Løsningsforslag, eksamen høst 2005, MA1103

Oppgave 1.

a) (i) $L = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2-x^3y^3}{x^2+y^2}$

Polarkoordinater: $x = r\cos\theta$
$y = r\sin\theta$

$L = \lim\limits_{r\to 0} \dfrac{r^2 - r^6\cos^3\theta\sin^3\theta}{r^2} = \lim\limits_{r\to 0} 1 - r^4\cos^3\theta\sin^3\theta = \underline{1}$,

(siden $\cos\theta$ og $\sin\theta$ er begrensede funksjoner)

(ii) $L = \lim\limits_{(x,y)\to(0,0)} \dfrac{4x^2y}{x^4+y^2}$

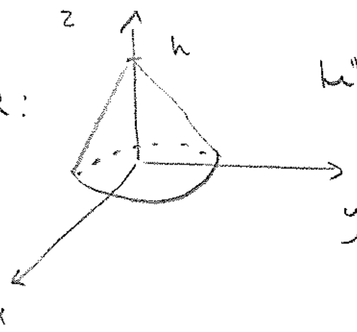
Langs y-aksen $(x=0)$: $\lim\limits_{(0,y)\to(0,0)} \dfrac{4\cdot 0^2 y}{0^2+y^2} = \underline{0}$

Langs linja $y = x^2$: $\lim\limits_{(x,x)\to(0,0)} \dfrac{4x^4}{x^4+x^4} = \lim\limits_{x\to 0} \dfrac{4x^4}{2x^4} = \underline{2}$

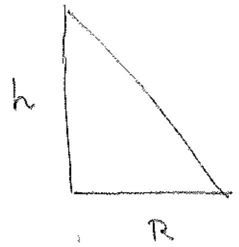
Grenseverdien eksisterer ikke.

b) En spiss kjegle med sirkelformet grunnflate $A = \pi R^2$ og høyde h:

R:



Likning: $z \le h - k(x^2+y^2)^{1/2}$,
hvor $k = \dfrac{h}{R}$



Volum: $V = \iiint\limits_R dV = \iint\limits_D \int_0^{h-\frac{h}{R}(x^2+y^2)^{1/2}} dz\, dA = \iint\limits_D h - \frac{h}{R}(x^2+y^2)^{1/2} dA$

$= \int_0^{2\pi}\int_0^R \left(h - \frac{h}{R}r\right) r\, dr\, d\theta = 2\pi\left[\frac{1}{2}hr^2 - \frac{1}{3}\frac{h}{R}r^3\right]_0^R$

$= 2\pi\left(\frac{1}{2}hR^2 - \frac{1}{3}hR^2\right) = \underline{\frac{1}{3}\pi R^3}$ (q.e.d.)

Oppgave 2

$$f(x,y) = e^{-2x^2-4xy-y^4}$$

a) Eksponensialfunksjonen er en voksende funksjon. Det er derfor tilstrekkelig å finne kritiske punkt for

$$g(x,y) = -2x^2 - 4xy - y^4.$$

$$\frac{\partial g}{\partial x} = -4x - 4y = 0 \quad \Rightarrow \quad \underline{x = -y}$$

$$\frac{\partial g}{\partial y} = -4x - 4y^3 = 0 \quad \Rightarrow \quad \underline{x = -y^3}$$

$$\left.\begin{matrix} x=-y \\ x=-y^3 \end{matrix}\right\} \quad y = y^3, \quad y = 0 \lor y = 1 \lor y = -1$$

Kritiske punkt: $(0,0)$, $(1,-1)$, $(-1,1)$ (for både f og g)

b) Vi klassifiserer de kritiske punkta ved å finne Hessienmatrisa til g.

$$\frac{\partial^2 g}{\partial x^2} = -4, \quad \frac{\partial^2 g}{\partial x \partial y} = -4 = \frac{\partial^2 g}{\partial y \partial x}, \quad \frac{\partial^2 g}{\partial y^2} = -12y^2$$

