MA 1201/6201, 21/5-2011

LØSNINGER:

OPPGAVE 1:

(a)
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 2 & 4 & 0 \\ 1 & 3 & -3 & 1 \end{array}\right] \begin{matrix} -2 & -1 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -2 & 6 & -2 \\ 0 & 1 & -2 & 0 \end{array}\right] 1 \cdot (-\tfrac{1}{2})$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 1 & -2 & 0 \end{array}\right] -1 \sim \left[\begin{array}{ccc|c} 1 & 0 & 5 & -1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{matrix} -5 & 3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \end{array}\right] ; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}$$

(b)
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \begin{matrix} -2 & -1 \end{matrix} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] 1 \cdot (-\tfrac{1}{2})$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] -1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \begin{matrix} -5 & +3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \qquad A^{-1} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}$$

Implisitt følger det at A er invertibel.

(c)
$$A\mathbf{x} = \mathbf{b} \Rightarrow \mathbf{x} = A^{-1}\mathbf{b}$$ når

A er invertibel. Dette gir for vårt ligningssystem i (a):

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 9-5 \\ -5+3 \\ -2+1 \end{bmatrix} = \underline{\begin{bmatrix} 4 \\ -2 \\ -1 \end{bmatrix}}$$

OPPGAVE 2:

(a) Vi har $(AB) = I$ og dermed $C(AB) = CI = C$. Den assosiative lov gir:

$$C(AB) = (CA)B = IB = B$$

Altså

$$\underline{C} = C(AB) = (CA)B = \underline{B}$$

(b) Vi har da:

$$(B^{-1}A^{-1})(AB) = B^{-1}((A^{-1}A)B) = B^{-1}(IB)$$

$$= B^{-1}B = I.$$

Videre:

$$(AB)(B^{-1}A^{-1}) = (A((BB^{-1})A^{-1}) = A(IA^{-1})$$

$$= AA^{-1} = I$$

Altså er $B^{-1}A^{-1} = (AB)^{-1}$.

Dette bygger på at både A^{-1} og B^{-1} eksisterer.

(c) Vi antar at A^{-1} eksisterer og at

$$A^{-1}A = I$$

Siden $\det(A^{-1})\det(A) = \det I = 1$, må $\det(A) \neq 0$.

OPPGAVE 3:

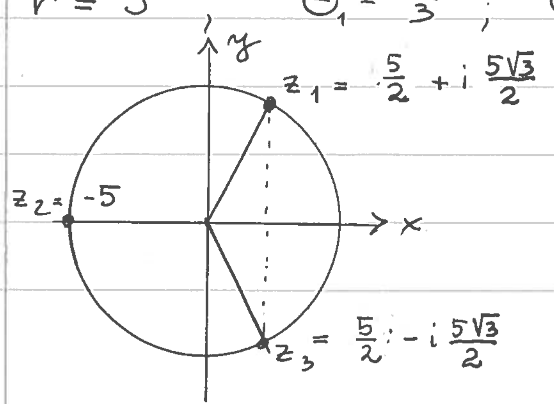
(a) $$-125 = 125(\cos\pi + i\sin\pi)$$

Vi setter så $z = r(\cos\theta + i\sin\theta)$ og får ligningen:

$$r^3(\cos 3\theta + i\sin 3\theta) = 125(\cos\pi + i\sin\pi)$$

Dette gir løsningene:

$r = 5$ $\quad \theta_1 = \frac{\pi}{3}$; $\quad \theta_2 = \frac{\pi}{3} + \frac{2\pi}{3} = \pi$; $\quad \theta_3 = \frac{\pi}{3} + \frac{4\pi}{3} = \frac{5\pi}{3}$



$$z_1 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$z_2 = 5(\cos\pi + i\sin\pi) = -5$$

$$z_3 = 5\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$$

(b) $(x-z_0)(x-\bar{z}_0) \equiv x^2-(z_0+\bar{z}_0)x + z_0\bar{z}_0$

$\equiv x^2 - (2\,\mathrm{Re}\, z_0)x + |z_0|^2$

$\mathrm{Re}\,\bar{z}_0$ og $|z_0|^2$ er reelle tall.

