Eksamen høst 2004

Opg 1

a)
$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & 7 & -11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & -7 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 4+\frac{3}{7} \\ 0 & -7 & 0 & -12 \\ 0 & 0 & 1 & \frac{1}{7} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{12}{7} \\ 0 & 0 & 1 & \frac{1}{7} \end{bmatrix}$$

Ligningssystemet får da løsningen

$$x = 1 \qquad y = \frac{12}{7} \qquad z = \frac{1}{7}$$

b)
$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

i) Ser at for $a = -4$ vil vi få ligningen $0 = -8$, noe som er en selvmotsigelse, så for $a = -4$ har vi ingen løsninger.

ii) For $a \neq \pm 4$ får vi en unik løsning

iii) For $a = 4$ vil siste rad være en rad med nuller, så vi vil få en fri variabel, & dermed uendelig mange løsninger.

Opg 2

$$\underline{u} \times \underline{v} = \begin{vmatrix} i & j & k \\ -3 & 2 & -1 \\ 0 & 5 & -1 \end{vmatrix} = (-2+5, 0-3, -15) = (3, -3, -15)$$

$$\overrightarrow{P_1P_2} = (-2,1,1) - (1,-1,2) = (-3,2,-1)$$
$$\overrightarrow{P_1P_3} = (1,4,1) - (1,-1,2) = (0,5,-1)$$

$$A = \frac{1}{2} \| \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} \| = \frac{1}{2}\sqrt{3^2 + (-3)^2 + (-15)^2}$$
$$= \frac{1}{2}\sqrt{9+9+225} = \frac{1}{2}\sqrt{243} = \frac{1}{2}\sqrt{3^5} = \frac{3^2}{2}\sqrt{3}$$
$$= \underline{\underline{\frac{9\sqrt{3}}{2}}}$$

Opg 3

$$u = 4\sqrt{2} + 4\sqrt{2}\,i$$
$$|u| = \sqrt{(4\sqrt{2})^2 + (4\sqrt{2})^2} = \sqrt{32+32} = \sqrt{64} = 8$$
$$u = 8\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = 8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

3. rødder av u:

$$u_0 = 2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$
$$u_1 = 2\left(\cos\frac{9\pi}{12} + i\sin\frac{9\pi}{12}\right) = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$u_2 = 2\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$$

Opg. 4

$$T_A(2,3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix} = (-3,2)$$

T_A er en rotasjon $90°$ mot klokka

Opg. 5

a) $$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 3 \\ -1 & \lambda \end{vmatrix} = \lambda(\lambda-4)+3 = \lambda^2 - 4\lambda + 3$$

$$\lambda = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm \sqrt{4}}{2} = \frac{4 \pm 2}{2} = \begin{cases} 3 \\ 1 \end{cases}$$

$\lambda_1 = 3$ & $\lambda_2 = 1$ er egenverdiene til A.

$\lambda_1 = 3$:

$$\begin{bmatrix} 3-4 & 3 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

$x_2 = t \quad x_1 = 3t$

$\Rightarrow \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ er en egenvektor til $\lambda_1 = 3$

$\lambda_2 = 1$

$$\begin{bmatrix} 1-4 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ er en egenvektor til $\lambda_2 = 1$

b) $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ er slik at $P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}$$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix}\left(\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\right) = \quad \%$$

$$= \frac{1}{2}\begin{bmatrix} 3^{k+1} & 1 \\ 3^k & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3^{k+1}-1 & -3^{k+1}+3 \\ 3^k-1 & -3^k+3 \end{bmatrix}$$

$$\frac{3^k-1}{2}A + \frac{3-3^k}{2}I = \frac{1}{2}\left(\begin{bmatrix} 4(3^k-1) & -3(3^k-1) \\ 3^k-1 & 0 \end{bmatrix} + \begin{bmatrix} 3-3^k & 0 \\ 0 & 3-3^k \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4\cdot 3^k - 4 + 3 - 3^k & -3^{k+1}+3 \\ 3^k - 1 & 3-3^k \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3^{k+1}-1 & -3^{k+1}+3 \\ 3^k-1 & 3^k+3 \end{bmatrix}$$

Ser at $A^k = \frac{3^k+1}{2}A + \frac{3-3^k}{2}I$

Opg6

$$x^2 + 4xy + y^2 = 1$$

$$\underline{x}^T A \underline{x} = 1 \qquad \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

La $\underline{x} = P\underline{x}'$

$$(P\underline{x}')^T A P\underline{x}' = 1$$

$$\underline{x}'^T P^T A P \underline{x}' = 1$$

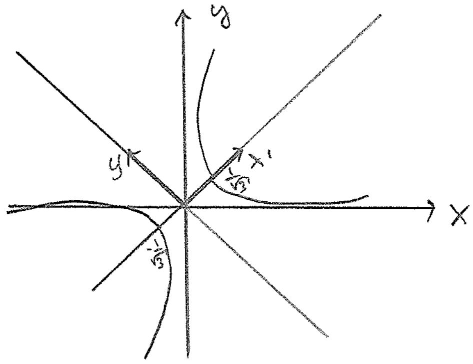
Siden P er ortogonal har vi $P^{-1} = P^T$

$$\underline{x}'^T D \underline{x}' = 1$$

$$3x'^2 - y'^2 = 1$$

$$\frac{x'^2}{\frac{1}{3}} - y'^2 = 1$$

Ser at dette gir en hyperbel.



Opg7

$\Rightarrow$: Anta A inverterbar. Dersom vi multipliserer med A^{-1} på begge sider av ligningen $A^2 = A$ får vi

$$\underbrace{A^{-1} \cdot A}_{I} \cdot A = \underbrace{A^{-1} \cdot A}_{I},$$

noe som gir at $A = I$, så vi ser at A inverterbar $\Rightarrow A = I$.

$\Leftarrow$: Anta at $A = I$. Siden I er inverterbar, er da A inverterbar, så vi ser at $A = I \Rightarrow A$ er inverterbar.

Opg8

$$\det(A_r) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ r & 1 & 1 & 1 \\ r & r & 1 & 1 \\ r & r & r & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1-r & 1-r & 1-r \\ 0 & 0 & 1-r & 1-r \\ 0 & 0 & 0 & 1-r \end{vmatrix} = (1-r)^3$$

A_r inverterbar $\Leftrightarrow \det A \neq 0 \Leftrightarrow (1-r)^3 \neq 0 \Leftrightarrow r \neq 1$