MA 1101

LØSNING TIL ØVING 14, H 2011

6.1 Oppg. # 15, s. 336

$$\int_{\frac{1}{2}}^{1} \frac{\arcsin x}{x^2}\, dx =$$

Delvis integrasjon gir:

$$\int \frac{\arcsin x}{x^2} dx = -\frac{1}{x} \arcsin x + \int \frac{1}{x} \frac{dx}{\sqrt{1-x^2}}$$

Substitusjonen: $x = \sin t$ gir $dx = \cos t\, dt$ og $\sqrt{1-x^2} = \cos t$ som videre gir:

$$\int \frac{1}{x} \frac{dx}{\sqrt{1-x^2}} = \int \frac{1}{\sin t} \cdot \frac{\cancel{\cos} t\, dt}{\cancel{\cos} t} = \int \frac{dt}{\sin t}$$

$$= \int \frac{dt}{2\cos\frac{t}{2}\sin\frac{t}{2}} = \frac{1}{2}\int \frac{dt}{\tan\frac{t}{2}\cos^2\frac{t}{2}} = \ln\left(\tan\frac{t}{2}\right) + C$$

$$= \ln\left(\frac{\sin t/2}{\cos t/2}\right) + C = \ln\left(\frac{\sin t}{2\cos^2 t/2}\right) + C$$

$$= \ln\left(\frac{\sin t}{1+\cos t}\right) + C = \ln\left(\frac{x}{1+\sqrt{1-x^2}}\right) + C$$

Altså:

$$\int_{\frac{1}{2}}^{1} \frac{\arcsin x}{x^2} dx = \left(-\frac{1}{x}\arcsin x\right)\Bigg|_{\frac{1}{2}}^{1} + \ln\left(\frac{x}{1+\sqrt{1-x^2}}\right)\Bigg|_{\frac{1}{2}}^{1}$$

$$= -\frac{\pi}{2} + 2\cdot\frac{\pi}{6} + \ln 1 - \ln\left(\frac{1/2}{1+\sqrt{\frac{3}{4}}}\right) = \underline{-\frac{\pi}{6} + \ln(2+\sqrt{3})}$$

Oppg. # 19, s. 337

$$I = \int \cos(\ln x)\, dx = x\cos(\ln x) - \int \cancel{x}(-\sin(\ln x))\cdot\frac{1}{\cancel{x}}\, dx$$

$$= x\cos(\ln x) + \int \underbrace{\sin(\ln x)\, dx}_{} = x\cos(\ln x)$$

$$+ x\sin(\ln x) - \int \cancel{x}\cos(\ln x)\frac{1}{\cancel{x}}\, dx$$

$$2I = x\cos(\ln x) + x\sin(\ln x) + K$$

$$I = \underline{\frac{x}{2}\left(\cos(\ln x) + \sin(\ln x)\right) + C}$$

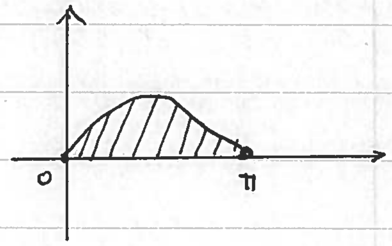
Oppg. # 29, s. 337

Vi skal bestemme arealet begrenset av:

$$y = e^{-x}\sin x \quad ; \quad y = 0 \quad \text{mellom}$$

$x = 0$ og $x = \pi$

$$A = \int_0^{\pi} e^{-x}\sin x\,dx$$



$$I = \int e^{-x}\sin x\,dx$$

$$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

$$= -e^{-x}\cos x - \left(e^{-x}\sin x - \int(-e^{-x})\sin x\,dx\right)$$

$$= -e^{-x}\cos x - e^{-x}\sin x - I \text{, som gir:}$$

$$2I = -e^{-x}(\cos x + \sin x) + K$$

$$\int_0^{\pi} e^{-x}\sin x\,dx = -\tfrac{1}{2}\left(e^{-x}(\cos x + \sin x)\right)\Big|_0^{\pi}$$

$$= -\tfrac{1}{2}\left(e^{-\pi}\cdot(-1) - (+1)\right) = \underline{\tfrac{1}{2}\left(1 + e^{-\pi}\right)}$$

6.2 Oppg. # 7, s. 345

$$\int \frac{dx}{a^2 - x^2}$$

Vi har da:

$$\frac{1}{a^2 - x^2} = \frac{A}{a-x} + \frac{B}{a+x}$$

$$= \frac{Aa + Ax + Ba - Bx}{a^2 - x^2} \quad \text{som gir:}$$

$A - B = 0$; 1. grads-ledd mangler.

