EKSAMEN FLERDIMENSJONAL ANALYSE

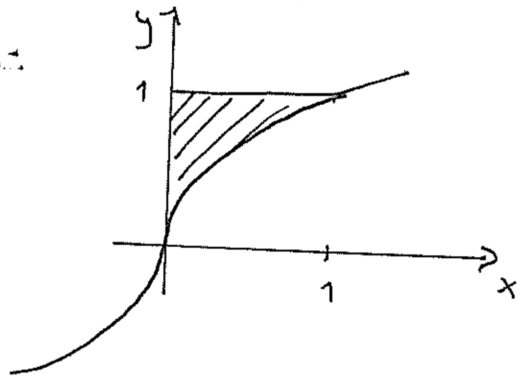
26. mai 2004

LØSNINGSFORSLAG

Oppgave 1

$$\iint_D f(x,y)\,dA = \int_0^1 \int_{x^{1/3}}^1 f(x,y)\,dy\,dx$$

a) D:



b)
$$\iint_D \sqrt{1-y^4}\,dA = \int_0^1 \int_0^{y^3} \sqrt{1-y^4}\,dx\,dy = \int_0^1 y^3\sqrt{1-y^4}\,dy$$

$$= -\frac{1}{4}\int_1^0 u^{1/2}\,du = -\frac{1}{6}\left[u^{3/2}\right]_1^0 = \underline{\underline{\frac{1}{6}}}$$

Oppgave 2

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} + \sqrt{|x|} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

a) Skal undersøke om $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

Gjør om til polarkoordinater:

$$f(r,\theta) = \frac{r^3\cos^2\theta\sin\theta}{r^2} + \sqrt{|r\cos\theta|} = r\cos^2\theta\sin\theta + \sqrt{|r\cos\theta|}$$

$$\lim_{r\to 0} f(r,\theta) = \lim_{r\to 0}\left(r\cos^2\theta\sin\theta + \sqrt{|r\cos\theta|}\right) = 0,$$

siden både $\cos\theta$ og $\sin\theta$ er begrensede funksjoner.

$\Rightarrow$ <u>f er kontinuerlig i $(0,0)$</u>

b) Her må vi bruke definisjonen av de partiellderivert:

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} : \quad \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{|h|}}{h} = \pm\infty,$$

$$\Rightarrow \underline{\underline{\frac{\partial f}{\partial x} \text{ eksisterer ikke i } (0,0)}}$$

$$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} : \quad \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0}{k} - 0 = 0$$

$$\Rightarrow \underline{\underline{\frac{\partial f}{\partial y} \text{ eksisterer i } (0,0), \text{ med verdi } 0.}}$$

Oppgave 3

$$\mathbb{F}(x,y,z) = (yze^x + yz)\hat{i} + (ze^x + xz - 2y)\hat{j} + yh(x)\hat{k}.$$

a) For at $\mathbb{F}$ skal være konservativt, må følgende være oppfylt:

1) $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$

2) $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$

3) $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

1) er allerede oppfylt

2) gir : $ye^x + y = yh'(x)$

$$\Downarrow$$

$$h'(x) = e^x + 1 \quad \Rightarrow \underline{h(x) = e^x + x + K}$$

3) gir : $\underline{e^x + x = h(x)}$

$$\Rightarrow \underline{\underline{h(x) = e^x + x}}$$

Finner en potensialfunksjon:

$$\varphi(x,y,z) = \int F_1\,dx = \int yze^x + yz\,dx = yze^x + yzx + k_1(y,z)$$
$$= \int F_2\,dy = \int ze^x + xz - 2y\,dy = yze^x + yzx - y^2 + k_2(x,z)$$
$$= \int F_3\,dz = \int ye^x + yx\,dz = yze^x + yzx + k_3(y,x)$$

Ser at vi kan velge $k_1(y,z) = k_3(y,x) = -y^2$, $k_2(x,z) = 0$.

