemann problem for (1.3) is observable at time T. Then, for all $\nu \in \mathbb{N}$, setting $\delta = 2^{-\nu}|u^* - u_*|$ and $u_\alpha = u_* + \alpha\delta$ for $\alpha = 0, \ldots, 2^\nu$, there exists a piecewise linear function $f^\nu\colon [u_*, u^*] \to \mathbb{R}$ such that $f_\nu(u_\alpha) = f(u_\alpha)$ for all α and*

$$\operatorname*{ess\,sup}_{[u_*, u^*]} |f'_\nu - f'| \leq \operatorname{Lip}(f')\delta\,, \tag{2.2}$$

where $\operatorname{Lip}(f')$ is a Lipschitz constant for the derivative f' on $[u_*, u^*]$.

This function f_ν represents a good reconstruction of the unknown flux f in the following sense: if $\hat{u}$ is a **BV** *function with values in $[u_*, u^*]$, and we denote by u^ν (resp. u_{obs}) the solution to the Cauchy problem for $\partial_t u + \partial_x f^\nu(u) = 0$ (resp. for (1.3)) with initial data $\hat{u}$, then*

$$\|u^\nu(T, \cdot) - u_{\text{obs}}(T, \cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq CT\delta \tag{2.3}$$

for a constant C which does not depend on δ.

We remark that (2.3) follows immediately from (2.2) and from the general stability results contained in [10] (see Theorem A.2). Here the relevant result is the procedure to construct a piecewise linear interpolation f^ν from the observed solutions, so that f^ν coincides with the original flux f at points $u_o = u_* < u_1 < \cdots < u_{2^\nu} = u^*$ of the interval $[u_*, u^*]$ and satisfies (2.2). We also stress that no assumptions are made in Theorem 2.1 concerning the regularity of the observed solutions or concerning their discontinuity structure. Furthermore, general solutions containing any finite number of shocks and centered rarefaction waves can appear without affecting the result of the reconstruction.

Next we study scalar conservation laws of the more general form (1.1) with a piecewise constant term $k(x)$. Since in general the existence of solutions to the Cauchy problem for (1.1) is much more difficult to prove than for (1.3) (see, e.g., [13] and references therein), we focus our attention on a specific class of conservation laws studied in [14, 15, 18, 19] for which existence of a solution to the Cauchy problem has been proved by Klingenberg and Risebro [15]. Namely, we assume:

(H1) • $k\colon \mathbb{R} \to (0,\infty)$ is piecewise constant and belongs to $\mathbf{BV}(\mathbb{R})$;
• $f\colon [u_1,u_2] \to [0,\infty)$ is of class $\mathbf{C}^2$, strictly concave and such that $f(u_1) = f(u_2) = 0$. In particular, $f > 0$ in (u_1,u_2) and there exists a unique $u^m \in (u_1,u_2)$ such that $f(u^m) = \max_{[u_1,u_2]} f$.

From [15, 18], we know that every Cauchy problem for (1.1) with flux functions k, f satisfying **(H1)**, and initial data in $\mathbf{BV}(\mathbb{R})$, admits a unique entropy solution in $\mathbf{C}([0,T];\mathbf{L}^1(\mathbb{R}))$, see Theorem A.3.

Example 2.2 (**Traffic flow on highways**). *A typical example of a system satisfying* **(H1)** *is the simple inhomogeneous variant of the classical Lighthill–Whitham–Richards model* [16, 17] *for car traffic flow on a highway, obtained by multiplying the flux function $f(u) = u(1-u)$ with a piecewise constant factor $k(x)$. In this model, u represents the density of cars on the highway and takes values in $[u_1,u_2] = [0,1]$ and $f(u)$ represents the flux of cars per unit of time. The function $k(x)$ represents specific features of the road considered in different spatial regions, e.g., regions in which cars have to reduce their speed or are allowed to increase it, all due to external factors.*

Motivated by Example 2.2 above, in the following we will say that a spatial region $I \subseteq \mathbb{R}$ is *congested* (resp. *fully congested*) if $u(x) \geq u^m$ (resp. $u(x) \equiv u_2$) for all $x \in I$.

We notice that if the flux functions satisfy assumptions **(H1)** and the solutions to any Riemann problem for (1.1) are observable, then we can always first consider a small region $[\alpha,\beta]$ where the road is homogeneous and use Theorem 2.1 on the interval $[u_1,u_2]$, with Riemann data centered in $x = (\alpha+\beta)/2$, to reconstruct $f(u)$ with any desired precision. Hence, without loss of generality, we can assume $f(u)$ to be a given function and focus our attention on the piecewise constant function $k(x)$. Under these assumptions, we can prove the following result, which provides an exact reconstruction procedure for the function $k(x)$ on any compact interval.

Theorem 2.3. *Let $T > 0$ and $J \subseteq \mathbb{R}$ be a fixed compact interval. Assume that the function f in* (1.1) *satisfies* **(H1)***, and that the solution to any Riemann problem for* (1.1) *is observable for all times $t \in (0,T]$. Then, there exists a unique piecewise constant function $k^J\colon J \to \mathbb{R}$ such that the following property holds. If we denote by u^J (resp. u_{obs}) the solution to the Cauchy problem for $\partial_t u + \partial_x\big(k^J(x)f(u)\big) = 0$ (resp. for* (1.1)*) with initial data $\hat{u} \in \mathbf{BV}(\mathbb{R})$ taking values in $[u_1,u_2]$, then*

$$u^J(t,x) = u_{\mathrm{obs}}(t,x)\,, \quad x \in J,\, t \in [0,T]\,. \tag{2.4}$$

Here, solutions must be observed on some interval $t \in (0,T]$ and not only at a single time $t = T$. The reason for this additional requirement is that, since the locations of the jumps in k are unknown, it is otherwise difficult to observe the speed of the waves appearing in the solution. However, the time interval $(0,T]$ can be taken arbitrarily small without interfering with our reconstruction procedure.

Finally, we focus our attention to the case of incomplete observability, i.e., when a part of the domain cannot be directly observed. First of all, we need to define a slightly different concept of observability which will be used in the remaining part of this section. Given a function $v\colon [0,\infty)\times\mathbb{R} \to \mathbb{R}$, we say that v is *partially observable* if $v(t,\cdot)\big|_{\mathbb{R}\setminus(a,b)}$ is observable for all $t \in [0,T]$.

Since Theorem 2.3 can be used to reconstruct $k(x)$ on every compact interval $J \subseteq (-\infty,a]$ and $J \subseteq [b,\infty)$, it is not restrictive to assume that $k(x)$ is known and constant in the observable region $\mathbb{R}\setminus(a,b)$. Moreover, we assume that in the unobservable region $[a, b]$ the changes in k can only be due to some sort of obstruction which reduces the speeds of propagation. In other words, we assume in the following that

(H2) $k(x) \equiv k_o$ in $\mathbb{R} \setminus (a, b)$ and $k(x) \leq k_o$ for all $x \in [a, b]$.

To prove our main results for the inverse problem with partial observability, we need to introduce a further hypothesis on the function $k(x)$ in the unobservable interval $[a, b]$.

(H3) $k(x)$ has exactly two jumps in $[a, b]$, i.e., there exist $k_1 \in (0, k_o)$ and $a \leq \xi_1 < \xi_2 \leq b$ such that

$$k(x) := \begin{cases} k_o & x \notin (\xi_1, \xi_2)\,, \\ k_1 & x \in (\xi_1, \xi_2)\,. \end{cases} \tag{2.5}$$

By applying Theorem A.3 we know that to each choice (k_1, ξ_1, ξ_2) in $(0, k_o) \times [a, b] \times [a, b]$ there corresponds a flux function $k(x)$, defined by (2.5), such that any Cauchy problem for (1.1) with **BV** initial data has a unique entropy solution.

For a scalar conservation law (1.1) satisfying hypotheses **(H1)**–**(H3)**, we consider two different inverse problems, corresponding to two possible applications to the traffic flow model described in Example 2.2: Reconstruction from initial data which is a stationary solution in $[a, \infty)$ and reconstruction from a constant initial data.

The first problem is the reconstruction of $k(x)$ in the case of an initial data $u(0, x)|_{[a,\infty)} = u_o(x)$ which is a stationary entropy solution of (1.1) in $[a, \infty)$ with values in $[u_1, u_2]$. In other words, we assume that the initial data is only prescribed in the half line $[a, \infty)$ and that it is given by a piecewise constant function u_o whose jumps are located in the same positions as the jumps in k and whose values satisfy Rankine–Hugoniot conditions with zero speed.

With this particular problem, we are attempting to describe the case of a physical system where some obstructions have appeared in the past and then the evolution has stabilized into a stationary solution. Using again the traffic flow model in Example 2.2, consider the case when an accident occurred in the unobservable interval (a, b) at some time in the past. The accident caused all cars to slow down until they overtook the section of the road obstructed by the vehicles involved, causing an increase in the density of cars localized only in some interval $J = [\xi_1, \xi_2] \subset (a, b)$, whose endpoints cannot be deduced from the density of cars in $[b, \infty)$, where the accident does not effect the dynamics. In this case, the only way to gather additional information is to change the number of cars entering at $x = a$ and to observe how this change affects the solution in the observable region $[b, \infty)$. In other words, this problem could be considered as an initial-boundary value problem in which we are free to choose suitable boundary data u_{bdry} at $x = a - \varepsilon$ for a fixed $\varepsilon > 0$, so that the observations in $[0, T] \times \big([a - \varepsilon, a] \cup [b, \infty)\big)$ of the solution to the initial-boundary value problem

$$\begin{aligned} &\partial_t u + \partial_x \big(k(x) f(u)\big) = 0\,, \text{ in } [0, T] \times [a - \varepsilon, \infty), \\ &u(0, x) = u_o(x), \quad x \in [a, \infty), \qquad\qquad u(t, a - \varepsilon) = u_{\mathrm{bdry}}(t), \quad t \in [0, T] \end{aligned}$$

allow the computation of $k(x)$.

However, such a problem can be reformulated in terms of an auxiliary Cauchy problem in the whole $[0, T] \times \mathbb{R}$, in which we are allowed to choose the initial data u_o in $(-\infty, a)$ instead of u_{bdry}. In this way, we are going to use the observed solution to the Cauchy problem for (1.1) with initial data

$$u(0, x) = u_o(x), \quad x \in \mathbb{R}\,,$$

in order to reconstruct $k(x)$. A posteriori, if we denote by $\hat{u}(t,x)$ the solution to such a Cauchy problem, $\hat{u}(t,x)|_{[a-\varepsilon,\infty)}$ provides a solution to the initial-boundary value problem with boundary data[1] $u_{\mathrm{bdry}}(t) = \hat{u}(t, a-\varepsilon+)$.

Our result for this first problem is that if the unobservable region $[a,b]$ is nowhere fully congested and if the observation interval $[0,T]$ is large enough, then we can choose a suitable initial data in $(-\infty,a)$ to reconstruct uniquely the function $k(x)$ in $[a,b]$, and hence in the whole $\mathbb{R}$ thanks to **(H2)**.

Theorem 2.4. *Assume that the conservation law satisfies* **(H1)**–**(H3)***, that f is a known function, that the initial data $u_o(\,\cdot\,)$ is a stationary solution on $[a,\infty)$ attaining values in $[u_1,u_2]$, and that for each choice of a* **BV** *initial data $u_o(\,\cdot\,)$ in $(-\infty,a)$ with values in $[u_1,u_2]$, the solution $u_{\mathrm{obs}}(t,x)$ to the corresponding Cauchy problem for* (1.1) *is partially observable.*
Then, if $u_o(\,\cdot\,) < u_2$ in $[a,b]$, there exists $T>0$ large enough and a unique choice of (k_1,ξ_1,ξ_2) such that, by denoting by $u_{(k_1,\xi_1,\xi_2)}$ the solution to (1.1) *with initial data u_o and with $k(x)$ defined in* (2.5)*, there holds*

$$u_{(k_1,\xi_1,\xi_2)}(t,x) = u_{\mathrm{obs}}(t,x), \quad (t,x)\in[0,T]\times(\mathbb{R}\setminus(a,b)) \,. \tag{2.6}$$

Remark 2.5. *In Theorem 2.4 we need the hypothesis that $u_o(x) < u_2$ for all $x\in[a,b]$, i.e., that no part of the unobservable region is fully congested, to complete the reconstruction procedure. This assumption needs some comments in view of possible applications, because it appears to require information on the initial state of the physical system that cannot be known based only on partial observability.*

As a preliminary fact, note that the assumption that $u_o(\,\cdot\,)$ is a stationary solution to (1.1) *in $[a,\infty)$, together with* **(H1)**–**(H3)***, implies that*

$$u_o(x) = \begin{cases} u_o(a), & \text{if } a<x<\xi_1\,,\\ \omega, & \text{if } \xi_1<x<\xi_2\,,\\ u_o(b), & \text{if } x>\xi_2\,, \end{cases} \tag{2.7}$$

for some constant $\omega\in[u_1,u_2]$, and that the jumps at $x=\xi_1$ and $x=\xi_2$ must be stationary. The Rankine–Hugoniot condition implies that there also holds

$$k_o f(u_o(a)) = k_1 f(\omega) = k_o f(u_o(b))\,, \tag{2.8}$$

where the quantities $k_o, u_o(a), u_o(b)$ are known and the quantities k_1,ω are unknown.

If $\{u_o(a),u_o(b)\}\subseteq(u_1,u_2)$, then $f>0$ in all the above equalities (2.8) *and also $\omega<u_2$ must hold. As a result, no part of the region (a,b) can be fully congested and Theorem 2.4 can be applied.*

In the case of either $u_o(a)=u_2$ or $u_o(b)=u_2$, the reconstruction procedure cannot be applied; indeed, it would be impossible to reconstruct the value k_1 attained by $k(x)$ in the interval $[\xi_1,\xi_2]$, because (2.8) *simply implies $f(\omega)=0$ independently of k_1. On a positive note, however, such an impossibility can also be immediately detected by the known values of $u_o(\,\cdot\,)$ in $x=a$ or $x=b$.*

It remains to consider the case of $u_o(a)=u_o(b)=u_1$. In this case, Theorem 2.4 applies if $\omega=u_1$ and fails if $\omega=u_2$. Since we cannot observe $u_o(\,\cdot\,)$ in $[\xi_1,\xi_2]$, it is not a priori possible to decide in which case we are. Trying to apply the reconstruction procedure to a problem where $u_o(x)=u_2$ in $[\xi_1,\xi_2]$ soon leads to the appearance of the "forbidden" state $u=u_2$ at $x=a$ so that the assumption of partial observability allows us a posteriori to detect the presence of a fully congested region inside (a,b).

