This is a translation of a similar note in Norwegian

The determinant of a matrix larger than 2×2

A square matrix A is invertible if the determinant is different from zero, i.e. $\det A \neq 0$, and, conversely, A is singular (not invertible) if $\det A = 0$.

The system $A\mathbf{x} = \mathbf{b}$ has a unique solution if $\det A \neq 0$ and either no solution or infinitely many solutions if $\det A = 0$.

In particular, the system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$ if $\det A \neq 0$ and infinitely many solutions if $\det A = 0$. (The system $A\mathbf{x} = \mathbf{0}$ will always have at least one solution, namely the trivial solution $\mathbf{x} = \mathbf{0}$).

2×2-matrices

As mentioned in the textbook, the determinant of a 2×2-matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is given by

$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{21} \end{vmatrix} = a_{11}a_{21} - a_{21}a_{12}.$$

Example 1.

$$\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = 1 \cdot (-2) - 2 \cdot 3 = -2 - 6 = -8.$$

Larger matrices

We compute the determinant of a matrix that is larger than 2×2 by

- starting with the product of the 1st entry in the top row and the determinant of the submatrix that is obtained by ignoring both the row and the column that the entry belongs to,
- then *subtract* the product of the 2nd entry in the top row and the determinant of the submatrix obtained by ignoring both the row and the column that the entry belongs to,
- then *add* the product of the 3rd entry in the top row and the determinant of the submatrix obtained by ignoring both row and the column that the entry belongs to,
- and so on for all the entries in the top row, alternating between addition and subtraction.

Example 2.

$$\begin{vmatrix} 4 & 5 & 8 \\ 7 & 1 & 3 \\ 6 & 2 & -2 \end{vmatrix} = 4\begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} - 5\begin{vmatrix} 7 & 3 \\ 6 & -2 \end{vmatrix} + 8\begin{vmatrix} 7 & 1 \\ 6 & 2 \end{vmatrix}$$
$$= 4 \cdot (-8) - 5 \cdot (-32) + 8 \cdot 8 = 192$$

Rules

The computations in the previous example would have been easier if we had some zeroes in the top row. We have some rules for computing with determinants that may help us to attain that:

1. The determinant of a matrix is the same as the determinant of the transpose matrix (i.e. $\det A = \det A'$).
2. The determinant changes sign if two rows or two colums are swapped.
3. If B is obtained by multiplying a single row or a column of A by a number k, then $\det B = k \det A$.
4. The determinant of a matrix remains unchanged if some multiple of a row or a column is added to another row or column.

Example 3.

$$
\begin{aligned}
\begin{vmatrix} 4 & 5 & 8 \\ 7 & 1 & 3 \\ 6 & 2 & -2 \end{vmatrix} &= \begin{vmatrix} -31 & 0 & -7 \\ 7 & 1 & 3 \\ -8 & 0 & -8 \end{vmatrix} && \text{Rule 4: } -5 \text{ times row 2 is added to row 1} \\
& && \text{and } -2 \text{ times row 2 is added to row 3} \\
&= -\begin{vmatrix} 0 & -31 & -7 \\ 1 & 7 & 3 \\ 0 & -8 & -8 \end{vmatrix} && \text{Rule 2: column 1 and 2 are swapped} \\
&= -\begin{vmatrix} 0 & 1 & 0 \\ -31 & 7 & -8 \\ -7 & 3 & -8 \end{vmatrix} && \text{Rule 1} \\
&= \begin{vmatrix} -31 & -8 \\ -7 & -8 \end{vmatrix} && \text{The } 3 \times 3 \text{ determinant is computed.} \\
&= 8 \begin{vmatrix} 31 & 1 \\ 7 & 1 \end{vmatrix} && \text{Rule 3: column 2 multiplied by } -1/8 \text{ (the} \\
& && \text{constant 8 is taken outside), and column} \\
& && \text{1 is multiplied by } -1 \text{ (the constant } -1 \text{ is} \\
& && \text{taken outside).} \\
&= 8 \cdot 24 = 192 && \text{The } 2 \times 2\text{-determinant is computed.}
\end{aligned}
$$

We could have saved ourselves the trouble of using rule 2 and rule 1 in the example above, by expanding the determinant along column 2 after having used rule 4 in the first step. To do that, we would use column 2 instead of row 1 and otherwise proceed as on the previous page. Then we would, however, have to be careful to begin the sum with a negative sign, then add the next term and finally subtract the last term. This happens because the use of rule 2 and 1 lead to a change of sign.

One can determine the sign of the first term by counting the number of rows (or columns) from the first row (or column) to the row (or column) one wishes to expand the determinant along. If the number is even, the sign is positive. If it is odd, the sign is negative.