Opg 1 Eksamen Vår 2006 (MA1201)

a)
$$\begin{bmatrix} 1 & -2 & 5 \\ 4 & -5 & 8 \\ -3 & 3 & -3 \end{bmatrix} \begin{matrix} -4R_1 \\ 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 3 & -12 \\ 0 & -3 & 12 \end{bmatrix} \begin{matrix} 1/3 \\ R_2 \end{matrix} \sim \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} 2R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{Redusert trappeform.}$$

b)
$$\begin{bmatrix} 1 & -2 & 5 & b_1 \\ 4 & -5 & 8 & b_2 \\ -3 & 3 & -3 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ 0 & -3 & 12 & 3b_1+b_3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & b_1 \\ 0 & 3 & -12 & -4b_1+b_2 \\ 0 & 0 & 0 & -b_1+b_2+b_3 \end{bmatrix}$$

Her må vi ha $b_1 = b_2 + b_3$

c)
$$\begin{bmatrix} 1 & -2 & 5 & -1 \\ 0 & 3 & -12 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x = 3t - 1$
$y = 4t$
$z = t$

$$\Leftrightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Opg 2

a) La $z = a + ib$

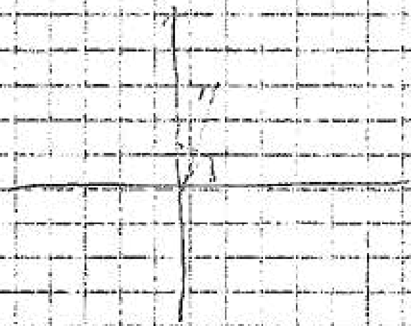
$$\frac{1}{2}(a+ib+a-ib) = \frac{1}{2}(2a) = a = \mathrm{Re}(z)$$

$$\frac{1}{2i}(a+ib-a+ib) = \frac{1}{2i}(2ib) = b = \mathrm{Im}(z).$$

b) La $z = -1$

$|z| = 1$

$\arg(z) = \pi$



4. røtter til $z = -1$

$$z_0 = \left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\pi/4\right)\right) = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

$$z_1 = \left(\cos\left(\frac{\pi}{4}+\frac{2\pi}{4}\right) + i\sin\left(\pi/4+\frac{2\pi}{4}\right)\right)$$

$$= \cos\left(\frac{3\pi}{4}\right) + i\sin\left(3\pi/4\right) = \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$z_2 = \left(\cos\left(\pi/4+\pi\right) + i\sin\left(\pi/4+\pi\right)\right)$$

$$= \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$$

$$z_3 = \cos\left(\frac{\pi}{4}+\frac{6\pi}{4}\right) + i\sin\left(\frac{\pi}{4}+\frac{6\pi}{4}\right) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$

Opg 3

(a) $|\lambda I - A| = \begin{vmatrix} \lambda-9 & 2 \\ 2 & \lambda-6 \end{vmatrix} = (\lambda-9)(\lambda-6)-4$

$= \lambda^2 - 15\lambda + 50 = 0$

$\Leftrightarrow (\lambda-5)(\lambda-10) = 0$

$\Rightarrow \lambda = 5$ or $\lambda = 10$

$\therefore$ egenvachene er $\lambda = 5$ og $\lambda = 10$

Vi finner egenvektorene:

$\underline{\lambda_1 = 5}$: $\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{1}{2} \\ 2 & -1 \end{pmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} x_1 = \frac{1}{2}t \\ x_2 = t \end{matrix} \quad \therefore \begin{bmatrix} x_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} t$

$\underline{\lambda_2 = 10}$

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{matrix} x_1 = -2t \\ x_2 = t \end{matrix} \quad \therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$

Normaliserte egenvectorer:

$v_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \qquad v_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

Vi setter nå $P = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

$$P^{-1} = P^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

Da blir
$$P^{-1}AP = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -2/\sqrt{5} \\ \frac{2}{\sqrt{5}} & 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$$

Da har vi funnet P.