$x^3+125 \equiv (x-z_1)(x-z_2)(x-z_3)$

$\equiv (x-z_1)(x-\bar{z}_1)\cdot(x-z_2) \equiv (x-z_2)(x^2-2(\mathrm{Re}\,z_1)x+|z_1|^2)$

$\equiv \underline{(x+5)(x^2-5x+25)}$

OPPGAVE 4:

(a) Velges $s=0$ og $t=1$ får vi:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$$

Altså ligger $(3,3,0)$ i planet.

De to vektorene

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \quad \text{og} \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -1 \end{bmatrix}$$

ligger som kjent i planet. De er dessuten ikke parallelle. Vi har:

$$\mathbf{v}_1 \cdot \mathbf{n} = (3,0,-1)(1,2,3) = 3+0-3 = 0$$

$$\mathbf{v}_2 \cdot \mathbf{n} = (-3,3,-1)(1,2,3) = -3+6-3 = 0$$

Altså er $\mathbf{n} \perp$ planet.

(b) Vi har at ligningen må ha formen

$$Ax + By + Cz + D = 0,$$

der (A,B,C) er en normalvektor til planet. Altså er ligningen av formen:

$$x + 2y + 3z + D = 0$$

Siden $(3,3,0)$ ligger i planet, må vi ha:

$$3 + 2\cdot 3 + 3\cdot 0 + D = 0 \Rightarrow D = -9.$$

M.a.o. får vi:

$$x + 2y + 3z - 9 = 0$$

Avstanden mellom planet og punktet $(x_1, y_1, z_1) = (0,0,0)$ blir:

$$\frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}} = \frac{|-9|}{\sqrt{1^2+2^2+3^2}} = \underline{\underline{\frac{9}{\sqrt{14}}}}$$

OPPGAVE 5:

(a) Vi löser ligningen:

$$\begin{vmatrix} (5-\lambda) & 2 \\ 2 & (2-\lambda) \end{vmatrix} = 0$$

eller $(5-\lambda)(2-\lambda) - 4 = \lambda^2 - 7\lambda + 10 - 4$

$= \lambda^2 - 7\lambda + 6 = 0$

Röttene blir da $\underset{\sim}{\overset{(i)}{\lambda_1 = 1}}$, $\underset{\sim}{\overset{(ii)}{\lambda_2 = 6}}$.

Egenvektorene blir:

(i) $\begin{aligned} 4x_1 + 2x_2 &= 0 \\ 2x_1 + x_2 &= 0 \end{aligned}$; $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ normalisert: $\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$

(ii) $\begin{aligned} -x_1 + 2x_2 &= 0 \\ 2x_1 - 4x_2 &= 0 \end{aligned}$; $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$$P = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

(b) $\mathbf{x}^T A \mathbf{x} = 5x^2 + 4xy + 2y^2$

$= \mathbf{x}'^T D \mathbf{x}' =$

$$[x', y'] \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \underset{\sim}{x'^2 + 6y'^2}$$

(c) $\begin{bmatrix} -\frac{28}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{28}{\sqrt{5}} & -\frac{4}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

$= \begin{bmatrix} -\frac{20}{5} & -\frac{60}{5} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = -4x' - 12y'$

I de nye koordinater blir derfor ligningen:

$$x'^2 + 6y'^2 - 4x' - 12y' + 4 = 0$$

Dette omformes til

$$(x'^2 - 4x' + 4) + 6(y'^2 - 2y' + 1) = 6$$

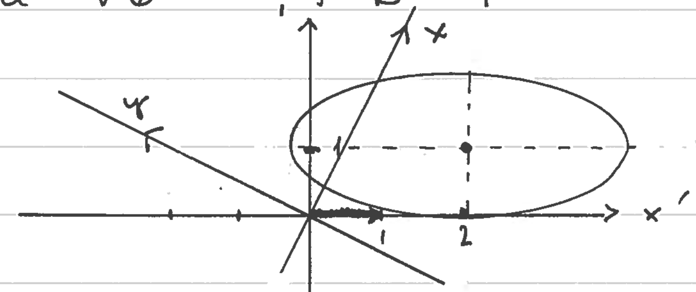
eller

$$(x'-2)^2 + 6(y'-1)^2 = 6$$

$$\frac{(x'-2)^2}{(\sqrt{6})^2} + \frac{(y'-1)^2}{1^2} = 1$$

Dette blir en ellipse i $x'y'$-systemet med sentrum i $(2,1)$ og halvakser:

