EKSAMEN MA1103, 4.05, LØSNINGSFORSLAG

Oppgave 1

$$f(x,y) = x^2 + 4y^2 - 2x + 6$$

a) Kritiske punkt: $\nabla f = 0$.

$$\frac{\partial f}{\partial x} = 2x - 2 = 0 \iff \underline{x = 1}$$

$$\frac{\partial f}{\partial y} = 8y = 0 \iff \underline{y = 0}$$

$\underline{\underline{(1,0) \text{ eneste kritiske punkt}}}$

Klassifisering

$$\frac{\partial^2 f}{\partial x^2} = \underline{2}, \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \underline{0}, \quad \frac{\partial^2 f}{\partial y^2} = \underline{8}$$

$$\mathcal{H}(1,0) = \begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}, \quad \det \mathcal{H} = 16 > 0, \quad A = 2 > 0$$

$\underline{\underline{(1,0) \text{ er et bunnpunkt}}}$

b) $E: \left(\frac{x}{2}\right)^2 + y^2 \leq 1$; elliptisk område.

$(x^2 + 4y^2 \leq 4;)$

Sjekker randen: $x^2 + 4y^2 = 4$.

$$f(\text{rand}) = 4 - 2x + 6 = 10 - 2x \qquad -2 \leq x \leq 2$$

$$f(x=2) = 10 - 4 = \underline{6}$$

$$f(x=-2) = 10 + 4 = \underline{14}$$

$$f(1,0) = 1 - 2 + 6 = \underline{5}$$

$\underline{\underline{f_{min} = 5}}$

$\underline{\underline{f_{max} = 14}}$

c) f positiv på hele E

ellipsekoordinater:

$x = 2r\cos\theta$

$y = r\sin\theta$

$dx\,dy = 2r\,dr\,d\theta$

$$\Rightarrow V = \iint_E f\,dA = \int_0^{2\pi}\int_0^1 (4r^2 - 4r\cos\theta + 6)2r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 8r^3 - 8r^2\cos\theta + 12r\,dr\,d\theta$$

$$= \int_0^{2\pi} 2 - \frac{8}{3}\cos\theta + 6\,.\,d\theta = \underline{\underline{16\pi}}$$

d) $\mathbb{F} = [y, x]$ Rand: $x = 2\cos\theta$, $y = \sin\theta$ $\hat{T}ds = [-2\sin\theta, \cos\theta]\,d\theta$

$$A = \int_C \mathbb{F}\cdot\hat{T}\,ds = \int_0^{2\pi} [\sin\theta, 2\cos\theta]\cdot[-2\sin\theta, \cos\theta]\,d\theta$$

$$= \int_0^{2\pi} -2\sin^2\theta + 2\cos^2\theta\,d\theta = \int_0^{2\pi} -(1-\cos 2\theta) + (1+\cos 2\theta)\,d\theta$$

$$= \int_0^{2\pi} 2\cos 2\theta\,d\theta = \underline{\underline{0}}$$

(Ved Greens teorem: $A = \oint_C \mathbb{F}\cdot\hat{T}\,ds = \iint_E \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\,dA = \iint_E 0\,dA = \underline{\underline{0}}$)

($\mathbb{F} = [y, x] = \nabla xy$; dvs konservativt, C lukket $\Rightarrow \int_C \mathbb{F}\cdot\hat{T}ds = 0$)

Oppgave 2

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} + 3x & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

a) Skal undersøke om $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.:

$y=0$: $\lim\limits_{(x,0)\to(0,0)} f(x,0) = \lim\limits_{(x,0)\to(0,0)} 3x = \underline{0}$

$x=0$: $\lim\limits_{(0,y)\to(0,0)} f(0,y) = \lim\limits_{(0,y)\to(0,0)} 0 = \underline{0}$

$y=x$: $\lim\limits_{(x,x)\to(0,0)} f(x,x) = \lim\limits_{(x,x)\to(0,0)} \frac{2x^2}{2x^2} + 3x = \underline{1} \neq 0$

$\Rightarrow$ $\underline{\underline{f \text{ er ikke kontinuerlig i } (0,0)}}$

b) i) <u>Svaret er nei</u>. Funksjonen i a) er et (mot)eksempel, med $(a,b) = (0,0)$ har vi $h(x) = 3x$, som er kont. for $x=0$, og $k(y) = 0$, som er kont. for $y=0$, men $f(x,y)$ er ikke kont. for $(x,y) = (0,0)$.

ii) <u>Svaret er ja</u>. Dersom $f(x,y)$ er kont. i (a,b), er den spesielt kont. langs veiene $f(x,b)$ og $f(a,y)$

