MA 1201 – Lineær algebra og geometri

Eksamen 15. desember 2006 – Løsningsforslag

1a)

$$A = \begin{bmatrix} 1 & 1 & -4 & 0 \\ 2 & 4 & -6 & 2 \\ -1 & -3 & 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Løsn. av lign. system: Fri variabel: $z = t$

$\rightsquigarrow x = -1 + 5t, \ y = 1 - t, \ z = t$, ~~[illegible]~~ t vilkårlig.

b)

$$\begin{bmatrix} 1 & 1 & -4 & 0 \\ 2 & 4 & -6 & 2 \\ -1 & -3 & (a^2-2) & a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & a^2-6 & a \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & -4 & 0 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & a^2-4 & a+2 \end{bmatrix} \qquad [a^2-4 = (a-2)(a+2)]$$

$a = 2$: ingen løsninger

$a = -2$: uendelig mange løsninger

$a \neq \pm 2$: nøyaktig én løsning.

2a) $-8 = 8(\cos\pi + i\sin\pi)$

3. røtter av -8: $2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 1 + \sqrt{3}\,i = w_1$

$2(\cos\pi + i\sin\pi) = -2 = w_2$

$2(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}) = 1 - \sqrt{3}\,i = w_3$

Løsn. av $(z-i)^3 = -8$:

$z_1 = w_1 + i = 1 + (1+\sqrt{3})i$

$z_2 = w_2 + i = -2 + i$

$z_3 = w_3 + i = 1 - \sqrt{3}\,i + i = 1 + (1-\sqrt{3})i$

2b) La $z = a + bi$, $a, b \in \mathbb{R}$.

Da er $\bar{z} = a - bi$

$z^2 = (a+bi)(a-bi) = (a^2 - b^2) + 2abi$

$(\bar{z})^2 = (a-bi)(a-bi) = (a^2 - b^2) - 2abi$

Ser at $z^2 = (\bar{z})^2 \Leftrightarrow 2abi = 0 \Leftrightarrow ab = 0 \Leftrightarrow a = 0$ eller $b = 0$

$\Leftrightarrow$ z rent imaginær eller z reell.

3a)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & 2 \\ 2 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2 - 4 = 0$$

$$\Leftrightarrow (\lambda - 1)^2 = 4$$

$$\Leftrightarrow (\lambda - 1) = \pm 2$$

$$\lambda_1 = 3, \quad \lambda_2 = -1$$

$\underline{\lambda_1 = 3:}$

$$(\lambda_1 I - A)\underline{x} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = -y$$

Egenvektorer: $t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ Velger $\underline{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$; $\|\underline{v}_1\| = 1$

$\underline{\lambda_2 = -1:}$

$$(\lambda_2 I - A)\underline{x} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow x = y$$

Egenvektorer: $t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Velger $\underline{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$; $\|\underline{v}_2\| = 1$

$$P = [\underline{v}_1 \ \underline{v}_2] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$ har determinant 1.

Får $$P^{-1}AP = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = D.$$

3 b) $x^2 - 4xy + y^2 = 9$

$$[x \ \ y]\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 9$$

$$\underline{x}^T A \underline{x} = 9$$

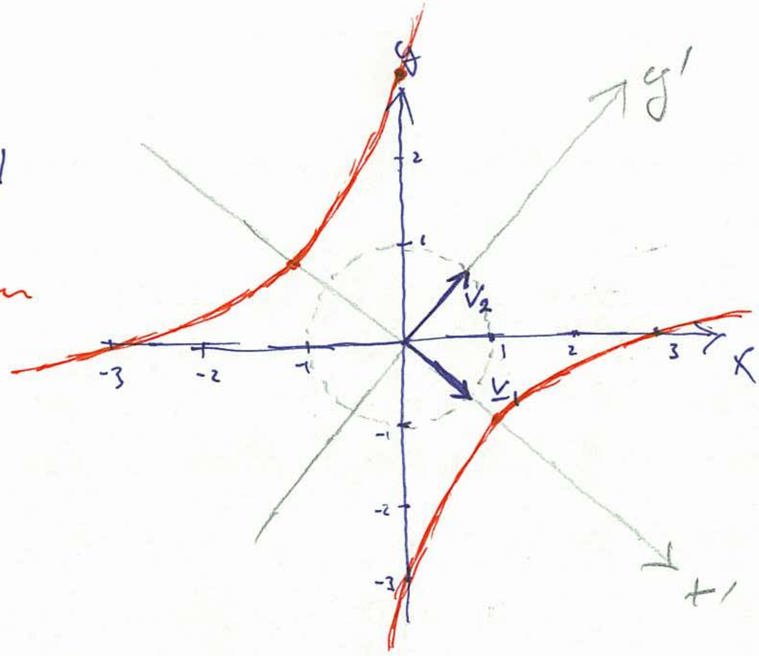
Koordinatskifte: $\underline{x} = P\underline{x}'$, $\underline{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$

$$(P\underline{x}')^T A (P\underline{x}') = 9 \iff \underline{x}'^T \underbrace{P^T A P}_{D} \underline{x}' = 9 \iff \underline{x}'^T D \underline{x}' = 9$$

$$3x'^2 - y'^2 = 9$$

$$\frac{x'^2}{(\sqrt{3})^2} - \frac{y'^2}{3^2} = 1$$

Kurven er en hyperbel:



4. Retningsvektor for l_1 : $\underline{v} = (1, 2, 2)$

Punkt på l_1 : $(2, 0, 1)$

Vektor fra $(1,1,1)$ til $(2,0,1)$: $\underline{u} = (1, -1, 0)$

Normalvektor for P: $\underline{u} \times \underline{v} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 0 \\ 1 & 2 & 2 \end{vmatrix} = (-2, -2, 3) = \underline{n}$

Ligning for P:

$$\underline{n} \cdot (x-1, y-1, z-1) = 0$$

$$(-2, -2, 3) \cdot (x-1, y-1, z-1) = 0$$

$$-2x + 2 - 2y + 2 + 3z - 3 = 0$$

$$\underline{\underline{-2x - 2y + 3z + 1 = 0}}$$

5. $T_A(3,-5,2) = \begin{bmatrix} -1 & 1 & 4 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{(0,0,0)}}$

$$\det(A^{(a)}) = \begin{vmatrix} a & 1 & 4 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 3a + 3$$

$T_{A^{(a)}}$ er 1-1 $\Leftrightarrow \det(A^{(a)}) \neq 0 \Leftrightarrow \underline{\underline{a \neq -1}}$

6. $A \cdot B$ ikke invertibar

$\Updownarrow$

$\det(A \cdot B) = 0$

$\Updownarrow$

$\det(A) \cdot \det(B) = 0$

$\Updownarrow$

$\det(A) = 0$ og/eller $\det(B) = 0$

$\Updownarrow$

A ikke invertibar og/eller B ikke invertibar

7. De sanne utsagnene er:

a)
d)
g)
i)