$a(A + B) = 1$; konstantledd $= 1$.

som gir: $A = B = \frac{1}{2a}$.

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\int \frac{dx}{a-x} + \frac{1}{2a}\int \frac{dx}{a+x} = -\frac{1}{2a}\ln|a-x|$$

$$+\frac{1}{2a}\ln|a+x| + C = \underline{\frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C}$$

Oppg. #15, s. 345

$$\int \frac{x^2+1}{6x-9x^2}\,dx =$$

Vi har her:

$$\frac{x^2+1}{-9x^2+6x} = \frac{x^2-\frac{2}{3}x+\frac{2}{3}x+1}{-9x^2+6x} = -\frac{1}{9}\cdot\frac{x^2-\frac{2}{3}x}{x^2-\frac{2}{3}x}$$

$$+\frac{\frac{2}{3}x+1}{-3x(3x+2)} = -\frac{1}{9}+\frac{A}{x}+\frac{B}{-9x+6}$$

Dette gir: $\dfrac{-9Ax+6A+Bx}{-9x^2+6x} = \dfrac{\frac{2}{3}x+1}{-9x^2+6x}$

og dermed: $\left.\begin{array}{ll}-9A+B & =\frac{2}{3}\\ 6A & =1\end{array}\right\}$ $\begin{array}{l}A=\frac{1}{6}\\ B=\frac{2}{3}+\frac{3}{2}=\frac{13}{6}\end{array}$

$$\int\frac{x^2+1}{6x-9x^2}\,dx = -\frac{1}{9}\int dx+\frac{1}{6}\int\frac{dx}{x}+\frac{13}{6}\int\frac{dx}{-9x+6}$$

$$= -\frac{1}{9}x + \frac{1}{6}\ln|x| - \frac{1}{9}\cdot\frac{13}{6}\ln|-9x+6| + C$$

$$= -\frac{1}{9}x + \frac{1}{6}\ln|x| - \frac{13}{54}\ln|9x-6| + C$$

$$= \underline{-\frac{1}{9}x + \frac{1}{6}\ln|x| - \frac{13}{54}\ln|3x-2| + C'}$$

Oppg. #27, s. 345

$$\int\frac{dx}{e^{2x}-4e^x+4}$$ Vi innfører $u=e^x$ og får $du = e^x dx$

$$= \int\frac{e^x\,dx}{e^x(e^x-2)^2} = \int\frac{du}{u(u-2)^2}$$

Delbrøkoppspaltning: $\dfrac{1}{u(u-2)^2} = \dfrac{A}{u}+\dfrac{B}{(u-2)^2}+\dfrac{C}{u-2}$

gir:

$$A(u-2)^2 + Bu + Cu(u-2)$$

$$= Au^2-4Au+4A+Bu+Cu^2-2Cu$$

$= 1$ gir: $\begin{array}{rl} A+\quad C & =0\\ -4A+B-2C & =0\\ 4A & =1\end{array}$

$$A = \frac{1}{4}, \quad B = \frac{1}{2}, \quad C = -\frac{1}{4}$$

$$\int \frac{du}{u(u-2)^2} = \frac{1}{4}\int \frac{du}{u} + \frac{1}{2}\int \frac{du}{(u-2)^2} - \frac{1}{4}\int \frac{du}{u-2}$$

$$= \frac{1}{4}\ln|u| - \frac{1}{2}\frac{1}{u-2} - \frac{1}{4}\ln|u-2| + C$$

$$= \frac{1}{4}\ln(e^x) - \frac{1}{2}\frac{1}{e^x-2} - \frac{1}{4}\ln|e^x-2| + C$$