Therefore, in the case $u_o(a)=u_o(b)=u_1$, which is the only one in applications where it would be impossible to know in advance if the assumption $u_o(\,\cdot\,)<u_2$ is satisfied, the conclusion of the theorem could be reformulated as follows: either there

[1]Here and in the following we use the convention that $\phi(a\pm) = \lim_{\epsilon\downarrow 0}\phi(a\pm\epsilon)$.

exists $T_1 > 0$ *such that* $u_{\text{obs}}(T_1, a+) = u_2$*, or there exists* $T_2 > 0$ *large enough and a unique choice of* (k_1, ξ_1, ξ_2) *such that*

$$u_{(k_1,\xi_1,\xi_2)}(t,x) = u_{\text{obs}}(t,x), \quad (t,x) \in [0,T_2] \times (\mathbb{R} \setminus (a,b)).$$

Remark 2.6. *In the application to the traffic flow model in Example 2.2, Theorem 2.4 covers for instance the case of initial data* $u_o(\,\cdot\,) \equiv 0$ *in* $[a,\infty)$*. In other words, among other cases, the reconstruction procedure in Theorem 2.4 allows us to recover* $k(x)$ *when we are considering a highway which is known to be empty at* $t = 0$*.*

The second problem we consider, under the assumption of partial observability, is the reconstruction of $k(x)$ in the case of initial data $u(0,x) \equiv \bar{u}_o$, for some constant $\bar{u}_o \in [u_1, u^m)$. In other words, we assume that at the initial time the whole spatial domain contains a constant state $\bar{u}_o$. With this particular problem, we are attempting to describe the case of a physical system in which, at time $t = 0$, the constant flux function $k_{\text{old}}(x) \equiv k_o$ is suddenly replaced by a piecewise constant function $k(x)$, due to the appearance of some obstructions in the system. Considering once again the traffic flow model in Example 2.2, you might think of a constant density of cars distributed in the whole highway and of a car accident occurring, at time $t = 0$, in some place inside the unobservable interval (a,b).

In this case, the initial data $u \equiv \bar{u}_o$ is not a stationary solution for (1.1) with discontinuous flux $k(x)f(u)$ and therefore the solution will immediately develop additional waves around the discontinuity points for k.

Our result for this problem is that, if we observe the solution long enough, then we can always reconstruct the function $k(x)$ in $[a,b]$, and hence in the whole $\mathbb{R}$, as before. Uniqueness of the resulting flux $k(x)$, on the other hand, only holds when the obstruction is large enough. This is not entirely surprising, because it is expected that the effect of a very small obstruction occurring in a very small spatial region $[\xi_1, \xi_2] \subseteq [a,b]$ gets canceled before reaching the observable region $\mathbb{R} \setminus (a,b)$. But it might also happen that the obstruction produces effects that can be detected in the observable region and still the data is insufficient to lead to a unique reconstruction: in the latter case, it is in general possible to provide infinitely many functions $k(x)$, all leading to the same solution in $\mathbb{R} \setminus (a,b)$.

Theorem 2.7. *Assume that the conservation law satisfies* **(H1)**–**(H3)***, that* f *is a known function, and that the solution* $u_{\text{obs}}(t,x)$ *to the Cauchy problem for* (1.1) *with a constant initial data* $u(0,\,\cdot\,) \equiv \bar{u}_o \in [u_1, u^m)$ *is partially observable.*

Then, either $u_{\text{obs}}(t,x) \equiv \bar{u}_o$ *for all* $(t,x) \in [0,\infty) \times (\mathbb{R} \setminus (a,b))$*, and hence we can assume* $k(x) \equiv k_o$ *for all* $x \in \mathbb{R}$*, or there exist* $T > 0$ *large enough and a choice of* (k_1, ξ_1, ξ_2) *such that, denoting* $u_{(k_1,\xi_1,\xi_2)}$ *the solution to* (1.1) *with* $k(x)$ *defined in* (2.5)*, there holds*

$$u_{(k_1,\xi_1,\xi_2)}(t,x) = u_{\text{obs}}(t,x), \quad (t,x) \in [0,T] \times (\mathbb{R} \setminus (a,b)). \tag{2.9}$$

Moreover, if there exists $T_1 \in (0,T)$ *such that*

$$u(T_1, a-) = \bar{u}_o < u(T_1, a+),$$

or

$$u(T_1, b+) = \bar{u}_o > u(T_1, b-) \quad \text{and} \quad \inf\left\{s \in (T_1, T)\ ;\ u(s,b) > u(T_1, b-)\right\} > T_1,$$

then the choice is unique.

A few comments are in order. First of all, we notice that Theorems 2.4 and 2.7 state that there exists an observation time $T > 0$ large enough so that the reconstruction procedure can be completed successfully. The reason for this is that we need enough waves to pass through the unobservable region $[a, b]$ and reach the observable region, before we can fully determine k. If, e.g., the constant k_1 in (2.5) is close to zero, the waves can take a very long time $\overline{T} \approx \mathcal{O}(1)\frac{b-a}{k_1}$ to pass through the unobservable region and therefore the reconstruction is not possible by only observing the solution in $[0, \tau]$ with $\tau < \overline{T}$. The technical Lemmas 4.1 (for Theorem 2.4) and 4.7 (for Theorem 2.7) show the properties satisfied by the observed solution u_{obs} at time $\overline{T}$, and characterize the minimal time T for which the reconstruction procedure can be completed.

Also, we want to emphasize some features of our results which should be taken into account in applications to practical problems. In Theorem 2.1 a single observation of the solution at a fixed time $T > 0$ is enough to recover a piecewise linear approximation of the flux $f(u)$. In Theorem 2.3 observations have to be performed on an interval of times $(0, T]$ to recover the flux function $k(x)$, but these additional data allow for an exact reconstruction of $k(x)$.

In both cases, T can be chosen arbitrarily small. Hence, we can always test a large number of initial data for a very short time to obtain an accurate reconstruction before a fixed time $\hat{T} > 0$. This is possible because we can observe the solution on the whole spatial domain and because we are free to select any initial data.

When the solution cannot be observed in the whole $\mathbb{R}$, as in Theorems 2.4 and 2.7, it becomes vital to study u_{obs} on a whole interval of times $[0, T]$, with T possibly very large as remarked above. At the same time, the choice of initial data becomes more important, because carefully chosen initial data can convey more information about the flux.

It is not surprising, therefore, that when there are unobservable regions, while we can observe the solution corresponding to any initial data of our choice, as in Theorem 2.4, we still can recover a unique exact reconstruction of $k(x)$, unless the unobservable region is fully congested.

On the other hand, when there are unobservable regions and the initial data cannot be freely chosen, like in Theorem 2.7, the amount of information that can be recovered from the solution is limited. In particular, in some cases we lose the uniqueness of the reconstructed flux $k(x)$, because the effect on the given initial data of many different small obstructions might pass equally undetected in the observable region.

Finally, we remark that the assumption **(H3)** on $k(x)$, by prescribing the exact number of discontinuities in the unobservable region, is strong. In terms of the traffic flow model presented in Example 2.2, this means that a single obstructed stretch of road can be present inside I. However, **(H3)** is really necessary for the inverse problem to be well-posed: in Section 3 we present a few examples where, one by allowing for three or more jumps in $k(x)$, immediately is led to the existence of infinitely many piecewise constant functions $\hat{k}(x)$, whose corresponding solutions coincide with u_{obs} in $[0, T] \times (\mathbb{R} \setminus [a, b])$. In other words, the reconstruction problem is in general ill-posed within the class of piecewise constant functions which do not satisfy **(H3)**.

3. Ill-posedness when $k(x)$ has more than two jumps

In this section, we show through a few examples that the problem with partial observability is in general ill–posed whenever the function $k(x)$ in (1.1) is allowed to have three or more jumps, i.e., when k satisfies **(H1)** and **(H2)** but not **(H3)**. Namely, we show that in several situations there exist infinitely many different functions k with three or more jumps which produce exactly the same solution in the observable region $\mathbb{R} \setminus (a, b)$. Considering the car traffic example, this means that in some situations there could be 2 or 3 or more small accidents in the region

(a,b) or a single larger one, and there would be no way to distinguish between them by just observing the situation in $\mathbb{R} \setminus (a,b)$. This is always the case, for instance, if the accident which is closer to the extreme $x = a$ reduces the flux more than the subsequent ones.

In view of these examples, and of the fact that there is no reason in applications to exclude obstructions which are larger close to $x = a$ than in the rest of the region, one can think to hypothesis **(H3)** as a way to single out an approximation of the real, and possibly very complex, structure of $k(\,\cdot\,)$ in (a,b) by means of a single obstruction. In turn, this approximation is “good” because the corresponding solution in $\mathbb{R}\setminus(a,b)$ coincides with the observed one for all times if the flux function satisfies **(H3)** or if we are in any of the cases below.

Example 3.1. *Consider the Cauchy problem for* (1.1) *with initial data* $u(0,x) \equiv u_1$ *for* $x \in (a,\infty)$. *In the highway Example 2.2, this initial data means that the road is initially empty. Assume that we have reconstructed a coefficient* $\kappa(x)$ *so that the solution* u^κ *to* $\partial_t u + \partial_x(\kappa(x)f(u)) = 0$ *coincides with* u_{obs} *in the observable region* $[0,T] \times \big(\mathbb{R} \setminus (a,b)\big)$, *and that*

$$\kappa(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \chi_1 + \chi_2]\,, \\ k_1, & x \in (\xi', \xi' + \chi_1]\,, \\ k_2, & x \in (\xi' + \chi_1, \xi' + \chi_1 + \chi_2]\,, \end{cases} \tag{3.1}$$

for suitable positive numbers χ_1, χ_2 *such that* $\chi_1 + \chi_2 \le b - a$, *for* $0 < k_1 < k_2 < k_o$ *and for a fixed* $\xi' \in [a, b - \chi_1 - \chi_2]$.

If $\chi_1 + \chi_2 < b - a$, *we claim that for every* $\varepsilon > 0$ *small enough, also the solutions* u^{κ_ε} *coincide with* u_{obs} *in* $[0,T] \times \big(\mathbb{R} \setminus (a,b)\big)$ *if we choose the coefficient* κ_ε *as follows*

$$\kappa_\varepsilon(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \chi_1 + \chi_2 + \varepsilon]\,, \\ k_1, & x \in (\xi', \xi' + \chi_1]\,, \\ k_\varepsilon, & x \in (\xi' + \chi_1, \xi' + \chi_1 + \chi_2 + \varepsilon]\,, \end{cases}$$

with $k_\varepsilon \in (k_2, k_o)$ *given by*

$$k_\varepsilon = \frac{\chi_2 + \varepsilon}{\dfrac{\chi_2}{k_2} + \dfrac{\varepsilon}{k_o}} = k_2 \left(1 + \varepsilon \, \frac{\dfrac{1}{k_2} - \dfrac{1}{k_o}}{\dfrac{\chi_2}{k_2} + \dfrac{\varepsilon}{k_o}} \right).$$

Indeed, independently of the choice of the initial data $u(0,\,\cdot\,)$ *in* $(-\infty, a]$ *(or of the boundary data* u_{bdry} *at* $x = a$*) the exact same solution will always be observed for* $x \ge b$. *This can be seen as follows. Fix any initial data in* $(-\infty, a]$ *attaining some value* ω *larger than* u_1. *Then the solution at time* $t = 0+$ *will contain a centered rarefaction wave traveling with speed* $k_o f'(u)$ *for* $u \in [u_1, \omega]$. *In particular, by* **(H1)** *states close to* u_1 *will travel with positive speed, i.e., towards the unobservable region* $[a,b]$. *Eventually, the centered rarefaction wave will cross completely* $[a,b]$ *and emerge at* $x = b$ *after having spent in* $[a,b]$ *a time*

$$T_u = \frac{b - a - (\chi_1 + \chi_2)}{k_o f'(u)} + \frac{\chi_1}{k_1 f'(u)} + \frac{\chi_2}{k_2 f'(u)}\,, \qquad u > u_1\,,$$

if the flux is κf, *and a time*

$$T'_u = \frac{b - a - (\chi_1 + \chi_2 + \varepsilon)}{k_o f'(u)} + \frac{\chi_1}{k_1 f'(u)} + \frac{\chi_2 + \varepsilon}{k_\varepsilon f'(u)}\,, \qquad u > u_1\,,$$

if the flux is $\kappa_\varepsilon f$. *It is easy to verify that the choice of* k_ε *implies* $T_u = T'_u$ *for all states* u *which pass* $[a,b]$, *proving that the solution restricted to* $\mathbb{R} \setminus (a,b)$ *is the same for both fluxes. Therefore, any function* $\kappa_\varepsilon(x)$ *provides a solution to our inverse problem.*

Example 3.2. *We now show that the loss of uniqueness cannot be avoided by prescribing the length of the "obstruction" interval $\chi_1 + \chi_2$. Indeed, let us consider the same problem as in Example 3.1 and the same possible flux function $\kappa(x)$ defined in (3.1) for suitable positive numbers χ_1, χ_2 such that $\chi_1 + \chi_2 \le b - a$, and for $0 < k_1 < k_2 < k_o$. It can be easily verified that, for any fixed $\rho > 0$ small enough, the solutions corresponding to flux functions*

$$\kappa_\rho(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \chi_1 + \chi_2]\,, \\ k_1, & x \in (\xi', \xi' + \chi_1 - \rho]\,, \\ k_\rho, & x \in (\xi' + \chi_1 - \rho, \xi' + \chi_1 + \chi_2]\,, \end{cases}$$

with $k_\rho \in (k_1, k_2)$ defined by

$$k_\rho = \frac{\chi_2 + \rho}{\dfrac{\chi_2}{k_2} + \dfrac{\rho}{k_1}} = k_2 \left(1 - \rho \, \frac{\dfrac{1}{k_1} - \dfrac{1}{k_2}}{\dfrac{\chi_2}{k_2} + \dfrac{\rho}{k_1}} \right)$$

once again coincide on $\mathbb{R} \setminus (a, b)$ with the ones found in Example 3.1.