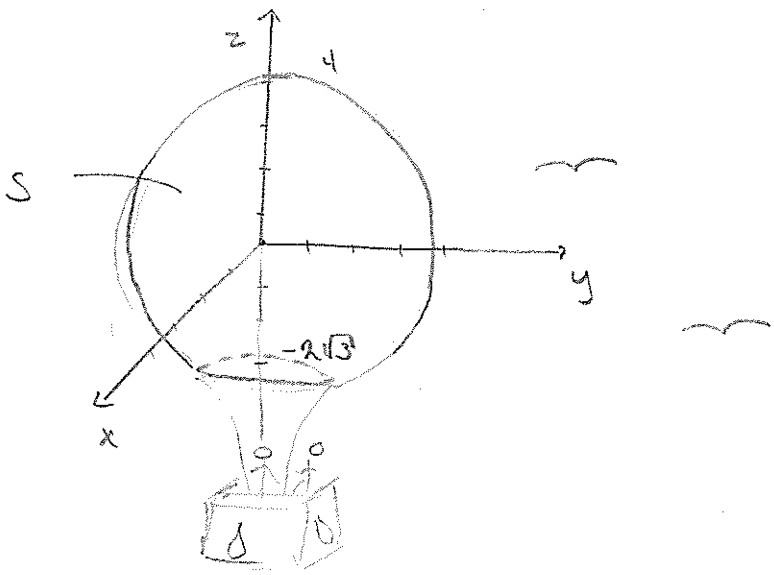
Oppgave 3

Ballong: $x^2+y^2+z^2=16$,

$z \geq -2\sqrt{3}$

a) $\mathbb{F}(x,y,z) = [x, y, z+4]$

Skisse:



Fluks:

$$\iint_S \mathbb{F}\cdot\hat{N}\,dS \qquad , \hat{N} = \frac{[x,y,z]}{4}$$

Kulekoordinater:

$x = 4\sin\varphi\cos\theta$

$y = 4\sin\varphi\sin\theta$

$z = 4\cos\varphi$

$dS = 4^2\sin\varphi\,d\varphi\,d\theta$

$$= \int_0^{2\pi}\int_0^{5\pi/6} [x,y,z+4]\cdot\frac{[x,y,z]}{4}\cdot 4^2\sin\varphi\,d\varphi\,d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}\int_0^{5\pi/6}(x^2+y^2+z^2+4z)\cdot 4^2\sin\varphi\,d\varphi\,d\theta$$

Grenser: $0 \leq \theta \leq 2\pi$

$0 \leq \varphi \leq \frac{5\pi}{6}$

$\cos\alpha = \frac{2\sqrt{3}}{4} = \frac{1}{2}\sqrt{3}$

$\underline{\alpha = \frac{\pi}{6}}$

$$= 4\int_0^{2\pi}\int_0^{5\pi/6}(16+4z)\sin\varphi\,d\varphi\,d\theta$$

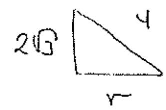
$$= 4^3\int_0^{2\pi}\int_0^{3\pi/6}(1+\cos\varphi)\sin\varphi\,d\varphi\,d\theta$$

$$= 4^3\cdot 2\pi\left[-\cos\varphi + \frac{1}{2}\sin^2\varphi\right]_0^{5\pi/6} = 4^3\cdot 2\pi\left(\frac{1}{2}\sqrt{3}+1+\frac{1}{8}\right) = \underline{\underline{2\pi(32\sqrt{2}+72)}}$$

b) Ved bruk av divergensteoremet: (Sjekk at alle forutsetninger holder)

- fluks gjennom bunn"flata" $(z = -2\sqrt{3})$

$$\iint_{Bunn} \mathbb{F} \cdot \hat{N}\, dS = \iint_{Bunn} [x, y, z+4] \cdot [0, 0, -1]\, dA$$



$r = \sqrt{16-12} = \underline{2}$

$\Rightarrow$ Bunnflata er en sirkel m/radius 2.

$$= \iint_{Bunn} -z-4\, dA = \iint_{Bunn} (2\sqrt{3} - 4)\, dA$$

$$= (2\sqrt{3} - 4)\pi\, 4 = \underline{4\pi \cdot 2\sqrt{3} - 4^2\pi}$$

- $\operatorname{div} \mathbb{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z} = \underline{3}$

$$\Rightarrow V = \frac{1}{3} \iiint_B \operatorname{div} \mathbb{F}\, dV = \frac{1}{3} \oiint_{S+bunn} \mathbb{F} \cdot \hat{N}\, dS = \frac{1}{3}\left(2\pi(32\sqrt{3} + 72) + 4\pi \cdot 2\sqrt{3} - 4^2\pi\right)$$

$$= \underline{\underline{\frac{4}{3}\pi \cdot 4^3 - \frac{4\pi \cdot 2\sqrt{3}}{3}}}$$

(=  + )

Oppgave 4

$$\mathbb{E} = \left[\frac{2x}{x^2+y^2+1} + ze^x,\ \frac{2y}{x^2+y^2+1} + 3z,\ e^x + 3y + 1\right]$$

$$\varphi = \int \frac{2x}{x^2+y^2+1} + ze^x\, dx = \ln(x^2+y^2+1) + ze^x + h_1(y,z)$$

$$\varphi = \int \frac{2y}{x^2+y^2+1} + 3z\, dy = \ln(x^2+y^2+1) + 3zy + h_2(x,z)$$

$$\varphi = \int e^x + 3y + 1\, dz = ze^x + 3yz + z + h_3(x,y)$$

$\underline{\underline{\varphi = \ln(x^2+y^2+1) + ze^x + 3zy + z}}$ er en potensialfunksjon for $\mathbb{E}$