$$= \underline{\frac{x}{4} - \frac{1}{2(e^x-2)} - \frac{1}{4}\ln|e^x-2| + C}$$

Oppg. # 29, s. 345

Vi skal skrive opp den rasjonale funksjonen

$$\frac{x^5+x^3+1}{(x-1)(x^2-1)(x^3-1)}$$

på delbrøkform – uten å beregne konstantene i tellerne. Vi merker oss først at $x^2-1 \equiv (x-1)(x+1)$ og at $(x^3-1) \equiv (x-1)(x^2+x+1)$ der $x^2+x+1 = x^2+x+\frac{1}{4}-\frac{1}{4}+1 = \left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ som ikke kan faktoriseres videre!

Vi har derfor: Siden graden av teller er < graden i nevner:

$$\frac{x^5+x^3+1}{(x-1)^3(x+1)(x^2+x+1)} \equiv \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}$$

$$+ \frac{D}{x+1} + \frac{Ex+F}{x^2+x+1}$$

6.3 Oppg. # 1, s. 352

$$\int \frac{dx}{\sqrt{1-4x^2}} = \frac{1}{2}\int \frac{2\,dx}{\sqrt{1-4x^2}} \qquad \begin{aligned} u &= 2x \\ du &= 2\,dx \end{aligned}$$

$$= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(2x) + C$$

Oppg. # 3, s. 352

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}$$

Vi substituerer $x = 3\sin u$ og får $dx = 3\cos u\,du$

$$= \int \frac{9\sin^2 u\,\cancel{3}\cos u\,du}{\cancel{3}\sqrt{1-\sin^2 u}} = 9\int \sin^2 u\,du$$

$$= 9\int (1-\cos 2u)/2\,du =$$

$$\frac{9}{2}\int 1\,du - \frac{9}{2}\int \cos 2u\,du$$

$$= \frac{9}{2}u - \frac{9}{4}\sin 2u + C = \frac{9}{2}\arcsin\frac{x}{3}$$

$$- \frac{9}{4}\,2\cdot\frac{x}{3}\sqrt{1-\frac{x^2}{9}} + C = \underline{\frac{9}{2}\arcsin\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C}$$

Oppg. # 12, s. 353

$$\int \frac{dx}{(a^2+x^2)^{3/2}}$$

Vi setter $x = a\sinh t$ og får: $\sqrt{1+\left(\frac{x}{a}\right)^2} = \cosh t$

$dx = a\cosh t\,dt$

$$= \frac{1}{a^3}\int \frac{dx}{\left(1+\left(\frac{x}{a}\right)^2\right)^{3/2}}$$

$$= \frac{1}{a^2}\int \frac{\frac{1}{a}\cosh t\,dt}{\cosh^3 t} = \frac{1}{a^2}\int \frac{dt}{\cosh^2 t} = \frac{1}{a^2}\tanh t + C$$

$$= \frac{1}{a^2}\,\frac{x}{a}\,\frac{1}{\sqrt{1+(x/a)^2}} + C = \underline{\frac{1}{a^2}\,\frac{x}{\sqrt{a^2+x^2}} + C}$$

6.5 # 3, s. 366

$$\int_0^\infty e^{-2x}\,dx = \lim_{T\to\infty}\int_0^T e^{-2x}\,dx = \lim_{T\to\infty}\left(-\frac{1}{2}e^{-2x}\right)\Bigg|_0^T = \lim\left(-\frac{1}{2e^{2T}} + \frac{1}{2}\right)$$

$$= \underline{\frac{1}{2}}$$

Oppg. # 2, s. 366

$$\int_3^{\infty} \frac{1}{(2x-1)^{2/3}}\,dx = \lim_{T\to\infty}\int_3^T \frac{dx}{(2x-1)^{2/3}} = \lim_{T\to\infty}\left(\frac{3}{2}(2x-1)^{1/3}\right)\Big|_3^T$$

$$= \lim_{T\to\infty}\frac{3}{2}\left[(2T-1)^{1/3} - (2\cdot 3-1)^{1/3}\right] = \underline{\infty} \quad \text{Divergens}$$