(b) $9x^2 - 4xy + 6y^2 = 45$ er det samme som

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 45$$

Vi skifter koodinater ve å sette

$$\begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Vi får ligningen

$$\begin{bmatrix} x' & y' \end{bmatrix} P^T A P \begin{bmatrix} x' \\ y' \end{bmatrix} = 45$$

i det nye koodinatsystemet

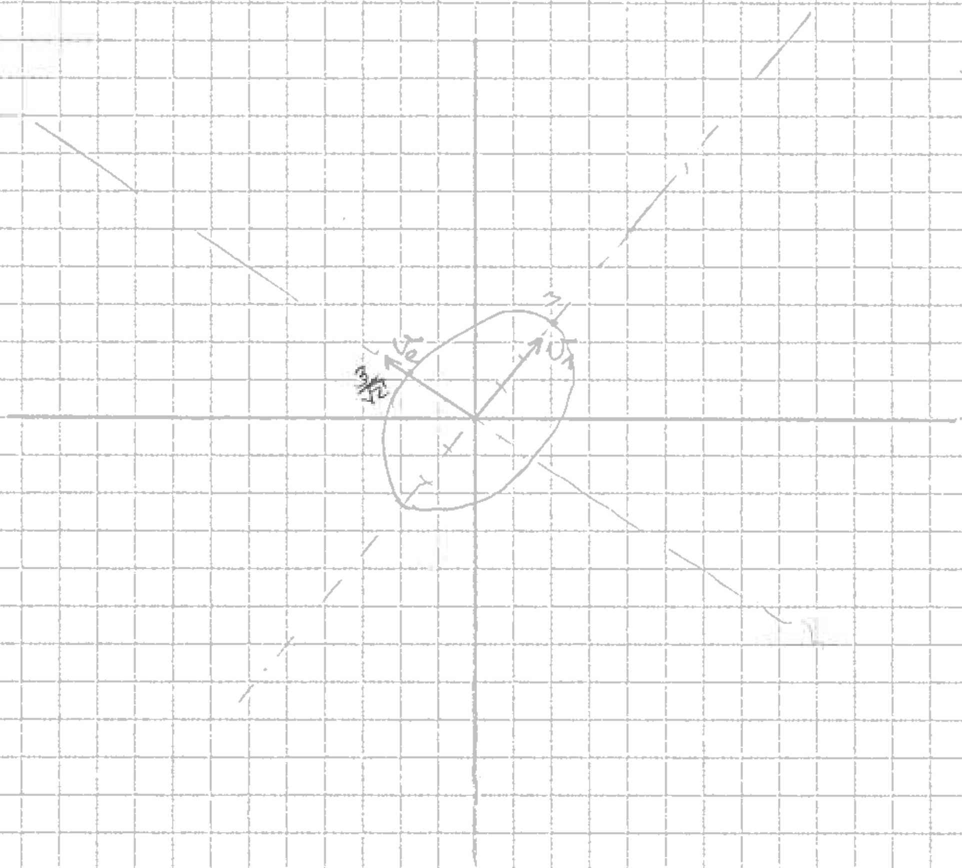
Vi forenkler ligningen og får

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 45$$

$$5x'^2 + 10y'^2 = 45$$

$$\frac{x'^2}{9} + \frac{y'^2}{9/2} = 1$$

Kjeglesnittet er altså en ellipse med følgende skisserte løsningsmengde.



Opg 4

(a) Ja

(b) Nei

(c) Nei

(d) Ja

Opg5

$\vec{PQ} = (-1, 2, 2) \qquad \vec{PR} = (1, 1, -1)$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 2 & 2 \\ 1 & 1 & -1 \end{vmatrix} = (4, 1, -3)$$

Arealet til $A = \frac{1}{2} \| \vec{PQ} \times \vec{PR} \|$

$$= \frac{1}{2} \sqrt{26}$$

Oppg 6 $\Rightarrow$: Anta at A er inverterbar. Fra ligningen $A^2 = A$ får vi

$$A^{-1}A^2 = A^{-1}A = I$$

$$\Rightarrow \quad A = I$$

$\Leftarrow$: Anta at $A = I$. Siden I er inverterbar er da A inverterbar.