$a = \sqrt{6}$, $b = 1$



For $\mathbf{x}' = [1,0]$, har vi $\mathbf{x} = \begin{bmatrix} \frac{1}{\sqrt5} & \frac{2}{\sqrt5} \\ -\frac{2}{\sqrt5} & \frac{1}{\sqrt5} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{\sqrt5} \\ -\frac{2}{\sqrt5} \end{bmatrix}$ i det opprinnelige systemet.

Vektoren $\mathbf{x}' = [1,0]$ ligger altså i 4. kvadrant i forhold til det opprinnelige koordinatsystemet.

MA 1201
Eks., 27/5 - 2011

(A)

LØSNINGER:

OPPGAVE 1:

Volumet av parallell-epipedet blir:

$$|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})| = \left|\begin{vmatrix} 0 & 2 & -2 \\ 1 & 2 & 0 \\ -2 & 3 & 1 \end{vmatrix}\right| = \left| -2\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix} \right|$$

$$= |-2\cdot 1 - 2(3+4)| = |-2-14| = \underline{16}$$

OPPGAVE 2:

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right] \begin{array}{c} -2 \quad 1 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right] \begin{array}{c} \\ 2 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \\ 1 \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Løsningen:

$$\begin{aligned} x &= 2 + y - w \\ y &= y \\ z &= 1 + w \\ w &= w \end{aligned}$$

eller

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}; \quad s,t \in \mathbb{R}.$$

OPPGAVE 3:

T_A kan skrives på matrise-form slik:

$$T_A\,\mathbf{x} = \underbrace{\begin{bmatrix} 1 & -2 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

B

$T_A^{-1}(w) = A^{-1}w = x$. Vi skal altså bestemme A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} -2 \quad -1 \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 5 & -3 & -2 & 1 & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ \frac{1}{5} \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 3 & -2 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ 2 \quad -3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{4}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & -\frac{3}{5} & -\frac{2}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{5} & \frac{1}{5} & -\frac{3}{5} & 1 \end{array}\right] \begin{array}{c} \\ \\ 4 \quad -3\cdot(-5) \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right] \qquad T_A^{-1} \sim \begin{bmatrix} 1 & -2 & 4 \\ -1 & 2 & -3 \\ -1 & 3 & -5 \end{bmatrix}$$

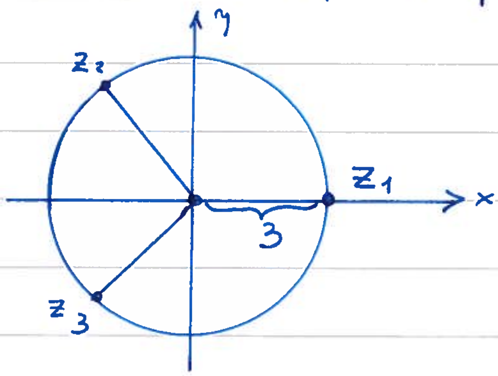
OPPGAVE 4:

(a) $27 = 27(\cos 0 + i \sin 0)$

$z^3 = 27 \qquad z = r(\cos\theta + i \sin\theta)$

gir: $r^3(\cos 3\theta + i \sin 3\theta) = 27(\cos 0 + i \sin 0)$

$r = 3. \qquad \theta_1 = 0, \qquad \theta_2 = \frac{2\pi}{3}, \qquad \theta_3 = \frac{4\pi}{3}$



