Løsning, MA1201 24. mai 2007

1a)

$$\begin{bmatrix} 1 & -2 & 0 & 1 \\ -1 & 3 & -1 & -2 \\ 3 & -4 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \end{bmatrix} \sim \underline{\underline{\begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}}$$

Løsninger av ligningssystemet:

$$\rightsquigarrow \left\{ \begin{array}{l} x \quad -2z = -1 \\ \quad y - z = -1 \end{array} \right\} \rightsquigarrow \left. \begin{array}{l} x = 2t - 1 \\ y = t - 1 \\ z = t \end{array} \right\} \text{ } t \in \mathbb{R} \text{ er en fri parameter.}$$

b) På denne oppgaven var det ment at nederste ligning skulle være $3x - 4y + (a^2-6)z = a-1$.

I så fall blir utregningen

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ -1 & 3 & -1 & -2 \\ 3 & -4 & a^2-6 & a-1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & a^2-6 & a-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & a^2-4 & a-2 \end{array}\right]$$

$$\Rightarrow \left\{ \begin{array}{ll} a \neq \pm 2 & : \text{unik løsning} \\ a = 2 & : \text{uendelig mange løsninger} \\ a = -2 & : \text{ingen løsninger} \end{array} \right.$$

Dersom man løser oppgaven slik den stod i oppgavesettet, får man

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ -1 & 3 & -1 & -2 \\ 3 & -4 & -(a^2-6) & a-1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 6-a^2 & a-4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 8-a^2 & a-2 \end{array}\right]$$

$$\Rightarrow \left\{ \begin{array}{l} \text{Unik løsning for } a \neq \pm\sqrt{8} = \pm 2\sqrt{2} \\ \text{Ingen løsninger for } a = \pm 2\sqrt{2}. \end{array} \right.$$

2a) $1+\sqrt{3}\,i = r(\cos\theta + i\sin\theta)$

$r = \sqrt{1+3} = 2$

$$\left.\begin{aligned} 2\cos\theta &= 1 \\ 2\sin\theta &= \sqrt{3} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} \cos\theta &= \tfrac{1}{2} \\ \sin\theta &= \tfrac{\sqrt{3}}{2} \end{aligned}\right\} \Rightarrow \theta = \frac{\pi}{3}$$

$$\Rightarrow \underline{\underline{1+\sqrt{3}\,i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)}}$$

$$\Rightarrow \underline{\underline{(1+\sqrt{3}\,i)^4}} = 2^4\left(\cos\left(4\cdot\frac{\pi}{3}\right) + i\sin\left(4\cdot\frac{\pi}{3}\right)\right) = \underline{\underline{16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)}}$$

b)

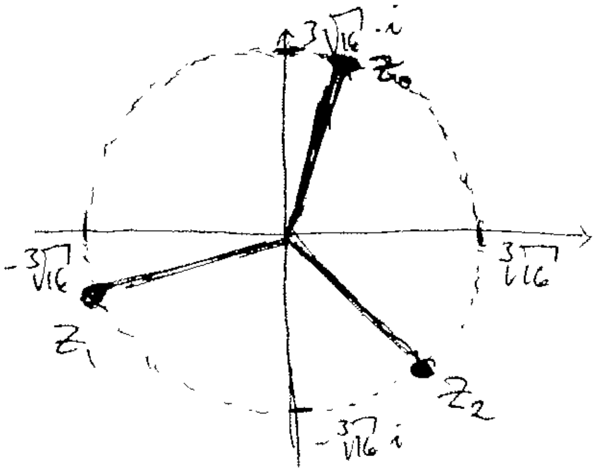
$$z_0 = \sqrt[3]{16}\left(\cos\frac{4\pi/3}{3} + i\sin\frac{4\pi/3}{3}\right) = \sqrt[3]{16}\left(\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}\right)$$

$$z_1 = \sqrt[3]{16}\cdot\left(\cos\left(\frac{4\pi}{9} + \frac{2\pi}{3}\right) + i\sin\left(\frac{4\pi}{9} + \frac{2\pi}{3}\right)\right)$$

$$= \sqrt[3]{16}\cdot\left(\cos\left(\frac{10\pi}{9}\right) + i\sin\left(\frac{10\pi}{9}\right)\right)$$

$$z_2 = \sqrt[3]{16}\cdot\left(\cos\left(\frac{4\pi}{9} + \frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{9} + \frac{4\pi}{3}\right)\right)$$

$$= \sqrt[3]{16}\cdot\left(\cos\left(\frac{16\pi}{9}\right) + i\sin\left(\frac{16\pi}{9}\right)\right)$$