Oppg. # 14, s. 367

$$\int_0^{\pi/2} \frac{dx}{\cos x} \qquad \int \frac{dx}{\cos x} = \int \frac{dx}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}$$

$$= 2\int \frac{\frac{1}{2}\frac{1}{\cos^2\frac{x}{2}}\,dx}{1-\tan^2\frac{x}{2}} \qquad u = \tan\frac{x}{2}, \quad du = \frac{1}{2}\frac{dx}{\cos^2\frac{x}{2}}$$

$$= 2\int \frac{du}{1-u^2} = \int \frac{du}{1+u} + \int \frac{du}{1-u}$$

fordi:

$$\frac{A}{1+u} + \frac{B}{1-u} = \frac{A - Au + B + Bu}{1-u^2} \quad \text{gir} \quad \left.\begin{array}{l} A+B=2 \\ -A+B=0 \end{array}\right\}$$

$$A = B = 1$$

$$\int \frac{du}{1+u} + \int \frac{du}{1-u} = \ln|1+u| - \ln|1-u| + C$$

$$= \ln\left|\frac{1+u}{1-u}\right| + C$$

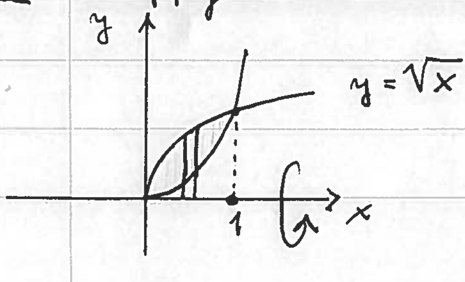
$$= \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$

$$\int_0^{\pi/2} \frac{dx}{\cos x} = \left(\ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|\right)\Big|_0^{\pi/2} = \ln\left|\frac{1+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{4}}\right| = \underline{\infty}$$

Divergens.

Bör studere: $\lim_{s\to\frac{\pi}{2}}\int_0^s \frac{dx}{\cos x}$ siden integranden $\to \infty$ når $x \to \frac{\pi}{2}-$.

7.1 Oppg # 3, s. 398

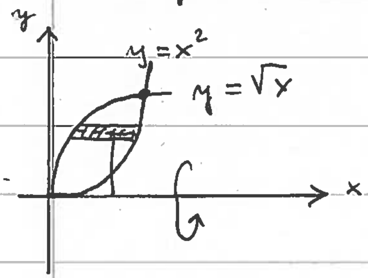


$\sqrt{x} = x^2$ gir $x = 0$ og $x = 1$

$$V = \pi\int_0^1 (x - x^4)\,dx = \pi\left(\frac{x^2}{2} - \frac{x^5}{5}\right)\Big|_0^1$$

$$= \pi\left(\frac{1}{2} - \frac{1}{5}\right) = \underline{\frac{3\pi}{10}}$$

Oppg. # 3, s. 398 (forts.)



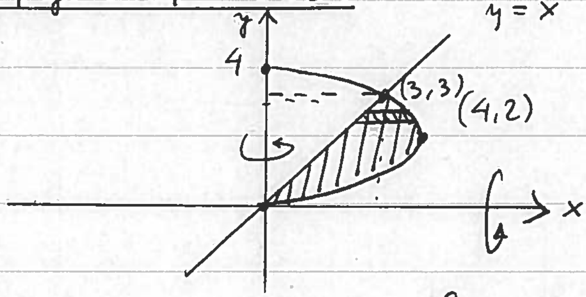
Sylinderskall - metoden:

$$\Delta V = 2\pi y(\sqrt{y} - y^2)\,\Delta y$$

$$V = 2\pi \int_0^1 (y^{3/2} - y^3)\,dy$$

$$= 2\pi \left[\frac{2}{5} y^{\frac{5}{2}} - \frac{y^4}{4}\right]\Bigg|_0^1 = 2\pi\left[\frac{2}{5} - \frac{1}{4}\right] = 2\pi\,\frac{3}{20} = \underline{\frac{3\pi}{10}}$$