Dermed: $\underline{\underline{\varphi(x,y,z) = yze^x + xyz - y^2}}$

b) For et konservativt vektorfelt $\mathbf{G}$ gjelder $\mathbf{G} = \nabla\varphi$, der φ er en skalarfunksjon;): $\mathbf{G} = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$

Dermed: $$\nabla \times \mathbf{G} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

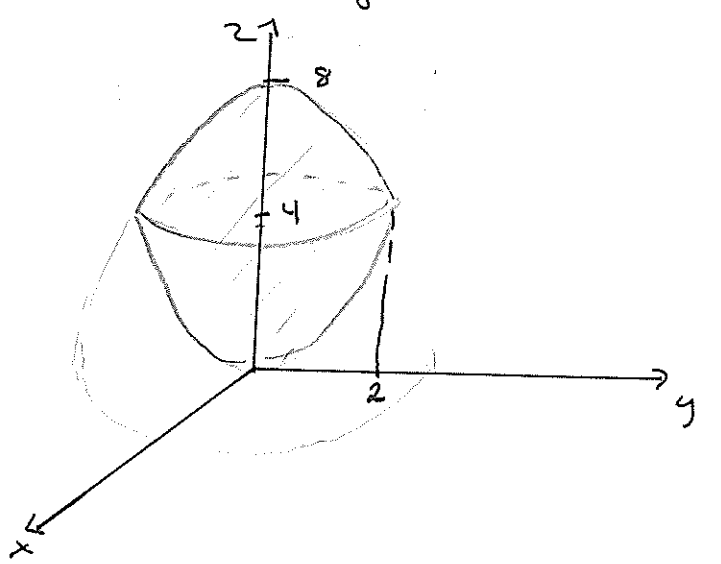
$$= \left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)\hat{i} + \left(\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}\right)\hat{j} + \left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right)\hat{k}$$
$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = \mathbf{0},$$

siden de blanda partiellderivert av φ e like.

Oppgave 4

B: $z \leq 8 - x^2 - y^2$
$z \geq x^2 + y^2$

a)



b) $\iint_S \mathbb{F}\cdot\hat{N}\,dS, \qquad \mathbb{F} = [x, y, z].$

Deler inn i to biter; S_1 der $z \geq 4$ og S_2 der $z \leq 4$.

S_1: $z = 8 - x^2 - y^2$, normalvektor $\hat{N} = \dfrac{[-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1]}{\| \text{—}\, " \,\text{—} \|}$

$\Rightarrow \hat{N}\,dS = [2x, 2y, 1]\,dx\,dy$

$\Rightarrow \iint_{S_1} \mathbb{F}\cdot\hat{N}\,dS = \iint_D 2x^2 + 2y^2 + (8 - x^2 - y^2)\,dA,$

der D er disken i xy-planet med radius 2

S_2: $z = x^2 + y^2$, normalvektor $\hat{N} = \dfrac{[\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1]}{\| \text{—}\, " \,\text{—} \|}$

$\Rightarrow \hat{N}\,dS = [2x, 2y, -1]\,dx\,dy$

$\Rightarrow \iint_{S_2} \mathbb{F}\cdot\hat{N}\,dS = \iint_D 2x^2 + 2y^2 - (x^2 + y^2)\,dA$

$\Rightarrow$ Total fluks: $\iint_S \mathbb{F}\cdot\hat{N}\,dS = \iint_D 2(x^2+y^2) + 8\,dA = \int_0^{2\pi}\int_0^2 (2r^2 + 8)\,r\,dr\,d\theta$

$= 2\pi\left[\frac{1}{2}r^4 + 4r^2\right]_0^2 = 2\pi(8 + 16) = \underline{\underline{48\pi}}$

c) Gauss teorem: $\oint\!\!\oint_S \mathbb{F}\cdot\hat{N}\,dS = \iiint_B \operatorname{div}\mathbb{F}\,dV,$

der S er den lukkede overflaten til B med enhetsnormal ut av B, og $\mathbb{F}$ er et glatt vektorfelt på B.

