Solving 2×2-systems with complex-conjugate eigenvalues

Martin Wanvik

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1 Complex numbers

We begin by recalling the formula for finding the roots of a degree 2 polynomial: the equation $\lambda^2 + a\lambda + b = 0$ has solutions

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

This yields a real number if and only if $a^2 - 4b \geq 0$; for instance, there does not exist a real number λ such that $\lambda^2 + 1 = 0$, because $\lambda^2 \geq 0$ for all real numbers λ. But this does not exclude the possibility that there could be other number systems where these kinds of polynomials have roots. Let us write $i = \sqrt{-1}$ (which is not a real number) - this is called the *imaginary unit*, and a number of the form ai is called an *imaginary number*. If $a^2 - 4b < 0$, we may write

$$\begin{aligned} \lambda &= \frac{-a \pm \sqrt{a^2 - 4b}}{2} = \frac{-a}{2} \pm \sqrt{-1}\frac{\sqrt{4b - a^2}}{2} \\ &= \frac{-a}{2} \pm i\frac{\sqrt{4b - a^2}}{2} \end{aligned}$$

We have used that $\sqrt{-a} = \sqrt{(-1) \cdot a} = \sqrt{-1}\sqrt{a}$. The number $\sqrt{4b - a^2}$ is real since $4b - a^2 > 0$, and hence we are dealing with numbers of the form $z = a + ib$, where a, b are real numbers. These are called *complex numbers*, because it has two parts, *the real part* $a =: \mathrm{Re}(z)$ and *the imaginary part* $b =: \mathrm{Im}\, z$. Two complex numbers $a + bi$ and $c + di$ are added in the same way as the vectors $\begin{bmatrix} a & b \end{bmatrix}'$ and $\begin{bmatrix} c & d \end{bmatrix}'$. An example is

$$(2 + 3i) + (1 + i) = (2 + 1) + (3 + 1)i = 3 + 4i$$

Multiplication is a bit (but not much) harder - we calculate the product $(a + bi)(c + di)$ in the same fashion as we would an ordinary expression of the form $(a + b)(c + d)$, while also remembering that $i^2 = -1$. The following calculation is an example:

$$\begin{aligned} (1 + i)(2 - 3i) &= 1 \cdot 2 + i \cdot 2 + 1 \cdot (-3i) - 3 \cdot i^2 = 2 + 2i - 3i - 3 \cdot (-1) \\ &= (2 - (-3)) + (2 - 3)i = 5 - i \end{aligned}$$

The complex numbers $\mathbb{C}$ includes/contains the real numbers $\mathbb{R}$, i.e. all numbers of the form $a + 0i$ (multiplication and addition also coincides with what we're used to).

In the above, we simply assumed that we could take the square root of -1 without any further commentary, and one might rightfully wonder if this really makes sense. For those who aren't entirely convinced we will briefly mention how one can actually *construct* the complex numbers. We start with the plane $\mathbb{R}^2$, with addition defined by the usual vector addition and multiplication defined by

$$\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac - bd \\ ad + bc \end{bmatrix}$$

Note that

$$\begin{bmatrix} a \\ 0 \end{bmatrix} \cdot \begin{bmatrix} c \\ 0 \end{bmatrix} = \begin{bmatrix} ac \\ 0 \end{bmatrix}$$

so multiplication of two real numbers stays the same. If we write $i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we see that

$$i^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

In other words, $i = \begin{bmatrix} 0 & 1 \end{bmatrix}' = \sqrt{-1}$.

Multiplication and addition of complex numbers obeys all the same laws that we use for multiplying and adding real numbers (this includes the commutative law, $z_1 z_2 = z_2 z_1$) We can also divide two complex numbers, provided the denominator is different from 0 (i.e. $0 + 0i$ to be more precise), but we won't need this operation here.

The complex numbers no longer form a line, as the real numbers do, but are located in a *plane*, called *the complex plane* $\mathbb{C}$.

Before we finish this section, we need to define one final operation, which does not have a real analogue, namely complex conjugation. If $z = a + bi$, we define the complex conjugate $\overline{z}$ of z by $\overline{z} = a - ib$. For example $\overline{1 + i} = 1 - i$ and $\overline{2 - 3i} = 2 + 3i$. The real part stays the same, while the imaginary part changes sign. This corresponds to reflection about the x-axis in the plane. Numbers that are located along the x-axis (i.e. the real numbers) stays put under this operation, since the conjugate has the same real part as the original number.