Oppg. #7, s. 398:



$x = y \quad ; f_2$

$x = 4y - y^2 \quad ; f_1$

Skjæringspunkter:

(0,0) og (3,3)

Volumet som framkommer når flatestykket roteres om x-aksen beregnes lettest ved sylinderskall-metoden:

$$\Delta V \approx 2\pi y\,(f_1(y) - f_2(y))\,\Delta y$$

som gir:

$$V = 2\pi \int_0^3 y \cdot ((4y - y^2) - y)\,dy$$

$$= 2\pi \int_0^3 (4y^2 - y^3 - y^2)\,dy = 2\pi \int_0^3 (3y^2 - y^3)\,dy$$

$$= 2\pi \left(y^3 - \frac{y^4}{4}\right)\Big|_0^3 = 2\pi\left[27 - \frac{81}{4}\right] = \underline{\frac{\pi}{2} \cdot 27}$$

Volumet ved rotering omkring y-aksen beregnes ved skivemetoden:

$$\Delta V = \pi f_1(y)^2 \Delta y - \pi f_2(y)^2 \Delta y$$

$$V = \pi \int_0^3 ((4y - y^2)^2 - y^2)\,dy$$

$$= \pi \int_0^3 ((16y^2 - 8y^3 + y^4) - y^2)\,dy$$

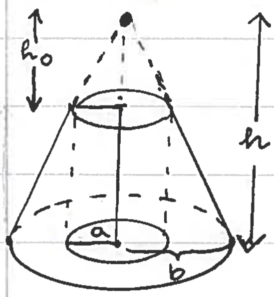
$$= \pi \int_0^3 (15y^2 - 8y^3 + y^4)\,dy$$

$$= \pi\left(5y^3 - 2y^4 + \frac{y^5}{5}\right)\Big|_0^3$$

$$= \pi\left(135 - 162 + \frac{1}{5}243\right) = \pi\left(-27 + \frac{243}{5}\right)$$

$$= \underline{\frac{108\pi}{5}}$$

Oppg #15, s. 398:



Volumet av kjeglen før hullet ble boret:

$$V_1 = \tfrac{1}{3}\pi b^2 \cdot h.$$

Volumet av toppsylinderen

$$V_0 = \tfrac{1}{3}\pi a^2 h_0 .$$

Vi har dessuten: $h_0/h = \frac{a}{b}$ $\therefore$ $h_0 = \frac{a}{b} h$

Altså: $V_0 = \frac{1}{3}\pi a^2 \cdot \frac{a}{b} \cdot h = \frac{1}{3}\pi \frac{a^3}{b} \cdot h$

Volumet av det sylindriske hullet:

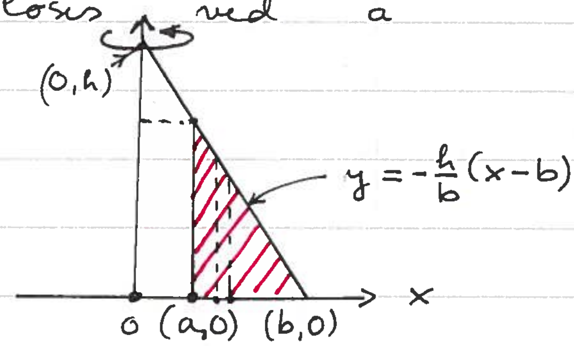
$$V_2 = \pi a^2 (h - h_0) = \pi a^2 (1 - \tfrac{a}{b}) h$$

Det søkte volum blir dermed:

$$V = V_1 - (V_0 + V_2) = \tfrac{1}{3}\pi h \left[b^2 - \tfrac{a^3}{b} - 3a^2 + 3\tfrac{a^3}{b} \right]$$

$$= \underline{\tfrac{1}{3}\pi h \left[b^2 - 3a^2 + 2\tfrac{a^3}{b} \right]}$$

Oppgaven kan også løses ved å rotere det skraverte området om y-aksen.