$(0,0)$: $\det \begin{bmatrix} -4 & -4 \\ -4 & 0 \end{bmatrix} = -16 < 0 \quad \Rightarrow \quad \underline{\text{sadelpunkt}}$

$(1,-1)$: $\det \begin{bmatrix} -4 & -4 \\ -4 & -12 \end{bmatrix} = 4 \cdot 12 - 16 > 0, \ A = -4 < 0 \Rightarrow \underline{\text{toppunkt}}$

$(-1,1)$: $\det \begin{bmatrix} -4 & -4 \\ -4 & -12 \end{bmatrix} = 4 \cdot 12 - 16 > 0, \ A = -4 < 0 \Rightarrow \underline{\text{toppunkt}}$.

c) Maks. og min til $g(x,y) = (x-y)^5$ langs $x^2+y^2=1$

Lagrange: $\begin{cases} \nabla g = \lambda \nabla(x^2+y^2) \\ x^2+y^2 = 1 \end{cases} \Rightarrow \left.\begin{matrix} 1) \ 5(x-y)^4 = \lambda 2x \\ 2) \ -5(x-y)^4 = \lambda 2y \end{matrix}\right\} \ \lambda 2x = -\lambda 2y, \ \underline{x = -y}$

$3) \ x^2 + y^2 = 1$

$x = -y$ innsatt i 3): $x^2 + x^2 = 1 \Rightarrow \underline{x = \pm \tfrac{1}{2}\sqrt{2}}$ $\quad (\Rightarrow y = \mp \tfrac{1}{2}\sqrt{2})$

$g(\tfrac{1}{2}\sqrt{2}, -\tfrac{1}{2}\sqrt{2}) = \sqrt{2}^5 = \underline{4\sqrt{2}}$, maksimalverdi

$g(-\tfrac{1}{2}\sqrt{2}, \tfrac{1}{2}\sqrt{2}) = \underline{-4\sqrt{2}}$, minimalverdi

Oppgave 3

a) Greens teorem:

R: lukket, regulært område i planet,

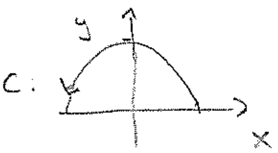
C: randen til R, positivt orientert, stykkevis glatt, lukket, krysser ikke seg selv

$\mathbb{F}$: glatt vektorfelt på R. (plant)

Da gjelder:

$$\oint_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA.$$

b) $$I = \int_C (-y^3+1)\,dx + (2x^3 + e^{y^2})\,dy \quad , \text{ der } C: \begin{array}{l} x = \frac{1}{\sqrt{2}}\cos t \\ y = \sin t \end{array} \; 0 \le t \le \pi$$



Vi ønsker å bruke Greens teorem med R: halv ellipse, begrenset av C og x-aksen. Randen til R er da C og C_1, der

C_1: $-\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}}$

$y = 0$

Med $\mathbb{F} = [-y^3+1, 2x^3+e^{y^2}]$ er alle forutsetninger i orden; så

$$\int_C (-y^3+1)\,dx + (2x^3+e^{y^2})\,dy + \int_{C_1} (-y^3+1)\,dx + (2x^3+e^{y^2})\,dy$$

$$= \iint_R 6x^2 + 3y^2\, dA$$

$$\Rightarrow I = \iint_R 6x^2 + 3y^2\, dA - \int_{C_1} (-y^3+1)\,dx + (2x^3+e^{y^2})\,dy$$

$\left(\begin{array}{l} x = \frac{1}{\sqrt{2}} r\cos\theta \\ y = r\sin\theta \end{array} \right)$

$$= \int_0^{\pi}\int_0^1 3r^2 \cdot \frac{1}{\sqrt{2}}\, r\,dr\,d\theta - \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} dx = \frac{3\pi}{\sqrt{2}}\left[\frac{1}{4} r^4\right]_0^1 - \frac{2}{\sqrt{2}}$$

$$= \underline{\underline{\frac{3\pi}{4\sqrt{2}} - \frac{2}{\sqrt{2}}}}$$

Oppgave 4

$\mathbb{F} = [e^x + xe^x,\ z\cos y,\ \sin y]$

Konservativt dersom det finnes $\varphi(x,y,z)$ slik at

$\nabla\varphi = \mathbb{F}$.

