[Here the universe comprises the real numbers in the open interval I, except possibly a. Also, the quantifiers $\forall \epsilon > 0$ and $\exists \delta > 0$ now contain some restrictive information.] Then, to negate this definition, we do the following (in which certain steps have been combined):

$$\begin{aligned}
&\lim_{x \to a} f(x) \neq L \\
\Longleftrightarrow\ &\neg[\forall \epsilon > 0\ \exists \delta > 0\ \forall x\ [(0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon)]] \\
\Longleftrightarrow\ &\exists \epsilon > 0\ \forall \delta > 0\ \exists x\ \neg[(0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon)] \\
\Longleftrightarrow\ &\exists \epsilon > 0\ \forall \delta > 0\ \exists x\ \neg[\neg(0 < |x - a| < \delta) \vee (|f(x) - L| < \epsilon)] \\
\Longleftrightarrow\ &\exists \epsilon > 0\ \forall \delta > 0\ \exists x\ [\neg\neg(0 < |x - a| < \delta) \wedge \neg(|f(x) - L| < \epsilon)] \\
\Longleftrightarrow\ &\exists \epsilon > 0\ \forall \delta > 0\ \exists x\ [(0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)]
\end{aligned}$$

Translating into words, we find that $\lim_{x \to a} f(x) \neq L$ if (and only if) there exists a positive (real) number ϵ such that for every positive (real) number δ, there is an x in I such that $0 < |x - a| < \delta$ (that is, $x \neq a$ and its distance from a is less than δ) but $|f(x) - L| \geq \epsilon$ [that is, the value of $f(x)$ differs from L by at least ϵ].

EXERCISES 2.4

1. Let $p(x), q(x)$ denote the following open statements.

$p(x)$: $x \leq 3$ $q(x)$: $x + 1$ is odd

If the universe consists of all integers, what are the truth values of the following statements?

a) $q(1)$ **b)** $\neg p(3)$ **c)** $p(7) \vee q(7)$
d) $p(3) \wedge q(4)$ **e)** $\neg(p(-4) \vee q(-3))$
f) $\neg p(-4) \wedge \neg q(-3)$

2. Let $p(x), q(x)$ be defined as in Exercise 1. Let $r(x)$ be the open statement "$x > 0$." Once again the universe comprises all integers.

a) Determine the truth values of the following statements.

i) $p(3) \vee [q(3) \vee \neg r(3)]$
ii) $p(2) \to [q(2) \to r(2)]$
iii) $[p(2) \wedge q(2)] \to r(2)$
iv) $p(0) \to [\neg q(-1) \leftrightarrow r(1)]$

b) Determine all values of x for which $[p(x) \wedge q(x)] \wedge r(x)$ results in a true statement.

3. Let $p(x)$ be the open statement "$x^2 = 2x$," where the universe comprises all integers. Determine whether each of the following statements is true or false.

a) $p(0)$ **b)** $p(1)$ **c)** $p(2)$
d) $p(-2)$ **e)** $\exists x\ p(x)$ **f)** $\forall x\ p(x)$

4. Consider the universe of all polygons with three or four sides, and define the following open statements for this universe.

$a(x)$: all interior angles of x are equal
$e(x)$: x is an equilateral triangle
$h(x)$: all sides of x are equal
$i(x)$: x is an isosceles triangle
$p(x)$: x has an interior angle that exceeds 180°
$q(x)$: x is a quadrilateral
$r(x)$: x is a rectangle
$s(x)$: x is a square
$t(x)$: x is a triangle

Translate each of the following statements into an English sentence, and determine whether the statement is true or false.

a) $\forall x\ [q(x) \veebar t(x)]$ **b)** $\forall x\ [i(x) \to e(x)]$
c) $\exists x\ [t(x) \wedge p(x)]$ **d)** $\forall x\ [(a(x) \wedge t(x)) \leftrightarrow e(x)]$
e) $\exists x\ [q(x) \wedge \neg r(x)]$ **f)** $\exists x\ [r(x) \wedge \neg s(x)]$
g) $\forall x\ [h(x) \to e(x)]$ **h)** $\forall x\ [t(x) \to \neg p(x)]$
i) $\forall x\ [s(x) \leftrightarrow (a(x) \wedge h(x))]$
j) $\forall x\ [t(x) \to (a(x) \leftrightarrow h(x))]$

5. Professor Carlson's class in mechanics is comprised of 29 students of which exactly

1) three physics majors are juniors;
2) two electrical engineering majors are juniors;
3) four mathematics majors are juniors;
4) twelve physics majors are seniors;
5) four electrical engineering majors are seniors;
6) two electrical engineering majors are graduate students; and
7) two mathematics majors are graduate students.

Consider the following open statements.

$c(x)$: Student x is in the class (that is, Professor Carlson's mechanics class as already described).