Skall-metoden gir:

$$\Delta V \approx 2\pi x \cdot f(x)\, \Delta x$$

Altså blir volumet:

$$V = 2\pi \int_a^b x \cdot \left(h - \frac{h}{b} x \right) dx$$

$$= 2\pi h \int_a^b \left(x - \frac{x^2}{b} \right) dx = 2\pi h \left(\frac{x^2}{2} - \frac{x^3}{3b} \right) \Big|_a^b$$

$$= 2\pi h \left(\frac{b^2}{2} - \frac{b^2}{3} - \frac{a^2}{2} + \frac{a^3}{3b} \right)$$

$$= 2\pi h \left(\frac{b^2}{6} - \frac{a^2}{2} + \frac{a^3}{3b} \right) = \underline{\frac{1}{3}\pi h \left(b^2 - 3a^2 + \frac{2a^3}{b} \right)}$$

7.3 Oppg. #1, s. 409:

Vi skal bestemme buelengden av kurven: $y = 2x - 1$ fra $x=1$ til $x=3$.

$y' = 2$ $\qquad L = \int_1^3 \sqrt{1 + y'^2}\,dx = \int_1^3 \sqrt{1+4}\,dx = \underline{2\sqrt{5}}$

Oppg. #13, s. 409:

Vi skal bestemme buelengden av kurven $y = x^2$ fra $x = 0$ til $x = 2$.

$y' = 2x$. Vi har derfor:

$$L = \int_0^2 \sqrt{1 + 4x^2}\,dx$$

Vi substituerer:

$$2x = \sinh t$$

og får $dx = \frac{1}{2}\cosh t \cdot dt$ og $\sqrt{1+4x^2} = \cosh t$

Dette gir: (s. 200, Adams/Essex)

$$\int \sqrt{1+4x^2}\,dx = \frac{1}{2}\int \cosh^2 t\,dt \overset{\downarrow}{=} \frac{1}{4}\int (1 + \cosh 2t)\,dt$$

$$= \frac{1}{4}t + \frac{1}{8}\sinh 2t + C = \frac{t}{4} + \frac{1}{4}\sinh t \cosh t + C$$

$$= \frac{1}{4}\ln(2x + \sqrt{1+4x^2}) + \frac{1}{4}\cdot 2x\sqrt{1+4x^2} + C$$

$$\int_0^2 \sqrt{1+4x^2}\,dx = \frac{1}{4}\ln(4+\sqrt{17}) - \frac{1}{4}\ln 1$$

$$+ \frac{1}{2}\cdot 2\sqrt{17} - 0 = \underline{\frac{1}{4}\ln(4+\sqrt{17}) + \sqrt{17}}$$

Hvordan bestemmes t ovenfor enklest mulig? Vi har:

$$\frac{e^t - e^{-t}}{2} = \sinh t = 2x$$

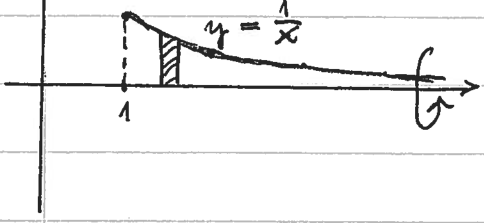
$$\frac{e^t + e^{-t}}{2} = \cosh t = \sqrt{1+\sinh^2 t} = \sqrt{1+4x^2}$$

Addisjon gir:

$$e^t = 2x + \sqrt{1+4x^2}$$

som gir: $t = \ln(2x + \sqrt{1+4x^2})$

Oppg. #37, s. 410



Kurven $y = \frac{1}{x}$; $1 \le x \le T$ roteres om x-aksen.