Forsøker å finne φ:

$$\frac{\partial\varphi}{\partial x} = F_1 \Rightarrow \varphi(x,y,z) = \int e^x + xe^x\,dx = e^x + xe^x - e^x + h_1(y,z)$$
$$= \underline{xe^x + h_1(y,z)}$$

$$\frac{\partial\varphi}{\partial y} = F_2 \Rightarrow \varphi(x,y,z) = \int z\cos y\,dy = \underline{z\sin y + h_2(x,z)}$$

$$\frac{\partial\varphi}{\partial z} = F_3 \Rightarrow \varphi(x,y,z) = \int \sin y\,dz = \underline{z\sin y + h_3(x,y)}$$

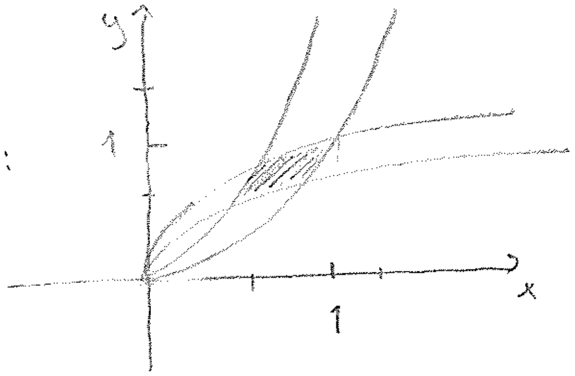
$\Rightarrow \underline{\varphi = xe^x + z\sin y}$ er en potensialfunksjon for $\mathbb{F}$,

så $\mathbb{F}$ er konservativt.

$$W = \int_C \mathbb{F}\cdot\hat{T}\,ds = \varphi(P_2) - \varphi(P_1) = \varphi(1, \tfrac{\pi}{2}, 2) - \varphi(0,0,0) = \underline{\underline{e + 2}}$$

Oppgave 5

a) D: $y = x^2$
$y = 2x^2$
$x = y^2$
$x = 4y^2$

D: 

Variabelskifte: $u = \frac{y}{x^2}$

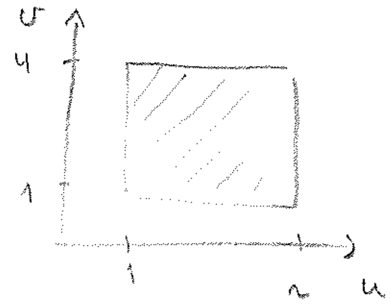
$v = \frac{x}{y^2}$

kurven $y = x^2$ sendes på $u = \frac{x^2}{x^2} = 1$

$y = 2x^2$ $\quad u = \frac{2x^2}{x^2} = 2$

$x = y^2$ $\quad v = \frac{y^2}{y^2} = 1$

$x = 4y^2$ $\quad v = \frac{4y^2}{y^2} = 4$

$\Rightarrow$ 

b) $A = \iint_D dx\,dy = \int_1^4\int_1^2 \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} \qquad \frac{\partial(u,v)}{\partial(x,y)} = \left|\begin{bmatrix} -\frac{2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & -\frac{2x}{y^3} \end{bmatrix}\right| = \frac{4}{x^2y^2} - \frac{1}{x^2y^2}$$

$$= \underline{\frac{3}{x^2y^2}}$$

Observere at $\frac{3}{x^2y^2} = 3\left(\frac{1}{xy}\right)^2 = 3(uv)^2 = 3u^2v^2$

$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3u^2v^2}$

$\Rightarrow A = \int_1^4\int_1^2 \frac{1}{3u^2v^2}\,du\,dv = \frac{1}{3}\left[-\frac{1}{u}\right]_1^2\left[-\frac{1}{v}\right]_1^4 = \frac{1}{3}\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) = \underline{\underline{\frac{1}{8}}}$