Example 3.3. *The previous examples can be easily generalized to the case of a flux function with four or more discontinuities. For instance, assuming that we have reconstructed the function*

$$\kappa(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \chi_1 + \chi_2 + \chi_3)\,, \\ k_1, & x \in (\xi', \xi' + \chi_1)\,, \\ k_2, & x \in (\xi' + \chi_1, \xi' + \chi_1 + \chi_2)\,, \\ k_3, & x \in (\xi' + \chi_1 + \chi_2, \xi' + \chi_1 + \chi_2 + \chi_3)\,, \end{cases}$$

for suitable positive constants χ_1, χ_2, χ_3 such that $\sum_i \chi_i \le b - a$ and for $0 < k_1 < k_2 < k_3 < k_o$, one can easily prove that the solutions to (1.1) with flux κf and initial data $u(0, x) \equiv u_1$ for $x \in (a, \infty)$ coincide (outside (a, b)) with the solutions to (1.1) with flux $\kappa_\varepsilon f$ and the same initial data, if we define

$$\kappa_\varepsilon(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \sum_i \chi_i)\,, \\ k_1, & x \in (\xi', \xi' + \chi_1)\,, \\ \hat{k}_\varepsilon, & x \in (\xi' + \chi_1, \xi' + \chi_1 + \chi_2 - \varepsilon)\,, \\ \tilde{k}_\varepsilon, & x \in (\xi' + \chi_1 + \chi_2 - \varepsilon, \xi' + \sum_i \chi_i)\,, \end{cases}$$

for $\varepsilon > 0$ small enough and for any $\hat{k}_\varepsilon, \tilde{k}_\varepsilon$ such that

$$\frac{\chi_3}{k_3} + \frac{\chi_2}{k_2} = \frac{\chi_3 + \varepsilon}{\tilde{k}_\varepsilon} + \frac{\chi_2 - \varepsilon}{\hat{k}_\varepsilon}\,.$$

In particular, by choosing

$$\tilde{k}_\varepsilon = \hat{k}_\varepsilon = \ell := k_2 \left(\frac{\chi_2/k_2}{\chi_2/k_2 + \chi_3/k_3} \right) + k_3 \left(\frac{\chi_3/k_3}{\chi_2/k_2 + \chi_3/k_3} \right) \in (k_2, k_3)$$

one obtains that the same solution u_{obs} in $\mathbb{R} \setminus (a, b)$ corresponding to $\kappa(x) f(u)$ can also be obtained as solution of the conservation law with flux $\bar{\kappa}(x) f(u)$ where

$$\bar{\kappa}(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \sum_i \chi_i)\,, \\ k_1, & x \in (\xi', \xi' + \chi_1)\,, \\ \ell, & x \in (\xi' + \chi_1, \xi' + \sum_i \chi_i)\,, \end{cases}$$

i.e., not only the available data are insufficient to distinguish between flux functions with a different number of discontinuities (in this case three or four jumps), but it is possible to construct infinitely many additional flux functions by applying the ideas

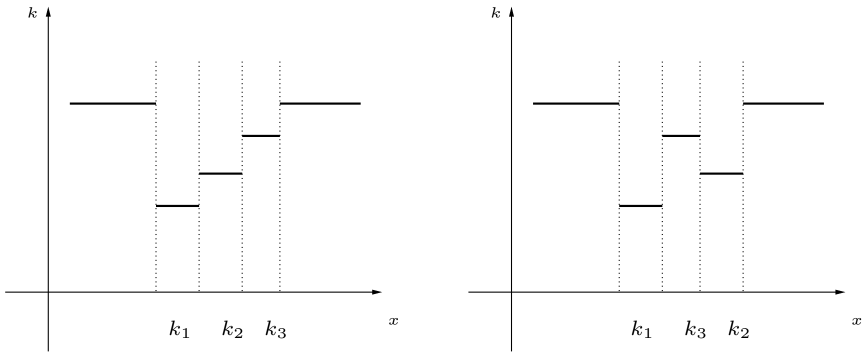

FIGURE 1. Two coefficients $k(x)$ producing the same solution outside (a, b).

of Examples 3.1 and 3.2 to $\bar\kappa$, and all these fluxes would give solutions coinciding with u_{obs} in the observable region $\mathbb{R} \setminus (a, b)$.
Finally, we remark that by repeating the same argument on $\bar\kappa$ and by defining

$$\ell' := k_1 \left(\frac{\chi_1/k_1}{\chi_1/k_1 + (\chi_2 + \chi_3)/\ell} \right) + \ell \left(\frac{(\chi_2 + \chi_3)/\ell}{\chi_1/k_1 + (\chi_2 + \chi_3)/\ell} \right) \in (k_1, \ell)$$

and

$$\bar\kappa'(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \sum_i \chi_i), \\ \ell', & x \in (\xi', \xi' + \sum_i \chi_i), \end{cases}$$

one also obtains a flux function which satisfies **(H3)** *and produces the same solution u_{obs} in the observable region $\mathbb{R} \setminus (a, b)$.*

Example 3.4. *As a last example of ill-posedness, we show that when four or more discontinuities are assumed to be present in $k(x)$, then not even imposing a priori the length of each discontinuity helps to recover uniqueness.*

Once again, consider the Cauchy problem for (1.1) *with the initial data $u(0, x) \equiv u_1$ for $x \in (a, \infty)$. Fix three positive numbers χ_1, χ_2, χ_3 such that $\sum_i \chi_i \leq b - a$, representing the length of the intervals in which $k(x) \neq k_o \equiv k|_{\mathbb{R}\setminus(a,b)}$ as in Example 3.3, and fix $\xi' \in [a, b - \sum_i \chi_i]$ representing the location of the first discontinuity of k. Assume that we reconstruct a flux function (see Figure 1)*

$$\kappa_1(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \sum_i \chi_i), \\ k_1, & x \in (\xi', \xi' + \chi_1), \\ k_2, & x \in (\xi' + \chi_1, \xi' + \chi_1 + \chi_2), \\ k_3, & x \in (\xi' + \chi_1 + \chi_2, \xi' + \sum_i \chi_i), \end{cases}$$

with $0 < k_1 < k_2 < k_3 < k_o$, so that the solution u^{κ_1} to $\partial_t u + \partial_x(\kappa_1(x) f(u)) = 0$ coincides with u_{obs} in the observable region $[0, T] \times \big(\mathbb{R} \setminus (a, b)\big)$. Then it is easy to verify that also the piecewise constant function defined by

$$\kappa_2(x) = \begin{cases} k_o, & x \notin (\xi', \xi' + \sum_i \chi_i), \\ k_1, & x \in (\xi', \xi' + \chi_1), \\ k_3, & x \in (\xi' + \chi_1, \xi' + \chi_1 + \chi_3), \\ k_2, & x \in (\xi' + \chi_1 + \chi_3, \xi' + \sum_i \chi_i), \end{cases}$$

i.e., obtained by switching the interval where $\kappa_1(x) = k_2$ and $\kappa_1(x) = k_3$, gives a solution u^{κ_2} which coincides with u_{obs} in $\mathbb{R} \setminus (a, b)$.

4. Technical proofs

Proof of Theorem 2.1. As already remarked in Section 2, once we have proved (2.2), the general stability result [10, Theorem 2.13] ensures that also (2.3) is satisfied. Hence, the proof reduces to the construction of the approximated flux f_ν which satisfies (2.2).

Let $T > 0$, $u_*, u^* \in \mathbb{R}$ such that $u_* < u^*$ and $c \in \mathbb{R}$ be fixed. Fix also $\nu \in \mathbb{N}$ and define δ and $\{u_o, \ldots, u_{2^\nu}\}$ as in the statement of Theorem 2.1. Of course, we start by definining $f_\nu(u_*) = c$.

Since we only assume to be able to observe the solution to (1.3) at time T, we have to choose carefully the initial data. In this case, let us consider the following family of Riemann data:

$$u_o^h(x) = \begin{cases} u_h, & x < 0, \\ u_{h+1}, & x > 0, \end{cases} \qquad h = 0, \ldots, 2^\nu - 1. \tag{4.1}$$

The strategy is to use the solution corresponding to each u_o^h to assign f_ν in u_{h+1}. First, we consider the particular case of a solution which at time T consists of a single wave, either an entropy shock wave or a centered rarefaction wave, joining the states u_h and u_{h+1}. This is the case when f has no inflection points in the interval $[u_h, u_{h+1}]$. Once we know how to deal with this easier case, we move to the general situation.

Step 1 (Shock). Fixed $h \geq 0$, let $\tilde{u}(\,\cdot\,) = u_{\text{obs}}(T, \,\cdot\,)$ be the solution to (1.3)–(4.1) at time T, consisting of a single shock wave joining u_h and u_{h+1}, and let $x_h \in \mathbb{R}$ be the location of the jump. Then, the propagation speed of this wave is given by $s_h = x_h/T$ and, by Rankine–Hugoniot conditions, there holds

$$f(u_{h+1}) = f(u_h) + s_h(u_{h+1} - u_h) = f(u_h) + \frac{\delta x_h}{T}\,.$$

Therefore, if $f_\nu(u_o), \ldots, f_\nu(u_h)$ are given so that $f_\nu(u_\alpha) = f(u_\alpha)$, we can define

$$f_\nu(u_{h+1}) = f_\nu(u_h) + \frac{\delta x_h}{T} = f(u_h) + \frac{\delta x_h}{T} = f(u_{h+1})\,.$$

Step 2 (Rarefaction). Fixed $h \geq 0$, let $\tilde{u}(\,\cdot\,) = u_{\text{obs}}(T, \,\cdot\,)$ be the solution to (1.3)–(4.1) at time T, consisting of a single centered rarefaction wave joining u_h and u_{h+1}, and let $I_h = [x_h, x_{h+1}]$ be the interval in which $\tilde{u}(\,\cdot\,)$ is not constant. Then, if $f_\nu(u_o), \ldots, f_\nu(u_h)$ are given so that $f_\nu(u_\alpha) = f(u_\alpha)$, we claim that

- if we replace this rarefaction wave with a shock wave separating the same states, whose jump is located at the point

$$\xi_h := \frac{\int_{u_h}^{u_{h+1}} x(u)\, du}{u_{h+1} - u_h} = \frac{\int_{u_h}^{u_{h+1}} x(u)\, du}{\delta} \tag{4.2}$$

  where $u \mapsto x(u)$ is the inverse of $x \mapsto \tilde{u}(x)$ on I_h, and traveling with speed ξ_h/T;
- and, if we define

$$f_\nu(u_{h+1}) = f_\nu(u_h) + \frac{\delta \xi_h}{T}, \tag{4.3}$$

then $f_\nu(u_{h+1}) = f(u_{h+1})$. Indeed, it is enough to recall that, by definition of centered rarefaction waves (see, e.g., [2]), the following equality holds

$$\int_{u_h}^{u_{h+1}} \frac{x(u)}{T}\, du = \int_{u_h}^{u_{h+1}} f'(u)\, du\,.$$

Therefore, we have

$$f_\nu(u_{h+1}) = f_\nu(u_h) + \frac{\delta \xi_h}{T} = f(u_h) + \int_{u_h}^{u_{h+1}} \frac{x(u)}{T}\, du$$

$$= f(u_h) + \int_{u_h}^{u_{h+1}} f'(u)\,du = f(u_{h+1})\,.$$

Note that the computation of the point ξ_h used to define $f_\nu(u_{h+1})$ can be done explicitly under the observability assumption. Indeed, $\tilde{u}(\,\cdot\,)$ is monotonically increasing in I_h. Therefore, once we know $\int_{x_h}^{x_{h+1}} \tilde{u}(x)dx$ and the value attained by $\tilde{u}$ at the points x_h, x_{h+1}, then we also know the value of the integral $\int_{u_h}^{u_{h+1}} x(u)\,du$ used to define ξ_h (see Lemma A.1), even without computing the expression of the inverse function $u \mapsto x(u)$.

Step 3 (General case). Fixed $h \geq 0$, assume $f_\nu(u_o), \ldots, f_\nu(u_h)$ are given so that $f_\nu(u_\alpha) = f(u_\alpha)$ and let $\tilde{u}(\,\cdot\,) = u_{\rm obs}(T,\,\cdot\,)$ be the solution to (1.3)–(4.1) observed at time T. In general, $\tilde{u}$ can consist of more than one single wave but, in any case, $\tilde{u}$ is monotonically increasing and it can contain only a finite number of different waves, thanks to the choice of the initial data and to the assumptions **(F)**.

Therefore, let $x_1 < \cdots < x_{M_1}$ be the locations of jumps of $\tilde{u}$ and let $I_1, \ldots, I_{M_2}$ the intervals in which $\tilde{u}$ has non-zero derivative. To define $f_\nu(u_{h+1})$, we simply proceed applying the construction in Step 1 to each shock and the one in Step 2 to each centered rarefaction wave which appears in $\tilde{u}(\,\cdot\,)$.

Namely, we first replace each rarefaction joining two states u^ℓ, u^r on the interval I_j $(j = 1, \ldots, M_2)$ with a shock centered at

$$\xi_j = \frac{\int_{u^\ell}^{u^r} x(u)\,du}{u^\ell - u^r} \in I_j\,.$$

In this way, we obtain a new piecewise constant function $\bar{u}(\,\cdot\,)$ whose jumps are located at points $y_1 < \cdots < y_M$, with $M = M_1 + M_2$ and $\{y_1, \ldots, y_M\} = \{x_1, \ldots, x_{M_1}, \xi_1, \ldots, \xi_{M_2}\}$. Let $v_1 < \cdots < v_{M+1}$ be the values attained by $\bar{u}$, i.e., let us assume

$$\bar{u}(x) = \begin{cases} v_1 = u_h, & \text{if } x < y_1\,, \\ \vdots & \\ v_\alpha, & \text{if } y_{\alpha-1} < x < y_\alpha\,,\ \alpha = 2, \ldots, M\,, \\ \vdots & \\ v_{M+1} = u_{h+1}, & \text{if } x > y_M\,. \end{cases}$$

By construction,

- $\bar{u}(\,\cdot\,)$ coincides with $\tilde{u}(\,\cdot\,)$ outside $\bigcup_{k=1}^{M_2} I_k$;
- on each interval I_k where $\tilde{u}$ has a rarefaction joining u^ℓ, u^r, $\bar{u}$ attains only the values u^ℓ, u^r and it jumps from u^ℓ to u^r at a point $\xi_k \in I_k$ such that
$$\frac{f(u^\ell) - f(u^r)}{u^\ell - u^r} = \frac{\xi_k}{T}$$
as follows from (4.2)–(4.3);
- all the values $v_1, \ldots, v_{M+1}$ are known, since they are attained by $\tilde{u}(\,\cdot\,)$ as adjacent states to shocks and rarefactions.