(a) Vi skal beregne volum som framkommer når $T \to \infty$: $\Delta V_T \approx \pi \frac{1}{x^2} \Delta x$ som gir

$$V_T = \pi \int_1^T \frac{dx}{x^2} = \pi\left(-(x)^{-1}\right)\Big|_1^T = \pi\left(1 - \frac{1}{T}\right)$$

$$V = \lim_{T\to\infty} \pi \int_1^T \frac{dx}{x^2} = \pi \int_1^\infty \frac{dx}{x^2} = \underline{\pi}$$

(b) Vi skal så beregne overflate-arealet av samme rotasjonslegeme.

$$\Delta A \approx 2\pi f(x)\sqrt{1+f'(x)^2}\,\Delta x$$

$f'(x) = -\frac{1}{x^2}$. Arealet blir da:

$$A_T = 2\pi \int_1^T \frac{1}{x}\sqrt{1 + \frac{1}{x^4}}\,dx$$

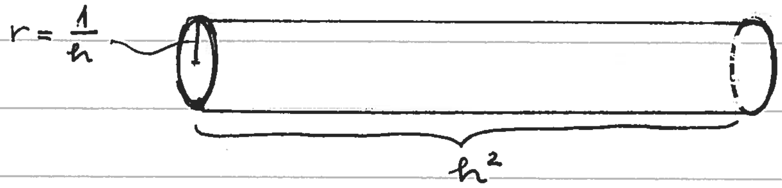
Vi har $\int_1^T \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > \int_1^T \frac{dx}{x} = \ln T$

Altså vil:

$$\lim_{T\to\infty} A_T = \lim_{T\to\infty} 2\pi \ln T = \underline{\infty}.$$

(c) Ovenstående område betegnes som Gabriel's horn eller Torricelli's trompet. Evangelista Torricelli var en italiensk matematiker som først oppdaget ovenstående paradoks: Man trenger en endelig mengde maling for å fylle hornet – men en uendelig mengde maling for å male det. En forklaring på denne sammenheng er jo at denne overflaten ikke har noen tykkelse, slik at beregning av den nödvendige malingsmengden som skal til blir et $\infty \cdot 0$ -uttrykk, som vi vet er "ubestemt".

Et enklere eksempel av samme natur:



Volumet av sylinderen blir:

$$V = \pi \cdot \frac{1}{h^2} \cdot h^2 = \pi$$

Overflaten blir:

$$A = 2\pi \cdot \frac{1}{h} \cdot h^2 = 2\pi h$$

Tenker vi oss sylinderen som en fast mengde modelleringskitt og strekker i lengderetningen vil volumet være fast lik π, mens arealet av overflaten går mot ∞. (Endeflatene kommer i tillegg til $2\pi h$!)

7.9 <u>Oppg. # 1, s. 452:</u>

Vi skal löse den separable differensialligningen: $\frac{dy}{dx} = \frac{y}{2x}$. Vi får da:

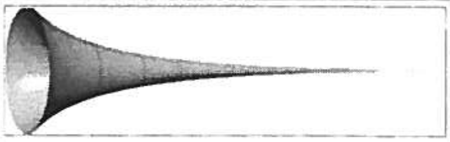
$$2\frac{dy}{y} = \frac{dx}{x} \quad \text{eller} \quad 2\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$2 \ln|y| = \ln|x| + \ln C \quad \text{som gir:}$$

$$y^2 = Cx \quad \text{eller} \quad \underline{y = Kx^{1/2}}$$

Gabriel's Horn

From Wikipedia, the free encyclopedia



3D illustration of Gabriel's Horn.

Gabriel's Horn (also called **Torricelli's trumpet**) is a geometric figure which has infinite surface area but encloses a finite volume. The name refers to the tradition identifying the Archangel Gabriel as the angel who blows the horn to announce Judgment Day, associating the divine, or infinite, with the finite. The properties of this figure were first studied by Italian physicist and mathematician Evangelista Torricelli.

