This is a direct translation of a similar note in Norwegian

1 Linear first-order differential equations

We want to solve the equation

$$x\frac{dy}{dx} - y = x$$

where $x > 0$. It can be rewritten as

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right) = 1,$$

and it is an example of a *linear first-order differential equation*. In general, these are equations that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

In our example, we have $P(x) = -1/x$ and $Q(x) = 1$. In order to solve such equations, we multiply both sides by a special function called an *integrating factor*. The purpose is to transform the equation so that the left hand side is the derivative of a suitable product. Let $A(x)$ be an antiderivative of $P(x)$. It turns out that the function $e^{A(x)}$ works as an integrating factor. If we multiply the equation by that function, we get

$$e^{A(x)}\frac{dy}{dx} + e^{A(x)}P(x)y = e^{A(x)}Q(x)$$

It follows from the product rule and the chain rule for differentiation, that the left hand side is in fact

$$\frac{d}{dx}\left(e^{A(x)}y\right).$$

(Exercise: verify this!). The differential equation is now

$$\frac{d}{dx}\left(e^{A(x)}y\right) = e^{A(x)}Q(x)$$

or, by antidifferentiation both sides and dividing by $e^{A(x)}$,

$$y(x) = e^{-A(x)}\int e^{A(x)}Q(x)\,dx$$

In our example above,

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right) = 1,$$

we have

$$A(x) = \int P(x)\,dx = -\int \frac{dx}{x} = -\ln x + C = \ln(1/x) + C$$

(There is no need for an absolute value, since we've assumed that $x > 0$). We choose $C = 0$, and hence the integrating factor is

$$e^{A(x)} = e^{\ln(1/x)} = \frac{1}{x}$$

Multiplying the original equation by this function, we get

$$\frac{1}{x}\frac{dy}{dx} + \left(-\frac{1}{x^2}\right) y = \frac{1}{x},$$

and, as mentioned, the left hand side is the derivative of the product y/x, i.e.

$$\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{1}{x}.$$

Integrating both sides gives

$$\frac{1}{x}y = \ln x + C$$

and hence the (general) solution is

$$y = x(\ln x + C)$$

Example 1. Solve the initial value problem

$$\frac{dx}{dt} + x = e^{2t},$$

where $x = 1$ when $t = 1$.

This equation is of the form

$$\frac{dx}{dt} + P(t)x = Q(t)$$

where $P(t) = 1$. To find an integrating factor for this equation, we first find an antiderivative of $P(t)$, and it is easy to see that t will do the job. Hence, we multiply the entire equation by e^t, giving

$$e^t\frac{dx}{dt} + e^t x = e^{3t}$$

or

$$\frac{d}{dt}(e^t x) = e^{3t}.$$

Integrating both sides we get $e^t x = (1/3)e^{3t} + C$ or $x = (1/3)e^{2t} + Ce^{-t}$. If $t = 0$, we see that $x = 1/3 + C$ and hence $C = 2/3$ if $x(0) = 1$. The solution is

$$x(t) = \frac{1}{3}\left(e^{2t} + 2e^{-t}\right)$$

Example 2. Solve the differential equation

$$\frac{1}{x}\frac{dy}{dx} + 2y = 1.$$

We rewrite the equation to the desired form $\frac{dy}{dx} + P(x)y = Q(x)$ by multiplying both sides by x, giving

$$\frac{dy}{dx} + 2xy = x.$$

An antiderivative of the function $P(x)$ is x^2, and we get e^{x^2} as an integrating factor. Multiplying both sides by it, we get

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2},$$

or

$$\frac{d}{dx}\left(e^{x^2}y\right) = xe^{x^2}$$

Antidifferentiation gives (we use the substitution rule with $u = x^2$ to evaluate the integral on the right hand side)

$$e^{x^2}y = \int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{x^2} + C$$

and hence

$$y = \frac{1}{2} + Ce^{-x^2}.$$