Now, set

$$y := \sum_{\alpha=1}^{M} \frac{v_{\alpha+1} - v_\alpha}{v_{M+1} - v_1}\, y_\alpha = \sum_{\alpha=1}^{M} \frac{v_{\alpha+1} - v_\alpha}{\delta}\, y_\alpha$$

and define

$$f_\nu(u_{h+1}) = f_\nu(u_h) + \frac{\delta y}{T}\,.$$

We claim that $f_\nu(u_{h+1}) = f(u_{h+1})$. Indeed,

$$\begin{aligned} f_\nu(u_{h+1}) = f_\nu(u_h) + \frac{\delta y}{T} &= f(u_h) + \sum_{\alpha=1}^{M} \frac{v_{\alpha+1} - v_\alpha}{T}\, y_\alpha \\ &= f(u_h) + \sum_{\alpha=1}^{M} [f(v_{\alpha+1}) - f(v_\alpha)] \\ &= f(u_h) + f(v_{M+1}) - f(v_1) = f(u_{h+1}) \,, \end{aligned}$$

where we have again used the Rankine–Hugoniot conditions and the particular choices of $\xi_1, \ldots, \xi_{M_2}$ as locations for the jumps in $\bar{u}$, which replace rarefactions in $\tilde{u}$.

At this point, we define f_ν on $[u_*, u^*]$ as the piecewise linear function joining the values obtained in the previous steps:

$$f_\nu(u) := f_\nu(u_h) + \frac{f_\nu(u_{h+1}) - f_\nu(u_h)}{\delta}\,(u - u_h), \qquad u \in [u_h, u_{h+1}] \,.$$

Finally, we are ready to prove (2.2). Given any point $u \in [u_*, u^*]$, there exists $\alpha \in \{0, \ldots, 2^\nu - 1\}$ such that $u \in [u_\alpha, u_{\alpha+1}]$. Setting $v_o = u_\alpha < \cdots < v_N = u_{\alpha+1}$ the values such that f is of class $C^{1,1}$ on each interval (v_j, v_{j+1}), we then have

$$\begin{aligned} |f'_\nu(u) - f'(u)| &= \Big| \frac{f(u_{\alpha+1}) - f(u_\alpha)}{\delta} - f'(u) \Big| \\ &= \Big| \sum_{j=1}^{N} \frac{f(v_j) - f(v_{j-1})}{\delta} - f'(u) \Big| \\ &= \Big| \sum_{j=1}^{N} \frac{f'(w_j)(v_j - v_{j-1})}{\delta} - f'(u) \Big| \\ &= \Big| \sum_{j=1}^{N} \frac{v_j - v_{j-1}}{\delta}\,(f'(w_j) - f'(u)) \Big| \\ &\le \mathrm{Lip}(f') \sum_{j=1}^{N} \frac{|v_j - v_{j-1}|}{\delta}\, |w_j - u| \\ &\le \mathrm{Lip}(f')\delta \,, \end{aligned}$$

where each w_j, $j = 1, \ldots, N$, is a suitable element in the interval (v_{j-1}, v_j). Passing to the essential supremum over u, the proof is complete. $\diamondsuit$

Proof of Theorem 2.3. Fix a compact interval J and denote by $\lambda = \max_{[u_1,u_2]} |f'(u)|$, and set

$$I = (\min J - \lambda T, \max J + \lambda T) \,.$$

Now, consider the piecewise constant initial data given by

$$u_o(x) = \begin{cases} \tilde{u}, & \text{if } x \in I, \\ u_1, & \text{if } x \notin I, \end{cases}$$

for a fixed $\tilde{u} \in (u_1, u^m)$.

By hyperbolicity there must exist a time $\tau > 0$ small enough such that the solution u_{obs} to the Cauchy problem for (1.1) with initial data $u(0, x) = u_o(x)$ can be obtained, up to time τ, by simply piecing together the solutions to the Riemann problems with data

$$\begin{cases} u_1, & \text{if } x < \min I, \\ \tilde{u}, & \text{if } x > \min I, \end{cases} \qquad\qquad \begin{cases} \tilde{u}, & \text{if } x < \max I, \\ u_1, & \text{if } x > \max I. \end{cases}$$

Since it is not restrictive to assume $\tau \leq T$, by the observabilty assumption, both solutions to the Riemann problems above are observable in $(0, \tau]$, and hence the whole solution to the Cauchy problem with data u_o is observable, up to time τ.

Relying on the explicit construction of the solutions to Riemann problems for (1.1), presented in [15] and briefly sketched in the Appendix, we can also give a better a priori description of the observed solution u_{obs} in $[0, \tau]$. Indeed, the particular choice of initial data u_o which is constant in I, implies that any Lax wave present in u_{obs} for $x \in J$ must have been generated by a discontinuity in the flux function k.

Moreover, the choice of a constant value $\tilde{u} < u^m$ in u_o ensures that, at each discontinuity point ξ for k, the solution u_{obs} contains not only a stationary jump located at $x = \xi$, but also a shock u-wave or a centered rarefaction u-wave with positive speed (here and in the following u-waves are Lax waves with constant values of k, see again the Appendix). Indeed, in a neighborhood of $x = \xi$ the conservation law (1.1) is equivalent to a Riemann problem for the auxiliary system (A.2) in the unknowns (k, u) with initial data

$$\begin{cases} (\tilde{u}, k^\ell), & \text{if } x < \xi, \\ (\tilde{u}, k^r), & \text{if } x > \xi, \end{cases}$$

for $k^\ell = k(\xi-)$ and $k^r = k(\xi+)$. Hence, the structure of the solution can be deduced by the construction of the Riemann solver for (A.2) with $u^\ell = u^r = \tilde{u} < u^m$ (see the explicit description of the Riemann solver given in the Appendix, in particular Cases 1 and 3).

Recalling that k has a finite number of jumps in J, by **(H1)**, there must be a time $\tau' > 0$ such that the solution u_{obs} in $(0, \tau']$ is obtained by piecing together the solutions of the Riemann problems for the auxiliary system (A.2) at jumps of the function k. In other words, at time τ' no interaction between waves generated in J has occurred yet. Without loss of generality, we can assume that $\tau' \leq \tau$.

We then observe the solution u_{obs} to (1.1), with initial data $u(0, x) = u_o(x)$, at times $t = \tau'/2$ and $t = \tau'$. Denote by $S = \{x_1, \ldots, x_M\}$ the set (possibly empty) of points $\xi \in J$ such that

$$u_{\text{obs}}(\tau'/2, \xi-) = u_{\text{obs}}(\tau', \xi-) \neq u_{\text{obs}}(\tau', \xi+) = u_{\text{obs}}(\tau'/2, \xi+) .$$

In other words, S is the set of stationary jumps in the solution u_{obs} and represents exactly the set of points of discontinuity for the flux $k(x)$. Notice that $k(x)$ can have no other jumps in J, because each jump in k generates a stationary discontinuity. This, in particular, implies that, if we find the values of k in the intervals $(x_{\alpha-1}, x_\alpha)$, then we have found exactly the correct function k which produces u_{obs}, and (2.4) is satisfied.

We introduce the notation

$$\kappa_o = k(x_1-) , \qquad \kappa_\alpha = k(x_\alpha+) , \ \alpha = 1, \ldots, M ,$$

for the values attained by $k(x)$ in J. From the admissibility of the jumps in u_{obs}, we deduce that for all $\alpha \in \{1, \ldots, M\}$ one must have

$$\kappa_{\alpha-1} f(u_{\text{obs}}(\tau', x_\alpha-)) = \kappa_\alpha f(u_{\text{obs}}(\tau', x_\alpha+)) ,$$

which is a set of M equations in the $M+1$ unknowns $\kappa_o, \ldots, \kappa_M$. To close the system we now need to find at least one of the κ_α. Indeed, if we can exactly identify one of the unknowns, then the system above becomes a system in M variables and M unknowns, which can be solved because of the choice $\tilde{u} > u_1$ in the initial data, which implies $f(u_{\text{obs}}(\tau', x_\alpha\pm)) > 0$ for all $\alpha = 1, \ldots, M$.

So we concentrate our attention on the interval $[x_M, \sup I]$ and we define

$$y = \sup \left\{ \xi > x_M \ ; \ u_{\text{obs}}(\tau', \xi) > \tilde{u} \right\} .$$

Having observed that in $[x_M, \sup I]$ there is a Lax u-wave propagating with positive speed, y is well defined and satisfies $y \le \sup I$. In particular, y is the location, at time $t = \tau'$, of a u-wave which got generated at $(t, x) = (0, x_M)$ from the discontinuity in k and which is now moving away from x_M. The speed σ of this wave can be simply computed as

$$\sigma = \frac{y - x_M}{\tau'} .$$

We want to find the value κ_M from the speed of this u-wave traveling in $[x_M, \sup I]$, so that the reconstruction of $k(x)$ is complete and so is the proof.

There are two cases, depending on whether $u_{\text{obs}}(\tau', \cdot\,)$ is continuous or discontinuous at y. For ease of notation, define

$$u(y-) := u_{\text{obs}}(\tau', y-) , \qquad\qquad u(y+) := u_{\text{obs}}(\tau', y+) = \tilde{u} .$$

If $u(y-) = u(y+)$, then the u-wave is a centered rarefaction. In this case,

$$\sigma = \kappa_M f'(\tilde{u}) \qquad \Longrightarrow \qquad \kappa_M = \frac{\sigma}{f'(\tilde{u})} ,$$

which is well defined because $\tilde{u} < u^m$.

If $u(y-) \ne u(y+)$, then the u-wave is a shock and $u(y-) = u_{\text{obs}}(\tau', x_M+)$. In this case,

$$\sigma = \kappa_M \frac{f(\tilde{u}) - f(u(y-))}{\tilde{u} - u(y-)} \qquad \Longrightarrow \qquad \kappa_M = \sigma \frac{\tilde{u} - u(y-)}{f(\tilde{u}) - f(u(y-))} ,$$

which is well defined because $\sigma > 0$ implies $f(\tilde{u}) \ne f(u(y-))$. $\diamondsuit$

Proof of Theorem 2.4. Most of the proof of Theorem 2.4 follows from a series of lemmas. The basic idea is that, since we assume only partial observability for the solutions of every Cauchy problem, we have to choose the initial data $u_o(\,\cdot\,)$ in $(-\infty, a)$, so that the observed solution gives enough data to reconstruct both the values attained by $k(x)$ and the locations of its discontinuities inside (a, b).

We start by recalling that, as in Remark 2.5, the assumption that the initial data in $[a, \infty)$ is a stationary solution to (1.1) with k and f satisfying **(H1)**–**(H3)** implies that $u_o(x)$ has the form (2.7) for some constant $\omega \in [u_1, u_2]$, and that there holds the relation

$$k_o f(u_o(a)) = k_1 f(\omega) = k_o f(u_o(b)) , \tag{4.4}$$

between the known quantities $k_o, u_o(a), u_o(b)$ and the unknown ones k_1, ω. From the analysis performed in [15], we also know that entropy admissibility of the stationary jumps located at $x = \xi_1$ and $x = \xi_2$ (see the so-called "smallest jump" admissibility condition (A.4)), implies that either $f'(u_o(a))f'(u_o(b)) > 0$, i.e., $u_m \notin (u_o(a), u_o(b))$, or $\omega = u_m$ and hence k_1 is immediately determined by (4.4). However, we have no way to determine from the observations in $\mathbb{R} \setminus (a, b)$ which case is occurring or which precise value ω is attained.

We now study the Cauchy problem with carefully selected initial data. Fix a positive real value $\tilde{x}$ and assume $u_o(\,\cdot\,)|_{[a,\infty)}$ to be a given stationary solution to (1.1). Let $\bar{v}_a$ be the state in $[u_1, u_2]$ characterized as the unique solution to

$$f(u_o(a)) = f(\bar{v}_a) , \qquad\qquad f'(u_o(a))f'(\bar{v}_a) \le 0 , \tag{4.5}$$

so that, in particular, $\bar{v}_a \in (u_1, u^m)$ if $u_o(a) \in (u^m, u_2)$, and $\bar{v}_a \in (u^m, u_2)$ if $u_o(a) \in (u_1, u^m)$. Define

$$\tilde{y} := \begin{cases} \dfrac{\max\{u_o(a), v_a\} - u_1}{f(u_o(a))} \, f'(u_1)(b - a + \tilde{x}), & \text{if } u_o(a) \in (u_1, u_2) , \\ 0, & \text{otherwise} , \end{cases}$$

and a piecewise constant function v_o as follows

$$v_o(x) = \begin{cases} u^m, & \text{if } x < a - \tilde{x} - \tilde{y}\,, \\ u_1, & \text{if } a - \tilde{x} - \tilde{y} < x < a - \tilde{x}\,, \\ u_o(a), & \text{if } a - \tilde{x} < x < a\,, \\ u_o(x), & \text{if } x > a\,. \end{cases} \tag{4.6}$$

The following lemma helps to understand the choice of the initial data $v_o(\,\cdot\,)$. Namely, we show that after some time, the corresponding solution is identically equal to the state u_1 in $[a, b]$, and only afterwards the real reconstruction procedure begins, with larger values of the state variable crossing the unobserved region. While this two–steps procedure is essential to remove the possible presence of the state u^m in the obstructed region $[\xi_1, \xi_2]$, which would prevent the passage of any further wave through that region, it also implies that the procedure might require a large time of observation to be completed if, e.g., $u_o(a)$ is close to u_1. More comments on this aspect can be found in Remark 4.5.

Lemma 4.1. *Assume that the conservation law* (1.1) *satisfies* **(H1)**–**(H3)** *and that $f(u)$ is a known function. Let $u(t, x)$ denote the solution to the Cauchy problem for* (1.1) *with initial data v_o given by* (4.6)*, and assume that $u_o(a) = v_o(a) \neq u^m$.*

Then, either $u_o(\,\cdot\,) = u_2$ somewhere in (a, b) or, by setting

$$\tilde{\tau} := \inf\{s > 0\ ;\ u(s, b) = u_1\}\ , \tag{4.7}$$

with $\tilde{\tau} < +\infty$ and $u(\tilde{\tau}, \,\cdot\,)|_{[a,b]} \equiv u_1$ and there exist times T_1, $T_2 > \tilde{\tau}$ such that

$$u_1 < u(T_1, b) < u^m < u(T_2, a) < u_2\,.$$

In the next lemma, we present a sufficient condition for finding a unique solution to the inverse problem with prescribed stationary initial data in (a, ∞).