It is hardly surprising that $\overline{\overline{z}} = z$. The roots of a degree 2 polynomial always come in pairs, where one root is the complex conjugate of the other. For instance, the polynomial $\lambda^2 - 4\lambda + 5$ has the roots

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm 2i$$

We also mention that if z is a complex number, then $z + \overline{z}$ is always real (if $z = a + ib$, we get that $z + \overline{z} = (a + bi) + (a - bi) = 2a$), and also that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.

2 Solving linear systems of differential equations with complex eigenvalues

Up to now, we have studied systems of differential equations of the form

$$\begin{bmatrix} dx_1/dt \\ dx_2/dt \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

or

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t) \tag{1}$$

and seen that if A has distinct, real eigenvalues, then the general solution can be written

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{u}_1 + c_2 e^{\lambda_2 t}\mathbf{u}_2 \tag{2}$$

where λ_1, λ_2 are the eigenvalues of A and $\mathbf{u}_1, \mathbf{u}_2$ are associated eigenvectors. We will now deal with the case when A has complex conjugate eigenvalues, and write down the general solution. It turns out that the solution can be written in exactly the same way, but then the solutions aren't necessarily real. Solving this problem will make up most of the remaining part of this note.

But let us begin by considering a more pressing question: if the eigenvalues λ_1, λ_2 are complex, what is $e^{\lambda t}$ supposed to mean? If $\lambda = a + bi$, we should have

$$e^{\lambda t} = e^{(a+ib)t} = e^{at}e^{ibt}$$

as this is what we would expect from an exponential function. Thus, it is enough for us to determine what e^{ib} is supposed to be, for a real number b. It turns out that

$$e^{ib} = \cos b + i \sin b \tag{3}$$

is the correct answer, but we won't even try to justify this (though we can mention that there are *very* good reasons for why the formula is the way it is, and also that it is fairly easy to check that the results we obtain are correct). Equation (3) is called Euler's formula.

The next problem is to find eigenvectors. The calculations are carried out in the same fashion as earlier, except that we are now working with complex numbers. It is probably a good time for an example, to illustrate the method:

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

We have $\operatorname{tr} A = 4$ and $\det A = 5$, so we get

$$\det(A - \lambda I) = \lambda^2 - 4\lambda + 5$$

The roots of this polynomial (the eigenvalues) are

$$\lambda = \frac{4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = 2 \pm i$$

We let $\lambda = 2 + i$ and solve the system $(A - \lambda I)\mathbf{w} = 0$, i.e.

$$A - \lambda I = \begin{bmatrix} 2-(2+i) & 1 \\ -1 & 2-(2+i) \end{bmatrix} = \begin{bmatrix} -i & -1 \\ -1 & -i \end{bmatrix}$$

We can solve the system by multiplying the second row by i, giving

$$\begin{bmatrix} -i & 1 \\ -i & -i^2 = -(-1) = 1 \end{bmatrix} \sim \begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}$$

If $\mathbf{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}'$, we must have $iw_1 - w_2 = 0$ or $iw_1 = w_2$, and we may choose $w_1 = 1$, giving $w_2 = i$. Hence

$$\mathbf{w} = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector associated to $\lambda = 2 + i$. It turns out that we don't need to compute an eigenvector associated to the other eigenvalue, $\overline{\lambda} = 2 - i$, separately - if $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ for *real vectors* $\mathbf{u}$ and $\mathbf{v}$ is an eigenvector associated to λ, then $\overline{\mathbf{w}} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated to the eigenvalue $\overline{\lambda}$. Hence

$$\overline{\mathbf{w}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

is an eigenvector associated to the eigenvalue $\lambda = 2 - i$. This is not difficult to see. With $\lambda = 2 - i$ we get that

$$A - \lambda I = \begin{bmatrix} 2 - (2 - i) & 1 \\ -1 & 2 - (2 - i) \end{bmatrix} = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

Multiplication by $-i$ in the second row yields

$$\begin{bmatrix} i & 1 \\ i & -i^2 = 1 \end{bmatrix} \sim \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}$$

hence the components $\mathbf{w} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}'$ must satisfy $iw_1 = -w_2$ or $-iw_1 = w_2$. If we choose $w_1 = 1$ we get $w_2 = -i$, and hence

$$\mathbf{w} = \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is an eigenvector associated to $\overline{\lambda}$, which is what we wanted to show.

Hence it is sufficient to find an eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$ associated to one of the eigenvalues λ, in which case $\overline{\mathbf{w}} = \mathbf{u} - i\mathbf{v}$ is an eigenvector associated to $\overline{\lambda}$.