Her: $\operatorname{div}\mathbb{F} = 1 + 1 + 1 = 3$

$\Rightarrow \oint\!\!\oint_S \mathbb{F}\cdot\hat{N}\,dS = \iiint_B 3\,dV = 3\int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dz\,dr\,d\theta$

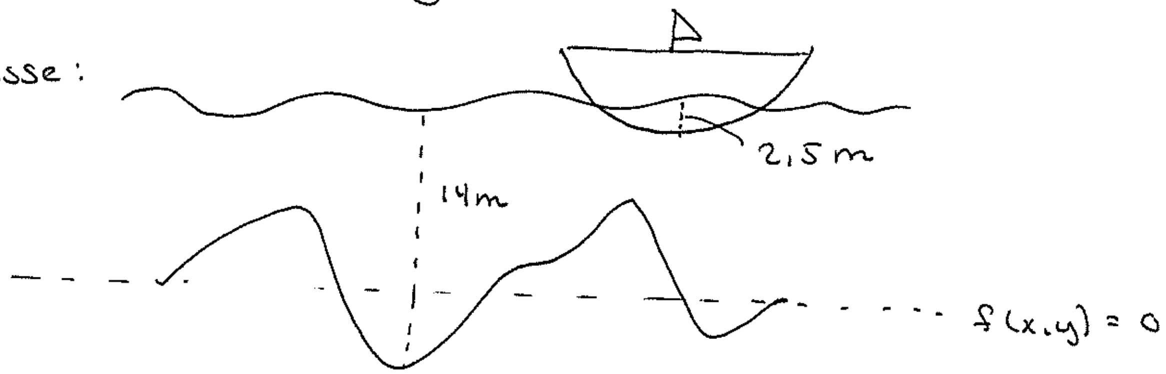
$= 6\pi\int_0^2 (8 - 2r^2)\,r\,dr = 6\pi\left[4r^2 - \frac{1}{2}r^4\right]_0^2 = 6\pi(16 - 8) = \underline{\underline{48\pi}}$

Oppgave 5

$D: \; x^2+y^2 \leq 4$

Havbunn: $f(x,y) = 5e\, xy\, e^{-1/2(x^2+y^2)}$

Skisse:



Må finne topp- og bunnpunkt for $f(x,y)$ på D.

Innenfor $x^2+y^2=4$:

$$\frac{\partial f}{\partial x} = 5e\left(y e^{-1/2(x^2+y^2)} - x^2 y e^{-1/2(x^2+y^2)}\right) = 0$$

$$\Rightarrow \quad y(1-x^2) = 0 \quad \Rightarrow \quad \underline{y=0} \quad \vee \quad \underline{x=\pm 1}$$

$$\frac{\partial f}{\partial y} = 5e\left(x e^{-1/2(x^2+y^2)} - x y^2 e^{-1/2(x^2+y^2)}\right) = 0$$

$$\Rightarrow \quad x(1-y^2) = 0 \quad \Rightarrow \quad \underline{x=0} \quad \vee \quad \underline{y=\pm 1}$$

Gir følgende kritiske punkt inne på disken:

$(0,0)$, $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$.

Finner funksjonsverdiene:

$f(0,0) = 0$
$f(1,1) = f(-1,-1) = 5$
$f(-1,1) = f(1,-1) = -5$

Må også undersøke randen.
Bruker parametriseringen $x = 2\cos\theta$, $y = 2\sin\theta$, $\theta \in [0, 2\pi]$.

$$g(\theta) = f(2\cos\theta, 2\sin\theta) = 5e \cdot 4\cos\theta\sin\theta \cdot e^{-2} = \frac{10}{e}\sin 2\theta$$

$\cancel{g'(\theta)} = \quad g(\theta)_{maks} = \frac{10}{e}, \quad g(\theta)_{min} = -\frac{10}{e}$

(siden $-1 \leq \sin 2\theta \leq 1$)

Siden $\frac{10}{e} < 5$ og $-\frac{10}{e} > -5$ er minimumsverdien

på området -5 mens maksimumsverdien er 5.

Vannstanden er 14 meter der $f(x,y) = -5$.

Det betyr at vannstanden er $14 - 10 = 4$ meter der

$f(x,y) = 5$. Og siden $4 > 2{,}5$ (som er båtens kjøl)

<u>kan den kjøre trygt over hele D.</u>