Lemma 4.2. *Assume that the conservation law* (1.1) *satisfies* **(H1)**–**(H3)***, that $f(u)$ is a known function and that the solution $u_{\text{obs}}(t, x)$ to the Cauchy problem for* (1.1) *with initial data v_o in* (4.6) *is partially observable in $[0, T] \times \big(\mathbb{R} \setminus (a, b)\big)$. Then the following holds: if there exists $\tilde{\tau} \in [0, T]$ such that $u_{\text{obs}}(\tilde{\tau}, \,\cdot\,)|_{[a,b]} \equiv u_1$ and if there exist $T_1, T_2 \in (\tilde{\tau}, T)$ such that*

$$u_1 = u_{\text{obs}}(\tilde{\tau}, b) < u_{\text{obs}}(T_1, b) < u^m < u_{\text{obs}}(T_2, a) < u_2\,, \tag{4.8}$$

then there exists a unique choice of (k_1, ξ_1, ξ_2) such that, denoting with $u_{(k_1,\xi_1,\xi_2)}$ the solution to the Cauchy problem for (1.1) *with initial data v_o in* (4.6) *and with $k(x)$ given by* (2.5)*, there holds*

$$u_{(k_1,\xi_1,\xi_2)}(t, x) = u_{\text{obs}}(t, x), \quad (t, x) \in [0, T] \times (\mathbb{R} \setminus (a, b))\,.$$

The combination of the previous results simplifies the proof of Theorem 2.4.

Proof of Theorem 2.4. We claim that under the assumptions of the theorem, we have $u_o(a) \neq u^m$. Indeed, we are assuming that $u_o(\,\cdot\,)$ is a stationary solution in (a, b), and hence $u_o(a) = u^m$ would imply $u_o(x) \equiv u^m$ and $k(x) \equiv k_o$ on $[a, \infty)$. But this contradicts the assumption **(H3)**, and hence it is not possible.

Since $u_o(a) \neq u^m$, we can choose to observe the solution corresponding to the initial data v_o in (4.6) and combine Lemma 4.1 and Lemma 4.2 to conclude. Indeed, by Lemma 4.1 we know that in finite time $t = \tilde{\tau}$, given by (4.7), the solution is constantly equal to u_1 in the unobservable region (a, b). Moreover, there exist $T_1, T_2 > \tilde{\tau}$ such that (4.8) holds, and hence Lemma 4.2 ensures the existence of a unique triple (k_1, ξ_1, ξ_2) giving a solution which satisfies (2.6). This concludes the proof. $\diamondsuit$

Remark 4.3. *We observe that in the proof of Theorem 2.4 we exclude the possibility of $u_o(a) = u^m$. It is clear that in that case the reconstruction is actually trivial: thanks to the assumption on $u_o(x)|_{[a,\infty)}$ being a stationary solution, the only possible flux function $k(x)$ has no jumps and it is constantly equal to k_o. Such a reconstructed flux is excluded from the proof just because it does not satisfy the assumption* **(H3)**.

It remains now to prove Lemma 4.1 and Lemma 4.2.

Proof of Lemma 4.1. In terms of [15], instead of (1.1) we can study the auxiliary system (A.2) for the unknowns (u, k). In this context, when dealing with piecewise constant initial data like (4.6) we call k-wave (resp. u-wave) any Lax elementary wave, i.e., shock waves or centered rarefaction waves, for the variable k (resp. u). It is known (see Theorem A.3) that to each choice (k_1, ξ_1, ξ_2) in $(0, k_o] \times [a, b] \times [a, b]$, there corresponds a unique entropy solution $u_{(k_1,\xi_1,\xi_2)}$ to the Cauchy problem with initial data $v_o \in \mathbf{BV}(\mathbb{R})$.

The choice of the initial data (4.6), allows us to write explicitly the solution for small times (see the description of the Riemann solver for (1.1) in the Appendix). We focus our attention first to the case $u_o(a) > u_1$, so that $\tilde{y} > 0$. Here, the solution to (1.1), (4.6) consists of a centered rarefaction u-wave, starting at $a - \tilde{x} - \tilde{y}$ and evolving with characteristic speeds in $[0, k_o f'(u_1)]$, followed by a shock u-wave, starting at $a - \tilde{x}$ and traveling with speed

$$\sigma = k_o \frac{f(u_o(a)) - f(u_1)}{u_o(a) - u_1} = k_o \frac{f(u_o(a))}{u_o(a) - u_1}, \tag{4.9}$$

and by the stationary solution $u_o(x)|_{[a,\infty)}$. And the structure of the solution is preserved at least as long as the shock u-wave remains in $(-\infty, a)$. Notice that in the case under consideration the shock has strictly positive speed σ, because $u_o(a) > u_1$, and that we can write for all $x \in \mathbb{R}$ and $t \in \left[0, \frac{\tilde{x}}{\sigma}\right]$

$$u(t,x) = \begin{cases} u^m, & \text{if } x < a - \tilde{x} - \tilde{y}, \\ \eta(x), & \text{if } a - \tilde{x} - \tilde{y} < x < a - \tilde{x} - \tilde{y} + \lambda_a t, \\ u_1 & \text{if } a - \tilde{x} - \tilde{y} + \lambda_a t < x < a - \tilde{x} + \sigma t, \\ u_o(a), & \text{if } a - \tilde{x} + \sigma t < x < a, \\ u_o(x), & \text{if } x > a, \end{cases} \tag{4.10}$$

where $\lambda_a := k_o f'(u_1)$ and $\eta(x)$ is the unique value such that $k_o f'(\eta(x)) = \frac{x - (a - \tilde{x} - \tilde{y})}{t}$.

We prove that $\tilde{\tau} < +\infty$ and that $u(\tilde{\tau}, \cdot)|_{[a,b]} \equiv u_1$, thanks to the choice of $\tilde{y}$. Indeed, the shock u-wave started at $a - \tilde{x}$ and traveling with speed σ will eventually reduce its speed when the it interacts with jumps of k, but it will always move with a speed $\sigma' \in [\frac{k_o f(u_o(a))}{M - u_1}, \frac{k_o f(u_o(a))}{m - u_1}]$, where

$$m := \min\left\{u_o(a), \bar{v}_a\right\}, \qquad M := \max\left\{u_o(a), \bar{v}_a\right\},$$

and $\bar{v}_a$ is the state characterized by (4.5), as in the definition of $\tilde{y}$. This immediately implies that the wave will reach $x = b$ at most in time

$$\frac{M - u_1}{k_o f(u_o(a))} (b - a + \tilde{x}) < +\infty$$

and that such a time gives an upper bound to $\tilde{\tau}$. Moreover, the choice of $\tilde{y}$ now implies that the rarefaction u-wave generated by the jump at $x = a - \tilde{x} - \tilde{y}$ is still traveling in $[a - \tilde{x} - \tilde{y}, a - \tilde{x}]$ when the shock emerges at $x = b$. This implies that $u_{\text{obs}}(\tilde{\tau}, \cdot)$ is a stationary solution for (1.1) in (a, b) with

$$u_{\text{obs}}(\tilde{\tau}, a) = u_{\text{obs}}(\tilde{\tau}, b) = u_1 < u_m .$$

Observing that $u_{\text{obs}}(\tilde{\tau}, \cdot)|_{[b,\infty)}$ contains a single shock wave traveling with positive speed, and therefore moving away from the unobservable region (a, b), this shock will not contribute anymore to the values attained by the solution in (a, b) for times $t \geq \tilde{\tau}$.

Consider now the case in which $u_o(a) = u_1$, and hence $\tilde{y} = 0$ in (4.6). In this case, the shock u-wave is not present at all and the solution $u(t, x)$ in (4.10) attains value $u_o(a) = u_1$ for $x \in [a - \tilde{x} - \tilde{y} + \lambda_a t, a]$ and times $t \in \left[0, \frac{\tilde{x}}{\lambda_a}\right]$, noticing that $\lambda_a > 0$ thanks to the assumption $u_o(a) \neq u^m$. Since the assumptions $u_o(\cdot) < u_2$ and u_o stationary solution in $[a, b]$ imply that $u_o \equiv u_1$, then we can conclude that $\tilde{\tau} = 0$ and that $u(\tilde{\tau}, \cdot)|_{[a,b]} \equiv u_1$ as before.

This completes the first step of the procedure, needed to remove the possible presence of congested regions. In the rest of the proof, we analyze the evolution of the solution for times larger than $\tilde{\tau}$ in order to reconstruct k in (a, b).

For times $t \geq \tilde{\tau}$, the rarefaction u-wave approaches the obstructed region and eventually reaches $x = \xi_1$ at time $t_{\xi_1} = \frac{\xi_1 - a + \tilde{x} + \tilde{y}}{\lambda_a}$, which is unknown since $\xi_1 \in [a, b]$ is unknown. Since $u(t_{\xi_1}, \xi_1+) = u_1$ from the previous analysis, after the interaction between the rarefaction wave and the stationary jump in k at x_1, part of the wave simply passes through the obstruction. The result for $x > \xi_1$ would then be a new rarefaction u-wave, propagating with a smaller characteristic speed. This new centered rarefaction u-wave is going to pass through $x = \xi_2$ at some later time t_{ξ_2}, and it keeps propagating towards $x = b$, because we also have that $u(t_{\xi_2}, \xi_2+) = u_1$.

Notice that, for times $t \geq t_{\xi_1}$, the value $u(t, \xi_1-)$ increases due to the incoming rarefaction wave and the solution $u(t, x)$ for $x \in [\xi_1, \xi_2]$ will be a smooth profile corresponding to a rarefaction u-wave joining u_1 with the value $u(t, \xi_1+)$ characterized by being the only state in (u_1, u^m) with the property

$$k_1 f(u(t, \xi_1+)) = k_o f(u(t, \xi_1-)) .$$

Since $k_1 < k_o$, the region $[\xi_1, \xi_2]$ becomes congested before the whole original u-rarefaction can pass through $x = \xi_1$. More precisely, setting $w := u(t, \xi_1-)$ the state for which $u(t, \xi_1+) = u^m$, then w is the maximal value of the conserved quantity that the obstructed region $[\xi_1, \xi_2]$ can accept. However, due to the continuous arrival of larger states from the left side, a shock u-wave appears at $x = \xi_1-$ and travels back towards $x = a$ with negative speed. Notice that along such a "reflected" discontinuity, the right state is always given by $w' \in (u^m, u_2)$ such that $k_o f(w') = k_1 f(u^m) = k_o f(w)$.

We sum up the discussion so far: Due to the propagation of the smaller states of the rarefaction wave, we find $T_1 > \tilde{\tau}$ such that $u(T_1, b) > u(\tilde{\tau}, b) = u_1$; due to the reflected shock which emerges at $x = \xi_1$, there exists $T_2 > \tilde{\tau}$ such that $u(T_2, a) > u^m$. Therefore, the lemma is proved. $\diamondsuit$

Proof of Lemma 4.2. Set

$$\tau_a := \inf\{t > \tilde{\tau} \; ; \; u_{\text{obs}}(t, a) > u^m\} ,$$

and

$$\tau_b := \inf\{t > \tilde{\tau} \; ; \; u_{\text{obs}}(t, b) > u_1\} .$$

These are known values, thanks to the partial observability assumption and we have $\tilde{\tau} < \tau_a \leq T_2$ and $\tilde{\tau} < \tau_b < T_1$.

Indeed, the description of the Riemann solver for (1.1) given in the Appendix, implies that in the case $u_o(a) > u_1$, so that $\tilde{y} > 0$, for small positive times u_{obs} consists of a centered rarefaction u-wave, starting at $a - \tilde{x} - \tilde{y}$ and evolving with characteristic speeds in $[0, k_o f'(u_1)]$, followed by a shock u-wave, starting at $a - \tilde{x}$ and traveling with speed σ, as in (4.9), and by the stationary solution $u_o(x)|_{[a,\infty)}$. And in the case $u_o(a) = u_1$ the structure is similar but without the shock u-wave.

Since we are assuming that $u(\tilde{\tau},\,\cdot\,)|_{[a,b]} \equiv u_1$, this means that at time $\tilde{\tau}$ the shock wave has already passed through the whole unobservable region and the rarefaction wave has not reached it yet. Then, it follows that $\tau_b > \tilde{\tau}$ is the first time when the centered rarefaction appears at the end of the unobservable region, while $\tau_a > \tilde{\tau}$ is the first time when a shock is reflected by the discontinuities of k inside (a, b) back towards $x = a$. Moreover, since k has the form (2.5), we also know that this shock u-wave emerging at time τ_a originated at $x = \xi_1$, when the rarefaction u-wave above interacted with the stationary jump of $k(x)$ and the state at $x = \xi_1+$ reached the value u^m. For later use, let us define $\tau_o \geq \tilde{\tau}$ the first time when the rarefaction u-wave originated at $a - \tilde{x} - \tilde{y}$ reaches $x = a$. Thanks to the partial observability assumption, τ_o can be considered a known value.

Let now $v := u(\tau_a, a-)$ and $w' := u(\tau_a, a+)$ be the states separated by the shock wave emerged at $x = a$. By the Rankine–Hugoniot conditions at the generating point of the shock u-wave, there must hold

$$k_o f(w') = k_1 f(u^m)\,,$$

which in turn implies

$$k_1 = \frac{k_o f(w')}{f(u^m)}\,. \tag{4.11}$$

Focusing our attention on the evolution of the rarefaction u-wave in (a, b), we know that $\tau_b - \tau_o$ must be equal to

$$\frac{\xi_1 - a}{k_o f'(u_1)} + \frac{\xi_2 - \xi_1}{k_1 f'(\omega)} + \frac{b - \xi_2}{k_o f'(u_1)}\,,$$

and ω is the unique solution in $(u_1, u_m]$ of

$$k_o f(u_1) = k_1 f(\omega). \tag{4.12}$$

Indeed, the wave must have traveled with speed $k_o f'(u_1)$ in $[a, \xi_1)$, with speed $k_1 f'(\omega)$ in (ξ_1, ξ_2), and again with speed $k_o f'(u_1)$ in $(\xi_2, b]$. Note that ω is now known from (4.12), because f is a known function and k_1 has been already found in (4.11). Therefore, we obtain

$$\xi_2 - \xi_1 = \frac{k_o k_1 f'(\omega)}{k_o f'(u_1) - k_1 f'(\omega)} \left[\frac{b - a}{k_o} - (\tau_b - \tau_o) f'(u_1)\right]\,, \tag{4.13}$$

where all quantities appearing at the right-hand side are known.