3a)

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-4 & 1 \\ -2 & \lambda-1 \end{vmatrix} = (\lambda-4)(\lambda-1)+2$$

$$= \lambda^2 - 5\lambda + 6$$

$$\lambda_1 = 3 \ , \ \lambda_2 = 2$$

$\underline{\lambda_1 = 3:}$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \Rightarrow x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \ , \ t \in \mathbb{R}$$

$\underline{\lambda_2 = 2:}$

$$\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \Rightarrow 2x_1 = x_2$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} \ , \ t \in \mathbb{R}.$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \ , \ P^{-1} = \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = D.$$

b) $A = PDP^{-1}$

$$\Rightarrow \underline{A^n} = (PDP^{-1})^n = PD^nP^{-1} =$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3^n & 0 \\ 0 & 2^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2\cdot 3^n & -3^n \\ -2^n & 2^n \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 2\cdot 3^n - 2^n & 2^n - 3^n \\ 2\cdot 3^n - 2^{n+1} & 2^{n+1} - 3^n \end{bmatrix}}}$$

4 $\overrightarrow{P_1P_2} = (-1, 2, 0)$

$\overrightarrow{P_1P_3} = (2, 2, -3)$

$$\left.\begin{matrix}\overrightarrow{P_1P_2} = (-1,2,0) \\ \overrightarrow{P_1P_3} = (2,2,-3)\end{matrix}\right\} \quad \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} i & j & k \\ -1 & 2 & 0 \\ 2 & 2 & -3 \end{vmatrix} = (-6, -3, -6)$$

Areal $= \frac{1}{2} \cdot \|(-6,-3,-6)\| = \frac{1}{2} \cdot \sqrt{36+9+36} = \frac{1}{2} \cdot \sqrt{81} = \underline{\underline{\frac{9}{2}}}$

Normal-vektor: $(-6, -3, -6)$.

Kan også bruke $(2, 1, 2)$, da disse er parallell.

Ligning for planet:

$(2, 1, 2) \cdot (x-3, y-1, z-2) = 0$

$2x - 6 + y - 1 + 2z - 4 = 0$

$\underline{\underline{2x + y + 2z = 11}}$

5

$$[T_A \circ T_B] = [T_A] \cdot [T_B] = A \cdot B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \underline{\underline{\begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 4 \\ 5 & 1 & 6 \end{bmatrix}}}$$

$$T_A \circ T_B(1, 1, -1) = \begin{bmatrix} 1 & 1 & 2 \\ 4 & 0 & 4 \\ 5 & 1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}},$$

$T_A \circ T_B$ er ikke 1-1, siden det finnes $\underline{v} \neq \underline{0} \in \mathbb{R}^3$ slik at $T_A \circ T_B(\underline{v}) = \underline{0}$.

(Kan også argumentere med at determinanten til standardmatrisen er 0.)

6 a) Anta $A^k = \underline{O}_{n\times n}$ for et ~~heltall~~ positivt heltall k.

Anta at A er inverterbar.

Da er $A^k \cdot (A^{-1})^k = A \cdot A \cdots A \cdot A^{-1} \cdots A^{-1} \cdot A^{-1} = I_n$.

Men siden $A^k = \underline{O}_{n\times n}$ betyr dette at

$$\underline{O}_{n\times n} \cdot (A^{-1})^k = \underline{I}_n$$

Dette er umulig, for $\underline{O}_{n\times n} \cdot M = \underline{O}_{n\times n}$ for alle $n\times n$-matriser M.

Vi har fått en selvmotsigelse.

Altså må A ikke være inverterbar.

b) Den motsatte implikasjonen,

"A ikke inverterbar $\Longrightarrow \exists$ pos. heltall k s.a. $A^k = \underline{O}_{n\times n}$"

er ikke sann.

Vi ser dette ved et moteksempel:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$\det A = 0 \Rightarrow A$ ikke inverterbar

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = A$$

$\vdots$

$A^k = A$ for alle pos. heltall k.

$A^k \neq \underline{O}_{n\times n}$ for alle pos. heltall k.