Contents

Mathematical definition

Gabriel's horn is formed by taking the graph of $y = \frac{1}{x}$, with the domain $x \geq 1$ (thus avoiding the asymptote at $x = 0$) and rotating it in three dimensions about the x-axis. The discovery was made using Cavalieri's principle before the invention of calculus, but today calculus can be used to calculate the volume and surface area of the horn between $x = 1$ and $x = a$, where $a > 1$. Using integration (see Solid of revolution and Surface of revolution for details), it is possible to find the volume V and the surface area A:

$$V = \pi \int_1^a \frac{1}{x^2} \mathrm{d}x = \pi \left(1 - \frac{1}{a}\right)$$

$$A = 2\pi \int_1^a \frac{\sqrt{1 + \frac{1}{x^4}}}{x} \mathrm{d}x > 2\pi \int_1^a \frac{\sqrt{1}}{x} \,\mathrm{d}x = 2\pi \ln a.$$

a can be as large as required, but it can be seen from the equation that the volume of the part of the horn between $x = 1$ and $x = a$ will never exceed π; however, it *will* get closer and closer to π as a becomes larger. Mathematically, the volume *approaches π as a approaches infinity*. Using the limit notation of calculus, the volume may be expressed as:

$$\lim_{a \to \infty} \pi \left(1 - \frac{1}{a}\right) = \pi.$$

This is so because as a approaches infinity, $1 / a$ approaches zero. This means the volume approaches $\pi(1 - 0)$ which equals π.

Oppg. #9, s. 452:

$$\frac{dy}{dt} = 2 + e^y$$

eller: $\frac{dy}{2+e^y} = dt$, $\int \frac{dy}{2+e^y} = \int dt$

Vi substituerer: $u = 2 + e^y$ og får
$du = e^y dy$. eller $dy = e^{-y} du = \frac{du}{u-2}$

Dette gir:

$$\int \frac{dy}{2+e^y} = \int \frac{du}{u(u-2)} = \int \frac{A}{u} du + \int \frac{B}{u-2} du$$

$$\frac{1}{u(u-2)} = \frac{A}{u} + \frac{B}{u-2} = \frac{Au - 2A + Bu}{u(u-2)}$$

$$= \frac{(A+B)u - 2A}{u(u-2)} \quad \text{gir} \quad \begin{cases} A + B = 0 & B = \frac{1}{2} \\ -2A = 1 & A = -\frac{1}{2} \end{cases}$$

$$\int \frac{A}{u} du + \int \frac{B}{u-2} du = -\frac{1}{2} \ln|u| + \frac{1}{2} \ln|u-2| + \ln|C|$$

$$= \frac{1}{2} \ln \left| C' \frac{u-2}{u} \right| = \underline{\frac{1}{2} \ln \frac{C' e^y}{2+e^y}} = \int dt = \underline{t + K}$$

$$\ln \frac{e^y}{2+e^y} = 2t + K' \quad \text{eller:} \quad \frac{e^y}{2+e^y} = e^{K'} e^{2t} = Ce^{2t}$$

$$e^y = Ce^{2t}(2 + e^y) \quad \text{eller} \quad e^y(1 - Ce^t) = 2Ce^{2t}$$

$$e^y = \frac{2Ce^{2t}}{1 - Ce^{2t}} = \frac{1}{\frac{1}{2}\frac{1}{C}e^{-2t} - \frac{1}{2}} = \frac{1}{Ke^{-2t} - \frac{1}{2}}$$

$$\underline{y = -\ln\left(Ke^{-2t} - \frac{1}{2}\right)}$$

Oppg. #18, s. 453:

Vi skal löse initialverdi-problemet:

$$y' + 3x^2 y = x^2 \qquad y(0) = 1$$

$p(x) = 3x^2$; $P(x) = \int 3x^2 dx = x^3 + K$

$$e^{P(x)} y' + 3x^2 y e^{P(x)} = x^2 e^{P(x)}$$

eller:

$$e^{x^3}y' + 3x^2 y e^{x^3} = x^2 e^{x^3}$$

$$\frac{d}{dx}(e^{x^3}y) = e^{x^3}x^2$$

$$e^{x^3}y = \int x^2 e^{x^3}dx = \frac{1}{3}\int 3x^2 e^{x^3} dx$$

$$= \frac{1}{3}e^{x^3} + K \quad \therefore \quad y = \frac{1}{3} + Ke^{-x^3}.$$

$y(0) = 1$ gir: $1 = y(0) = \frac{1}{3} + K \cdot 1$

$K = \frac{2}{3}$ $\qquad \underline{y = \frac{1}{3} + \frac{2}{3}e^{-x^3}}$