We want to use τ_a and the states v, w' observed in (τ_a, a) to determine ξ_1. Let $w \in [u_1, u^m)$ be the unique solution of $f(w) = f(w')$. We know from the structure of the Riemann solver that the shock u-wave separating v and w' originated at $x = \xi_1$ when the rarefaction wave traveling with speed $k_o f'(w)$ interacted with the stationary jump of $k(x)$ and u_{obs} at $x = \xi_1+$ reached u^m. Then, we can conclude that the interaction at $x = \xi_1$ which generated the reflected shock occurred at the time

$$\bar{\tau}(\xi_1) := \frac{\xi_1 - a + \tilde{x} + \tilde{y}}{k_o f'(w)} \leq \tau_a\,.$$

Notice that, due to the structure of the rarefaction wave, we have that $u_1 < w \leq v = u(\tau_a, a-)$. Indeed, it is not possible to have $w = u_1$ as the reflected state because in such a case we would have $w' = u_2$ and

$$k_1 = \frac{k_o f(u_2)}{f(u^m)} = 0\,,$$

which is not possible for a function k satisfying **(H3)**. On the other hand, the limit case $w = v$ happens when the reflection occurs at time τ_a and hence it is equivalent to having $\xi_1 = a$. Since ξ_2 is uniquely determined as well, by using (4.13), the proof is complete.

It remains to consider the case $u_1 < w < v$. This means we can assume $\xi_1 > a$ and, hence, there hold both $\bar{\tau}(\xi_1) < \tau_a$ and $\bar{\tau}(\xi_1) > \frac{\tilde{x}+\tilde{y}}{k_o f'(w)}$, because the latter is the time at which the rarefaction front passes at $x = a$, before getting reflected. Moreover, the wave observed in $x = a$ at $t = \tau_a$ is exactly the (forward) generalized characteristic $\xi(t)$ associated to $u = u_{\mathrm{obs}}$, emanating from the point $(\bar{\tau}(\xi_1), \xi_1)$ (see [5]). Due to the particular structure of our problem, this curve can be found as the solution of the backward Cauchy problem for

$$\dot{\xi}(t) = k_o \, \frac{f\left(u(t, \xi(t)-)\right) - f\left(w'\right)}{u(t, \xi(t)-) - w'}\,, \tag{4.14}$$

with data

$$\xi(\tau_a) = a\,. \tag{4.15}$$

Thanks to the regularity of f in $[u_1, u_2]$ and of u in $\Omega = (-\infty, \xi_1)$, the problem (4.14)–(4.15) has a unique Carathéodory solution defined in $(\bar{\tau}(\xi_1), \tau_a]$, since we know that only at $t = \bar{\tau}(\xi_1)$ the solution reaches the boundary $\partial\Omega$. Therefore, ξ_1 satisfies the relation:

$$\xi_1 = a - \int_{\bar{\tau}(\xi_1)}^{\tau_a} \dot{\xi}(t)\, dt = a - k_o \int_{\bar{\tau}(\xi_1)}^{\tau_a} \frac{f\left(u(t, \xi(t)-)\right) - f\left(w'\right)}{u(t, \xi(t)-) - w'}\, dt\,. \tag{4.16}$$

Setting

$$\chi(\xi_1) := \xi_1 - a + k_o \int_{\bar{\tau}(\xi_1)}^{\tau_a} \frac{f\left(u(t, \xi(t)-)\right) - f\left(w'\right)}{u(t, \xi(t)-) - w'}\, dt\,,$$

we can combine

$$\begin{aligned} \frac{d\chi}{d\xi_1} &= 1 - k_o \, \frac{f\left(u(\bar{\tau}, \xi_1)\right) - f\left(w'\right)}{u(\bar{\tau}, \xi_1) - w'} \, \frac{d\bar{\tau}}{d\xi_1} \\ &= 1 - \frac{1}{f'(w)} \, \frac{f\left(w\right) - f\left(w'\right)}{w - w'} = 1 > 0\,, \end{aligned}$$

with $\chi(a) < 0$ and $\chi(a - \tilde{x} - \tilde{y} + k_o f'(w)\tau_a) = k_o f'(w)\left(\tau_a - \frac{\tilde{x}+\tilde{y}}{k_o f'(w)}\right) > 0$, to conclude that there exists a unique value ξ_1 such that $\chi(\xi_1) = 0$, i.e., a unique location ξ_1 where the reflected shock has been generated. Finally, using (4.13), ξ_2 is uniquely determined as well and the proof is complete. $\diamondsuit$

Remark 4.4. *It is worth noticing that given f, the expression for ξ_1 can be explicitly obtained from* (4.16)*. To fix ideas, let $[u_1, u_2] = [0, 1]$ and $f(u) = u(1-u)$, as in Example 2.2. Then the ordinary differential equation solved by ξ_1 reduces to*

$$\dot{\xi}(t) = k_o \left(1 - w' - u(t, \xi(t)-)\right)$$

which implies that (4.16) *can be written in the form*

$$\xi_1 = a - k_o(\tau_a - \bar{\tau})(1 - w') - k_o \int_{\bar{\tau}}^{\tau_a} u(t, \xi(t)-)\, dt.$$

Since in this case, the solution u_{obs} is given by (4.10) *with $\eta(x) = \frac{1}{2} - \frac{\xi_1 - a + \tilde{x}}{2k_o t}$, the integral can be computed explicitly and ξ_1 can be retrieved as a root of a polynomial of degree three.*

Remark 4.5. *The choice of the initial data $v_o(\,\cdot\,)$ in* (4.6) *for the proof of Theorem 2.4 needs a few comments. With such a choice, the reconstruction procedure consists in "emptying" the unobservable region (a, b) before starting to send new waves that allow to identify exactly the location and size of the obstruction. The first part of the procedure cannot be avoided when $\max\{u_o(a), u_o(b)\} > u^m$, because* *in this case no rarefaction wave can pass through the congested part of* (a, b) *to collect the information needed for the reconstruction. However, this makes the process slower whenever no congested region is present.*

An alternative choice when $\max\{u_o(a), u_o(b)\} < u^m$ *is the following. The assumption that* $u_o(\,\cdot\,)|_{[a,\infty)}$ *is a stationary solution to* (1.1)*, implies that the flux function* $k(\,\cdot\,)$ *attains a value* $k_1 \in [k_o \frac{f(u_o(a))}{f(u^m)}, k_o)$ *in* $[\xi_1, \xi_2]$*. If we knew that* $k_1 > k_o \frac{f(u_o(a))}{f(u^m)}$*, then a more effective choice of the initial data would be*

$$w_o(x) = \begin{cases} u^m, & \text{if } x < a - \tilde{x}\,, \\ u_o(a), & \text{if } a - \tilde{x} < x < a, \\ u_o(x), & \text{if } x > a\,, \end{cases}$$

for any choice of $\tilde{x} > 0$*. In the solution to the Cauchy problem for* (1.1) *with initial data* $w_o(\,\cdot\,)$ *there is no shock wave emptying the unobservable region, but only the rarefaction wave connecting the states* $u_o(a)$ *and* u^m*. Hence, it would still be possible to proceed as in the proof of Lemmas 4.1–4.2 and to find a unique flux function* $k(\,\cdot\,)$ *with the properties required in Theorem 2.4. In addition, the process could be completed in a shorter time.*
The problem is that a priori we cannot exclude that $k_1 = k_o \frac{f(u_o(a))}{f(u^m)}$*, or equivalently that* $u_o(x) = u^m$ *for* $x \in [\xi_1, \xi_2]$*, and in this case no wave in the solution would pass through the congested region* $[\xi_1, \xi_2]$*. Thus, repeating the previous reconstruction procedure would only give* k_1 *and* ξ_1*, but not* ξ_2*.*

A way to combine the best aspects of both approaches is to use $w_o(\,\cdot\,)$ *as initial data and wait to see if at some time* $\tau > 0$ *a shock u-wave appears at* $x = a$*, separating the states* $u_{\text{obs}}(\tau, a-)$ *and* $u_{\text{obs}}(\tau, a+) > u^m$ *with*

$$f(u_{\text{obs}}(\tau, a+)) = f(u_o(a))\,.$$

If this happens, then we realize a posteriori that the region $[\xi_1, \xi_2]$ *was originally congested. Therefore, relying on the fact that* $u_{\text{obs}}(\tau, \,\cdot\,)|_{[a,\infty)}$ *is a stationary solution for* (1.1) *with* $u_{\text{obs}}(\tau, a) > u^m$*, we can restart the procedure for times* $t \geq \tau$ *with the initial data* $v_o(\,\cdot\,)$ *given by* (4.6) *and complete the reconstruction process.*
On the other hand, if $f(u_{\text{obs}}(\tau, a+)) > f(u_o(a))$ *or if there exists* $\tau' > 0$ *such that* $u_{\text{obs}}(\tau', b) > u_o(b)$*, then we can deduce that no congested area was present in* (a, b) *at time* $t = 0$ *and the initial data* $w_o(\,\cdot\,)$ *will be sufficient to complete the reconstruction procedure.*

Proof of Theorem 2.7. The proof of Theorem 2.7 follows from two lemmas. First of all, note that if $k_1 \neq k_o$, then $u \equiv \bar{u}_o$ is not a stationary solution for (1.1). Hence, at time $t = 0+$ the jumps in $k(x)$ produce waves with non-zero speed at one or both sides of each jump (see the Appendix). Our first lemma deals with the evolution of the solution to (1.1) when $u(0, x) \equiv \bar{u}_o$ and k is of the form (2.5).

Lemma 4.6. *Assume that the conservation law* (1.1) *satisfies* **(H1)**–**(H3)** *and that* $f(u)$ *is a known function. Fix a constant* $\bar{u}_o \in [u_1, u^m)$ *and denote by* $u(t, x)$ *the solution to the Cauchy problem for* (1.1) *with* $u(0, x) \equiv \bar{u}_o$ *for* $x \in \mathbb{R}$*. Then, the following facts hold:*

(i): *If* $k_o f(\bar{u}_o) > k_1 f(u^m)$*, then there exist* $T_1, T_2 > 0$ *such that*

$$u(T_1, b+) = \bar{u}_o > u(T_1, b-) \text{ and } u(T_2, a-) = \bar{u}_o < u(T_2, a+)\,. \tag{4.17}$$

(ii): *If* $k_o f(\bar{u}_o) \leq k_1 f(u^m)$*, then either* $u(t, \,\cdot\,) \equiv \bar{u}_o$ *in* $\mathbb{R} \setminus (a, b)$ *for all* $t > 0$ *or there exist* $0 < T_1 < T_2$ *such that*

$$u(T_1, b+) = \bar{u}_o > u(T_1, b-) \text{ and } u(T_2, b) > u(T_1, b-)\,. \tag{4.18}$$

Moreover, the former case does not happen if $k_o f(\bar{u}_o) = k_1 f(u^m)$*.*

The second lemma gives sufficient conditions for the existence of a triple (k_1, ξ_1, ξ_2) such that the solution to (1.1) with constant initial data $u(0,x) \equiv \bar{u}_o$ and flux $k(x)f(u)$, $k(x)$ being given by (2.5), coincides with the observed solution $u_{\rm obs}$ in the observable region.

Lemma 4.7. *Assume that the conservation law* (1.1) *satisfies* **(H1)**–**(H3)***, that* $f(u)$ *is a known function and that the solution* $u_{\rm obs}(t,x)$ *to the Cauchy problem for* (1.1) *with constant initial data* $u(0,x) \equiv \bar{u}_o \in [u_1, u^m)$ *is partially observable in* $[0,T] \times \big(\mathbb{R} \setminus (a,b)\big)$.

(i): *If there exist* $T_1, T_2 \in (0,T)$ *such that*

$$u_{\rm obs}(T_1, b+) = \bar{u}_o > u_{\rm obs}(T_1, b-) \text{ and } u_{\rm obs}(T_2, a-) = \bar{u}_o < u_{\rm obs}(T_2, a+)\,, \tag{4.19}$$

then there exists a unique choice of (k_1, ξ_1, ξ_2) *such that if* $u_{(k_1,\xi_1,\xi_2)}$ *denotes the solution of the Cauchy problem for* (1.1) *with initial data* $u(0,x) \equiv \bar{u}_o$ *and* $k(x)$ *given by* (2.5)*, we have that*

$$u_{(k_1,\xi_1,\xi_2)}(t,x) = u_{\rm obs}(t,x), \quad (t,x) \in [0,T] \times (\mathbb{R} \setminus (a,b))\,.$$

(ii): *If there exist* $0 < T_1 < T_2 < T$ *such that*

$$u_{\rm obs}(T_1, b+) = \bar{u}_o > u_{\rm obs}(T_1, b-) \text{ and } u_{\rm obs}(T_2, b) > u_{\rm obs}(T_1, b-)\,, \tag{4.20}$$

then there exists a choice of (k_1, ξ_1, ξ_2) *such that if* $u_{(k_1,\xi_1,\xi_2)}$ *denotes the solution of the Cauchy problem for* (1.1) *with initial data* $u(0,x) \equiv \bar{u}_o$ *and* $k(x)$ *given by* (2.5)*, we have that*

$$u_{(k_1,\xi_1,\xi_2)}(t,x) = u_{\rm obs}(t,x), \quad (t,x) \in [0,T] \times (\mathbb{R} \setminus (a,b))\,.$$

Moreover, if

$$\inf\{s \in (T_1, T_2)\ ;\ u(s,b) > u(T_1, b-)\} > T_1\,,$$

then the choice is also unique.

Now the proof of Theorem 2.7 is immediate.

Proof of Theorem 2.7. Let $u_{\rm obs}(t,x)$ denote the solution of the Cauchy problem for (1.1) with constant initial data $u(0,x) \equiv \bar{u}_o \in [u_1, u^m)$ and a flux function kf with k given by (2.5). Even if we do not know the values of k_1, ξ_1, ξ_2, we know that either $k_o f(\bar{u}_o) > k_1 f(u^m)$ or $k_o f(\bar{u}_o) \le k_1 f(u^m)$. In the former case, Lemma 4.6 ensures that there exist $T_1, T_2 > 0$ such that (4.17) holds for $u_{\rm obs}$. Hence, we can apply part *(i)* of Lemma 4.7 to find the triple (k_1, ξ_1, ξ_2) which gives a solution satisfying (2.9). Similarly, in the latter case, Lemma 4.6 ensures that either $u(t,\cdot\,) \equiv \bar{u}_o$ in $\mathbb{R} \setminus (a,b)$ for all $t > 0$, or there exist $0 < T_1 < T_2$ such that (4.17) holds for $u_{\rm obs}$. In particular, if $u(t,\cdot\,) \not\equiv \bar{u}_o$ for all times $t > 0$, part *(ii)* of Lemma 4.7 gives (k_1, ξ_1, ξ_2) such that (2.9) holds.

The uniqueness part follows from Lemma 4.7 as well, under the hypotheses of Theorem 2.7, completing the proof. $\diamondsuit$

It remains to prove Lemma 4.6 and Lemma 4.7.