The solution (2) can now be written

$$\mathbf{x}(t) = c_1 e^{\lambda t} \mathbf{w} + c_2 e^{\overline{\lambda} t} \overline{\mathbf{w}} \tag{4}$$

where c_1, c_2 are arbitrary complex constants. In general $\mathbf{x}(t)$ is a complex vector, which is not what we want. It turns out that $\mathbf{x}(t)$ is a real vector for all t if and only if $c_1 = \overline{c_2}$ (recall that $z + \overline{z}$ is always real, and that $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$), so the answer becomes

$$\mathbf{x}(t) = 2\,\mathrm{Re}(c_1 e^{\lambda t} \mathbf{w}\ ) = \mathrm{Re}(C e^{\lambda t} \mathbf{w}) \tag{5}$$

where $C = 2c_1$ is an arbitrary complex constant. If we write $C = C_1 - iC_2$ (the negative sign is not strictly necessary and only there to make things convenient for us), equation (5) becomes

$$\begin{aligned} \mathbf{x}(t) &= \mathrm{Re}((C_1 - iC_2) e^{\lambda t} \mathbf{w}) = \mathrm{Re}(C_1 e^{\lambda t}\ \mathbf{w}) + \mathrm{Re}(-iC_2 e^{\lambda t} \mathbf{w}) \\ &= C_1\,\mathrm{Re}(e^{\lambda t} \mathbf{w}) + C_2\,\mathrm{Re}(-i e^{\lambda t} \mathbf{w}) \end{aligned}$$

Note that if $z = a + ib$, we have

$$\operatorname{Re}(-iz) = \operatorname{Re}(-i(a+ib)) = \operatorname{Re}(-(ia-b)) = b = \operatorname{Im} z$$

so we can write

$$\mathbf{x}(t) = C_1 \operatorname{Re}(e^{\lambda t}\mathbf{w}) + C_2 \operatorname{Im}(e^{\lambda t}\mathbf{w}) \tag{6}$$

where C_1, C_2 are *real constants*. It is thus sufficient for us to calculate the real part and the imaginary part of $e^{\lambda t}\mathbf{w}$, which isn't too difficult; we write $\lambda = a + ib$ and $\mathbf{w} = \mathbf{u} + i\mathbf{v}$, which gives

$$\begin{aligned} e^{\lambda t}\mathbf{w} &= e^{at}(\cos bt + i \sin bt)(\mathbf{u} + i\mathbf{v}) \\ &= e^{at}(\mathbf{u}\cos bt + i\mathbf{u}\sin bt + i\mathbf{v}\cos bt - \mathbf{v}\sin bt) \\ &= e^{at}\ (\mathbf{u}\cos bt - \mathbf{v}\sin bt) + ie^{at}(\mathbf{u}\sin bt + \mathbf{v}\cos bt) \end{aligned}$$

Hence

$$\operatorname{Re} e^{\lambda t}\mathbf{w} = e^{at}(\mathbf{u}\cos bt - \mathbf{v}\sin bt)$$

and

$$\operatorname{Im} e^{\lambda t}\mathbf{w} = e^{at}(\mathbf{u}\sin bt + \mathbf{v}\cos bt)$$

We state what we've found so far in the following theorem:

Theorem 1. *If A is a 2×2-matrix with complex-conjugate eigenvalues $\lambda = a \pm bi$, with associated eigenvectors $\mathbf{w} = \mathbf{u} \pm i\mathbf{v}$, then any solution to the system*

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}(t)$$

can be written

$$\mathbf{x}(t) = C_1 e^{at}(\mathbf{u}\cos bt - \mathbf{v}\sin bt) + C_2 e^{at}(\mathbf{u}\sin bt + \mathbf{v}\cos bt) \tag{7}$$

where C_1, C_2 are (real) constants.

As an example, let us complete our earlier calculation; we had

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$$

with eigenvalues $\lambda = 2 \pm i$ (i.e. $a = 2$ and $b = 1$) and associated eigenvectors

$$\mathbf{w} = \mathbf{u} \pm i\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \pm i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We mention in passing that since $a > 0$, the origin is an unstable spiral in this example. According to (7), the solution can be written

$$\begin{aligned} \mathbf{x}(t) &= C_1 e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t\right) + C_2 e^{2t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\ \cos t\right) \\ &= e^{2t}\begin{bmatrix} C_1\cos t + C_2 \sin t \\ -C_1 \sin t + C_2 \cos t \end{bmatrix} \end{aligned}$$

We have

$$\mathbf{x}(0) = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

so the values of C_1 og C_2 are precisely the coordinates for the point which the solution passes through at $t = 0$. This guarantees that we have found a solution passing through any point in the plane.