Proof of Lemma 4.6. In terms of [15], we can study the Cauchy problem for (1.1) by studying the auxiliary system (A.2) for the unknowns (k,u). In this case, the initial data $u(0,\cdot\,) \equiv \bar{u}_o$ is written

$$(u_{in}, k_{in}) = (u,k)(0,x) = \begin{cases} (\bar{u}_o, k_o) & \text{if } x < \xi_1\,, \\ (\bar{u}_o, k_1) & \text{if } \xi_1 < x < \xi_2\,, \\ (\bar{u}_o, k_o) & \text{if } x > \xi_2\,, \end{cases} \tag{4.21}$$

for the resonant system (A.2). In the following, we call u-waves any Lax elementary wave for (A.2), i.e., shock waves or centered rarefaction waves, propagating with constant k, and k-waves the stationary jumps between states satisfying (A.3) (see the Appendix). When considering the Cauchy problem (A.2)–(4.21), both jumps in the initial data (u_{in}, k_{in}) create for $t > 0$ a stationary k-wave and some u-waves. Namely, since $\xi_1 < \xi_2$, for small times the solution to the Cauchy problem is given by the juxtaposition of the solutions to the Riemann problems in $x = \xi_1$ and $x = \xi_2$ and it can be described, in terms of the Riemann solver described in the Appendix, as follows:

(i) Assume $k_o f(\bar{u}_o) > k_1 f(u^m)$ and consider first the jump in $x = \xi_1$. In this case, the Riemann problem is solved by a shock u-wave with negative speed σ^-, a stationary k-wave and a centered rarefaction u-wave with positive speeds. In terms of the original system, this means that at $x = \xi_1+$ the variable passes from $\bar{u}_o$ to the larger u^m, because there is more incoming u than the obstructed region can carry. Moreover, a shock wave appears in $x = \xi_1-$ and propagates back towards $x = a$ with speed σ^-. In terms of the traffic flow model presented in Example 2.2, such a solution can be interpreted as a queue of cars forming at $x = \xi_1-$ and traveling back towards $x = a$, followed by a region of congested traffic in $x = \xi_1+$ due to the continuous arrival of more cars than the obstructed highway can carry.

Consider now the jump from k_1 to k_o at $x = \xi_2$. The Riemann problem for (A.2) is solved by a stationary k-wave followed by a shock u-wave traveling with positive speed σ^+ towards $x = b$. In terms of the original system, this means that at $x = \xi_2+$ a smaller value $u' \in [u_1, \bar{u}_o]$, such that $k_o f(u') = k_1 f(\bar{u}_o)$, emerges from the discontinuity and therefore a shock u-wave between u' and $\bar{u}_o$ forms. From the point of view of Example 2.2, this means that cars at $x = \xi_2-$ have to slow down due to the obstruction and a region with smaller car density u' appears at $x = \xi_2+$.

As t increases, the u-shock traveling with speed σ^- simply propagates in (a, ξ_1) and reaches $x = a$ at time $T_2 = \frac{a-\xi_1}{\sigma^-} > 0$, as requested by the second part of (4.17).

On the other hand, the rarefaction u-wave created at $x = \xi_1$ eventually interacts with the stationary k-wave in $x = \xi_2$ and keeps propagating in (ξ_2, b) as a rarefaction u-wave, but with larger positive speeds. In particular, this rarefaction u-wave in (ξ_2, b) is now separating the states in the interval $[u', \bar{u}_o)$ and its front travels with speed $f'(u')$ larger than the speed σ^+ of the shock u-wave generated at $t = 0$ in $x = \xi_2$, due to the entropy admissibility of the shock. Hence, the u-rarefaction could start interacting with the u-shock, before they reach $x = b$. However, the u-shock cannot be completely canceled by the u-rarefaction and, therefore, there must exist $T_1 > 0$ such that the first part of (4.17) is verified as well.

(ii) Assume now $k_o f(\bar{u}_o) \leq k_1 f(u^m)$, and consider first the jump at $x = \xi_1$. In this case, the Riemann problem is solved by a stationary k-wave and a centered rarefaction u-wave traveling with positive speed. In terms of the original system, this means that the incoming quantity $\bar{u}_o$ from $x = \xi_1-$ does not completely fill the region in (ξ_1, ξ_2). As a consequence, u only increases from $\bar{u}_o$ to a larger value $u'' \leq u^m$ such that $k_o f(\bar{u}_o) = k_1 f(u'')$. In the terminology of Example 2.2, this means that the incoming cars do not completely fill the road in (ξ_1, ξ_2) and therefore no queue appears at $x = \xi_1-$.

Considering the jump from k_1 to k_o at $x = \xi_2$, the Riemann problem for (A.2) is solved again by a stationary k-wave followed by a shock u-wave traveling with positive speed $\bar{\sigma}$ towards $x = b$. As before, in terms of the original system, this means that at $x = \xi_2+$ a smaller value $u' \in [u_1, \bar{u}_o]$, such that $k_o f(u') = k_1 f(\bar{u}_o)$, emerges from the discontinuity and therefore a shock u-wave between u' and $\bar{u}_o$ forms.

As t increases, the rarefaction u-wave exiting $x = \xi_1$ will eventually interact with the stationary k-wave in $x = \xi_2$ and will keep propagating as a rarefaction u-wave, but with larger positive speed in (ξ_2, b). As before, this rarefaction u-wave is faster than the shock u-wave traveling with speed $\bar{\sigma}$ and, hence, the u-waves could start interacting before reaching $x = b$. This interaction opens up two different scenarios:

- Either the whole interaction between rarefaction and shock takes place in (ξ_2, b), resulting in a complete cancellation of the two waves (this is the case whenever
$$\frac{\xi_2 - \xi_1}{k_1 f'(\bar{u}_o)} + \frac{b - \xi_2}{k_o f'(u')} \leq \frac{b - \xi_2}{\bar{\sigma}},$$
where $\bar{\sigma}$ denotes the speed of the shock separating u' and $\bar{u}_o$).
- Or there exists $\tau_1 > 0$ such that $u(\tau_1, b) < \bar{u}_o$.

In the former case, $u(t, \cdot)|_{\mathbb{R}\setminus(a,b)} \equiv \bar{u}_o$ for all $t > 0$. In the latter case, we simply set $T_1 = \tau_1$ and
$$T_2 = \inf\{s \geq T_1 \ ; \ u(s, b-) > u(T_1, b)\} + \varepsilon,$$
for any fixed $\varepsilon > 0$ small. Observe that if $k_o f(\bar{u}_o) = k_1 f(u^m)$ the last case holds, and hence $u'' = u^m$, which completes the proof. ♢

Proof of Lemma 4.7. By assuming that f and k satisfy **(H1)**–**(H3)**, one immediately obtains some properties of the solution u_{obs} to (1.1) with constant initial data $u(0, x) \equiv \bar{u}_o$. In particular, from the description of the Riemann solver for (1.1) given in the Appendix, one can see that $k_1 < k_o$ in (2.5) implies that $u_{\text{obs}}(t, x)$ for small $t > 0$ can only contain the following Lax waves: a shock wave propagating from the second discontinuity point $x = \xi_2$ towards $x = b$, a rarefaction wave propagating from the first discontinuity point $x = \xi_1$ with positive speeds and, possibly, a shock wave propagating from $x = \xi_1$ towards $x = a$. The presence of the latter shock depends on the value k_1, which is unknown to the observer. We can now proceed to the proof of the lemma.

(i) Assume that at time $t = T_1$ a jump in $u_{\text{obs}}(T_1, \cdot)$ appears at $x = a$. This jump corresponds to a shock u-wave arriving from $x = \xi_1$ with negative speed. Let $\bar{u}_o = u_{\text{obs}}(T_1, a-)$ and $v_o = u_{\text{obs}}(T_1, a+)$ denote the densities separated by the shock with propagation speed σ_a. Then we can immediately deduce
$$k_1 = k_o \frac{f(v_o)}{f(u^m)},$$
and thus
$$\xi_1 = a - \sigma_a T_1.$$
Since the shock cannot have interacted with any other wave, due to the fact that both $\bar{u}_o$ and k are constant in $[a, \xi_1)$, these values represent the only possible choice of k_1, ξ_1 which generates the observed shock.

Hence, to complete the reconstruction of $k(x)$ it only remains to find ξ_2. This can be done by using the second condition in (4.19). At $t = T_2$, a new wave appears in $x = b$ and it is a shock. Let σ_b be the positive speed of this shock and $v_1 = u_{\text{obs}}(T_1, b-)$ and $\bar{u}_o = u_{\text{obs}}(T_1, b+)$ be the states separated by the shock. If
$$k_o \frac{f(v_1)}{f(\bar{u}_o)} = k_1, \tag{4.22}$$
then we can conclude that the shock has reached $x = b$ without interacting with any other wave, and find ξ_2 as
$$\xi_2 = b - \sigma_b T_2.$$
Otherwise, if (4.22) does not hold, v_1 is not the original left state of the shock. Hence, the observed shock is in fact the result of the interaction between the shock

generated at $t = 0+$ at $x = \xi_2$ and the faster rarefaction wave generated at $t = 0+$ at $x = \xi_1$. In this case, to find ξ_2 we exploit the following relation

$$\xi_2 - \xi_1 = \left(T_2 - \frac{b - \xi_2}{k_o f'(v_1)}\right) k_1 f'(w_1)\,, \tag{4.23}$$

where w_1 is such that $w_1 < u^m$ and $k_o f(v_1) = k_1 f(w_1)$. Observe that (4.23) states that the rarefaction wave observed in (T_2, b) has traveled with speed $k_1 f'(w_1)$ in (ξ_1, ξ_2) and with speed $k_o f'(v_1)$ in (ξ_2, b). This completes the proof when (4.19) holds.

(ii) Set now

$$\tau := \inf\{s \in \ (T_1, T_2)\ ;\ u(s, b) > u(T_1, b-)\}\ .$$

If $\tau > T_1$, then the shock wave generated at $x = \xi_2$ has reached the observable region $[b, \infty)$ without interacting with the centered rarefaction wave generated at $x = \xi_1$. Hence, we can repeat the procedure followed in *(i)* and let $\bar{u}_o = u_{\mathrm{obs}}(T_1, b+)$, $v_o = u_{\mathrm{obs}}(T_1, b-)$ be the densities separated by the shock which reaches $x = b$ at $t = T_1$ and σ_b be its positive propagation speed. Then,

$$k_1 = k_o\, \frac{f(v_o)}{f(\bar{u}_o)}\,,$$

and

$$\xi_2 = b - \sigma_b T_1\,.$$

To find ξ_1 we now use the fact that for $\tau < T_2$ the rarefaction appears at $x = b$. Indeed, we know that the rarefaction wave taking the value v_o has traveled with speed $k_o f'(v_o)$ in (ξ_2, b) and with speed $k_1 f'(w_o)$ in (ξ_1, ξ_2), where w_o is such that $w_o < u^m$ and $k_o f(v_o) = k_1 f(w_o)$. Hence, we can exploit the relation

$$\xi_2 - \xi_1 = \left(\tau - \frac{b - \xi_2}{k_o f'(v_o)}\right) k_1 f'(w_o)\,,$$

which gives ξ_1.

On the other hand, if $\tau = T_1$, then the wave observed in (τ, b) is a rarefaction wave followed by an adjacent shock wave, and this means that the interaction between the faster rarefaction wave and the slower shock wave in $(\xi_2, b]$ has already begun. In this case, we can still proceed as above: let $\bar{u}_o = u_{\mathrm{obs}}(T_1, b+)$, $v_1 = u_{\mathrm{obs}}(T_1, b-)$ be the densities separated by the shock and σ_b be its propagation speed and set

$$k_1 = k_o\, \frac{f(v_1)}{f(\bar{u}_o)}\,,$$

$$\xi_2 = b - \sigma_b T_1\,,$$

and

$$\xi_2 - \xi_1 = \left(T_1 - \frac{b - \xi_2}{k_o f'(v_1)}\right) k_1 f'(w_1)\,,$$

where w_1 is again such that $w_1 < u^m$ and $k_o f(v_1) = k_1 f(w_1)$. These values of (k_1, ξ_1, ξ_2) provide a solution $u_{(k_1, \xi_1, \xi_2)}$ which coincides with u_{obs} outside (a, b), but they are not in general the only ones with such a property. $\diamondsuit$

5. Conclusions

In this paper we have presented some new results concerning inverse problems for scalar conservation laws of the form (1.1). Namely, for homogeneous equations (1.3), we have presented a reconstruction procedure to find piecewise linear interpolations f_ν of any piecewise $C^{1,1}$ flux f having a finite number of inflection points, under the unique assumption that solutions to Riemann problems are observable at a fixed time $T > 0$. No a priori assumption is requested on the smoothness of the observed solution, or on its jumps structure. The reconstructed flux is accurate in the following sense: solutions to Cauchy problems for the conservation law with flux f_ν are close in $\mathbf{L}^1$ to the solutions for the conservation law with exact flux f.

For general inhomogeneous equations (1.1), we have first proved that being able to observe in $[0,T]\times\mathbb{R}$ the solutions to Cauchy problems, for an arbitrarily small time T, is sufficient to obtain a piecewise linear approximation of f, and the precise form of $k(x)$ for x in any compact interval $J\subseteq\mathbb{R}$.

Then, motivated by applications to traffic flow models, we have studied the same inverse problem when the solutions are only observable in part of the domain, due to the presence of some inaccessible spatial region $I\subseteq\mathbb{R}$, and the goal is to reconstruct $k(x)$ also inside I. In this case, even assuming the observation of the solution for a long time interval $[0,T]$, the function k in the unobservable region can only be recovered under the strict assumption that k has no more than two jumps in I. Unfortunately, this is not just a mathematical obstacle or a limitation in the results we have presented: The examples in Section 3 show that if three or more jumps are present, then in many situations we end up with an infinite number of piecewise constant functions $k(x)$ which all give the same solution in the observable region $\mathbb{R}\setminus I$.

In view of these examples, it seems clear that inverse problems for inhomogeneous conservation laws, when only partial observability of the solution is assumed, are in general ill-posed. Therefore, the next steps in the study of inverse problems should focus the attention either on specific inhomogeneous scalar models or on homogeneous systems of hyperbolic conservation laws. In the former case, one can hope that physical features of the particular model considered help in order to obtain well-posedness. In the latter case, one can try to exploit the front-tracking algorithm to choose initial data which are particularly well suited for the reconstruction, in the spirit of Theorem 2.1. This problem is much more difficult for systems than for the single equation, due to the possibly complicate wave structure of the solutions, but some positive result could be possible, at least in the case of Temple class systems, which have coinciding shock and centered rarefaction waves. Alternatively, one could try to adapt the least square method, used in [11], to the generalized differentiability structures which have been introduced for systems of conservation laws by Bressan and Marson [3], and look for a reconstruction of the flux as the minimizer of a cost like (1.2). Also with this approach, however, the case of systems is substantially more difficult than the case of the single equation, and it is not clear which regularity can be expected from the cost functional.

Appendix

Here we collect some auxiliary results which have been used in the paper. We start from a simple property from standard calculus which have been exploited in the proof of Theorem 2.1.

Lemma A.1. *If $\gamma\colon [a,b]\to\mathbb{R}$ is a continuous and strictly monotone function, then*

$$\int_{\gamma(a)}^{\gamma(b)} \gamma^{-1}(s)\,ds = \gamma(b)b-\gamma(a)a-\int_a^b \gamma(t)\,dt\,. \tag{A.1}$$

Next, we recall a well-posedness result for scalar conservation laws that we have exploited to show that (2.2) implies (2.3). The proof can be found in [10, Theorem 2.3].

Theorem A.2. *Let f,g be Lipschitz continuous functions, and assume $\hat{u}\in\mathbf{BV}(\mathbb{R})$. Denote by u^f and u^g, respectively, the solutions to the Cauchy problems for*

$$\partial_t u+\partial_x f(u)=0\,,\qquad\qquad \partial_t u+\partial_x g(u)=0\,,$$

with initial data $u(0,x) = \hat{u}(x)$. Then, there exists a positive constant C such that for all $T \geq 0$

$$\|u^f(T,\,\cdot\,) - u^g(T,\,\cdot\,)\|_{\mathbf{L}^1(\mathbb{R})} \leq CT\mathrm{Lip}(f-g)\,.$$

Finally, we offer a brief description of the Riemann solver for (1.1) defined and studied in [15, 18], which we have used extensively in the proofs of Theorems 2.3–2.7. First of all, taking flux functions $k(x)$ and $f(u)$ that satisfy **(H1)**, we observe that the inhomogeneous equation (1.1) can be studied by considering an auxiliary system of conservation laws

$$\begin{cases} \partial_t u + \partial_x(kf(u)) = 0, \\ \partial_t k = 0, \end{cases} \tag{A.2}$$

which represents the conservation of the quantity $v = (u,k)$ with flux $g(v) = (kf(u), 0)$. The aim of this auxiliary system $\partial_t v + \partial_x g(v) = 0$ is to help in the study of the behavior of the solution to (1.1) at discontinuities of $k(x)$. However, such an auxiliary system is non-strictly hyperbolic, since waves of the second family (i.e., related to the second equation) all have null speed, while waves of the first family (i.e., related to the original scalar equation) can have positive or negative speeds depending on the sign of f'. Hence, the system (A.2) requires some additional attention.

The properties we need to know here are the following:

- a solution to any Riemann problem for (A.2) can be constructed by following [8];
- the solution is unique, provided an additional "entropy" condition holds at jumps of k (the precise condition will be discussed below);
- the construction allows one to build a converging front-tracking approximation for general Cauchy problems following [15].

Namely, the construction proceeds as follows. The Rankine–Hugoniot conditions for (A.2) can be written as

$$kf(u) - k'f(u') = \lambda(u-u')\,, \qquad\qquad 0 = \lambda(k-k')$$

for a discontinuity separating states (u,k),(u',k') and traveling with speed λ. In other words, either $k = k'$ and we have a discontinuity in u only, or $\lambda = 0$ and the states separated by the stationary jump satisfy

$$kf(u) = k'f(u')\,. \tag{A.3}$$

In particular, all discontinuities in k give origin to a stationary jump in the solution, with neither k nor u being continuous across the jump.
Under the assumption **(H1)**, for a fixed state u and fixed constants k, k' in general there exist two solutions $u' = v_1$ and $u' = v_2$ to (A.3), and they satisfy $v_1 \leq u^m \leq v_2$. The admissibility condition (or "entropy" condition) mentioned above, which is needed to select a single state u' at the stationary jumps, is the following: the admissible state is the one which realizes $\min\{|u-v_1|, |u-v_2|\}$. In general, fix a jump of $k(x)$ and denote the states adjacent to the discontinuity and satisfying (A.3) by k^-, u^- and k^+, u^+. Then the solution connecting these states is *entropy admissible* if and only if u^+, u^- satisfy

$$|u^+ - u^-| = \min\left\{|v-v'|\ ;\ k^+f(v) = k^-f(v')\right\}, \tag{A.4}$$

i.e., if they minimize the quantity $|v - v'|$ among all pairs satisfying (A.3). In [8] it was shown that this "smallest jump" condition is equivalent to a viscous profile entropy condition for the auxiliary system (A.2), justifying the use of the word entropy also in the context of (1.1).

Now assume that k has a discontinuity at $x = 0$ and that we are given a Riemann initial data

$$v_o(x) = \begin{cases} (u^\ell, k^\ell), & \text{if } x < 0, \\ (u^r, k^r), & \text{if } x > 0, \end{cases} \tag{A.5}$$

for suitable constants $u^\ell, u^r \in [u_1, u_2]$ and $k^\ell, k^r > 0$. Following the conventions of [8, 15], we will call u-waves the Lax waves with constant k (equivalently, the Lax waves of the first family for (A.2)) and k-waves the Lax stationary waves where k changes and (A.3) holds (equivalently, the Lax waves of the second family for (A.2)). Then, the solution to (A.2)–(A.5) can be constructed as follows:

Case 1. Assume $k^\ell < k^r$ and $u^\ell \le u^m$. Then, if $u^r \le u^m$ or $k^\ell f(u^\ell) < k^r f(u^r)$, the solution is given by a stationary k-wave between (k^ℓ, u^ℓ) and (k^r, v), with v satisfying $v < u^m$ and $k^\ell f(u^\ell) = k^r f(v)$, followed by a u-shock or a centered u-rarefaction with positive speed, between the states $(k^r, v), (k^r, u^r)$. On the other hand, if $u^r > u^m$ and $k^\ell f(u^\ell) \ge k^r f(u^r)$, the solution is given by a u-shock, traveling with negative speed, between the states (k^ℓ, u^ℓ) and (k^ℓ, w), with $w > u^m$ and $k^\ell f(w) = k^r f(u^r)$, followed by a k-wave separating (k^ℓ, w) and (k^r, u^r).

Case 2. Assume $k^\ell < k^r$ and $u^\ell > u^m$. Then, if $u^r \le u^m$ or $k^r f(u^r) > k^\ell f(u^m)$, the solution is given by a centered u-rarefaction with positive speed, between the states $(k^\ell, u^\ell), (k^\ell, u^m)$, followed by a k-wave between (k^ℓ, u^m) and (k^r, v'), with v' satisfying $v' < u^m$ and $k^\ell f(u^m) = k^r f(v')$, followed by a u-shock or a u-rarefaction with positive speed, separating the states $(k^r, v'), (k^r, u^r)$. On the other hand, if $u^r > u^m$ and $k^r f(u^r) \le k^\ell f(u^m)$, the solution is given by a u-shock or a u-rarefaction with positive speed, between the states (k^ℓ, u^ℓ) and (k^ℓ, w'), with $w' \ge u^m$ and $k^\ell f(w') = k^r f(u^r)$, followed by a k-wave separating (k^ℓ, w') and (k^r, u^r).

Case 3. Assume $k^\ell > k^r$ and $u^r \le u^m$. Then, if $u^\ell \ge u^m$ or $k^\ell f(u^\ell) > k^r f(u^m)$, the solution is given by a u-shock or a centered u-rarefaction with negative speed, separating the states (k^ℓ, u^ℓ) and (k^ℓ, v''), with v'' satisfying $v'' > u^m$ and $k^r f(u^m) = k^\ell f(v'')$, followed by a k-wave between (k^ℓ, v'') and (k^r, u^m), followed by a u-rarefaction with positive speed, between the states $(k^r, u^m), (k^r, u^r)$. On the other hand, if $u^\ell < u^m$ and $k^\ell f(u^\ell) \le k^r f(u^m)$, the solution is given by a k-wave between the states (k^ℓ, u^ℓ) and (k^r, w''), with $w'' < u^m$ and $k^r f(w'') = k^\ell f(u^\ell)$, followed by a u-shock or a u-rarefaction with positive speed, separating the states $(k^r, w''), (k^r, u^r)$.

Case 4. Assume $k^\ell > k^r$ and $u^r > u^m$. Then, if $u^\ell \ge u^m$ or $k^\ell f(u^\ell) \ge k^r f(u^r)$, the solution is given by a u-shock or a centered u-rarefaction with negative speed, between the states (k^ℓ, u^ℓ) and (k^ℓ, v'''), with v''' satisfying $v''' > u^m$ and $k^r f(u^r) = k^\ell f(v''')$, followed by a stationary u-wave separating (k^ℓ, v''') and (k^r, u^r). On the other hand, if $u^\ell < u^m$ and $k^\ell f(u^\ell) < k^r f(u^r)$, the solution is given by a k-wave between the states (k^ℓ, u^ℓ) and (k^r, w'''), with $w''' < u^m$ and $k^r f(w''') = k^\ell f(u^\ell)$, followed by a u-shock with positive speed, between the states $(k^r, w'''), (k^r, u^r)$.

This construction of a solution to every the Riemann problem (A.2)–(A.5) provides a Riemann solver which allows to solve the Cauchy problem too, by means of a standard front-tracking algorithm [2, 10]. The precise proof of the next theorem, and in particular of the compactness of the approximation which allows to apply Helly's theorem, can be found in [15].

Theorem A.3. *Let f, k be flux functions satisfying* **(H1)**. *Then, for every initial data $\hat u \in \mathbf{BV}(\mathbb{R})$, the Cauchy problem for*

$$\partial_t u + \partial_x\big(k(x) f(u)\big) = 0\,,$$

with initial data $u(0,x) = \hat{u}(x)$ admits a weak solution u such that, for every time $t \geq 0$, $u(t,\cdot) \in \mathbf{L}^1(\mathbb{R})$ is obtained as the uniform $\mathbf{L}^1_{\mathrm{loc}}$ limit of a front-tracking approximation $u^\delta(t,\cdot)$, constructed using the Riemann solver described above.

References

[1] F. Bachmann, *Analysis of a scalar conservation law with a flux function with discontinuous coefficients.* Advances in Differential Equations **11–12** (2004), 1317–1338. 1

[2] A. Bressan, *Hyperbolic Systems of Conservation Laws. The One-dimensional Cauchy Problem.* Oxford University Press, Oxford, 2000. 1, 1, 4, 5

[3] A. Bressan & A. Marson, *A variational calculus for discontinuous solutions of systems of conservation laws.* Comm. Partial Differential Equations **20** (1995), no. 9–10, 1491–1552. 5

[4] C. Castro & E. Zuazua, *Flux identification for 1-d scalar conservation laws in the presence of shocks.* Math. Comp. **80**, 276, (2011), 2025–2070. 1

[5] C. M. Dafermos, *Hyperbolic Conservation Laws in Continuum Physics.* Third edition. Springer, Berlin, 2005. 1, 4

[6] S. Diehl, *On scalar conservation laws with point source and discontinuous flux function.* SIAM J. Math. Anal. **26** (1995), no. 6, 1425–1451. 1

[7] S. Diehl, *Scalar conservation laws with discontinuous flux function. I. The viscous profile condition.* Comm. Math. Phys. **176** (1996), no. 1, 23–44. 1

[8] T. Gimse & N. H. Risebro, *Riemann problem with a discontinuous flux function.* In Proc. 3rd Internat. Conf. Hyperbolic Problems, Uppsala, 488–502. Studentlitteratur, Lund, 1991. 5, 5, 5

[9] T. Gimse & N. H. Risebro, *Solution of the Cauchy problem for a conservation law with a discontinuous flux function.* SIAM J. Math. Anal. **23** (1992), no. 3, 635–648. 1

[10] H. Holden & N. H. Risebro, *Front Tracking for Hyperbolic Conservation Laws.* Second edition. Springer, Berlin, 2007. 1, 1, 2, 4, 5, 5

[11] F. James & M. Sepúlveda, *Convergence results for the flux identification in a scalar conservation law.* SIAM J. Control Optim. **37** (1999), no. 3, 869–891. 1, 5

[12] H. Kang & K. Tanuma, *Inverse problems for scalar conservation laws.* Inverse Problems **21** (2005), no. 3, 1047–1059. 1

[13] K. H. Karlsen, N. H. Risebro & J. D. Towers, *L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients.* Skr. K. Nor. Vidensk. Selsk. 2003, no. 3, 1–49. 2

[14] R. A. Klausen & N. H. Risebro *Stability of conservation laws with discontinuous coefficients.* J. Differential Equations. **157** (1999), no. 1, 41–60. 1, 2

[15] C. Klingenberg & N. H. Risebro, *Convex conservation laws with discontinuous coefficients. Existence, uniqueness and asymptotic behavior.* Comm. Partial Differential Equations **20** (1995), no. 11-12, 1959–1990. 1, 2, 4, 4, 4, 4, 5, 5, 5

[16] M. J. Lighthill & G. B. Whitham, *On kinematic waves II. A theory of traffic flow on long crowded roads.* Proc. Roy. Soc. London Ser. A **229** (1955), 317–345. 1, 1, 2.2

[17] P. I. Richards, *Shock waves on the highway.* Operations Res. **4** (1956), 42–51. 1, 1, 2.2

[18] N. H. Risebro, *An introduction to the theory of scalar conservation laws with spatially discontinuous flux functions.* In Applied Wave Mathematics, ed. E. Quak, Springer Verlag 2009, 395–464. 2, 5

[19] J. D. Towers, *Convergence of a difference scheme for conservation laws with a discontinuous flux.* SIAM J. Numer. Anal. **38** (2000), no. 2, 681–698. 1, 2

(Holden)
Department of Mathematical Sciences, Norwegian University of Science and Technology, NO–7491 Trondheim, Norway,
and
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway
E-mail address: holden@math.ntnu.no
URL: www.math.ntnu.no/~holden

(Priuli)
Dipartimento di Matematica, Università degli Studi di Roma Tor Vergata, Via della Ricerca Scientifica 1, I–00133 Roma, Italy.
E-mail address: priuli@mat.uniroma2.it
URL: http://www.mat.uniroma2.it/~priuli/

(Risebro)
Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, NO–0316 Oslo, Norway.
E-mail address: nilshr@math.uio.no
URL: www.math.uio.no/~nilshr