$\underline{z_1 = 3}$

$z_2 = 3(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3})$

$z_3 = 3(\cos\frac{4\pi}{3} + i \sin\frac{\pi}{3})$

$\underline{z_2 = -\frac{3}{2} + i\frac{3\sqrt{3}}{2}}$

$z_3 = -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$

(b) $(z - z_2)(z - z_3) = (z - z_2)(z - \bar{z}_2) = z^2 - (2\,\mathrm{Re}\, z_2)z + |z_2|^2$

$= z^2 + 3z + 9.$

Dette gir:

$\underline{x^3 - 27 = (x-3)(x^2 + 3x + 9)}$

c

OPPGAVE 5:

(a) $(I-A)^2 = (I-A)(I-A) = I - IA - AI + A^2$

$= I - A - A + A^2 = I - A - A + A = I - A$

siden $A^2 = A$.

(b) $(I-A)(I + A + A^2 + \cdots + A^{k-1})$

$= I + IA + IA^2 + \cdots + IA^{k-1}$

$\quad - AI - A^2 - A^3 - \cdots - A^k$

$= I + \cancel{A} + \cancel{A^2} + \cdots + \cancel{A^{k-1}}$

$\quad - \cancel{A} - \cancel{A^2} - \cdots - \cancel{A^{k-1}} - A^k = I - A^k = I$

Altså er $I + A + A^2 + \cdots + A^{k-1}$ høyre-invers til $I-A$. I følge teorien er en høyre-invers også venstre-invers. Altså har vi:

$$(I-A)^{-1} = I + A + A^2 + \cdots + A^{k-1}.$$

OPPGAVE 6:

(a) $\begin{vmatrix} (2-\lambda) & -2 \\ -2 & (-1-\lambda) \end{vmatrix} = (\lambda-2)(\lambda+1) - 4$

$= \lambda^2 - \lambda - 6 = 0 \qquad \lambda = \frac{1}{2}[1 \pm \sqrt{1 - 4\cdot 1(-6)}]$

$\qquad\qquad\qquad\qquad\quad = \frac{1}{2}[1 \pm 5]$

$\lambda_1 = 3 \quad , \quad \lambda_2 = -2$

Egenvektorene:

$\lambda_1 = 3: \begin{cases} -1x_1 - 2x_2 = 0 \\ -2x_1 - 4x_2 = 0 \end{cases} \qquad x_1 = -2x_2 ; \ \mathbf{w}_1 = s\begin{bmatrix} 2 \\ -1 \end{bmatrix} ; \ \mathbf{v}_1 = \begin{bmatrix} \frac{2}{\sqrt5} \\ -\frac{1}{\sqrt5} \end{bmatrix}$

$\lambda_2 = -2: \begin{cases} 4x_1 - 2x_2 = 0 \\ -2x_1 + x_2 = 0 \end{cases} \qquad x_2 = 2x_1 ; \ \mathbf{w}_2 = s\begin{bmatrix} 1 \\ 2 \end{bmatrix} ; \ \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt5} \\ \frac{2}{\sqrt5} \end{bmatrix}$

$\mathbf{x} = \begin{bmatrix} \frac{2}{\sqrt5} & \frac{1}{\sqrt5} \\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5} \end{bmatrix} \mathbf{x}'$ gir med $\mathbf{x}' = \begin{bmatrix} x' \\ y' \end{bmatrix}$: $\quad \mathbf{x}'^T \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{x}'$

$= 3x'^2 - 2y'^2$

Videre har vi:

$$-4x + 10y = [-4 \ \ 10]\begin{bmatrix}x\\ y\end{bmatrix}$$

$$= [-4 \ \ 10]\begin{bmatrix}\frac{2}{\sqrt5} & \frac{4}{\sqrt5}\\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}\begin{bmatrix}x'\\ y'\end{bmatrix} = \left[-\frac{18}{\sqrt5} \ \ \frac{16}{\sqrt5}\right]\begin{bmatrix}x'\\ y'\end{bmatrix} = -\frac{18}{\sqrt5}x' + \frac{16}{\sqrt5}y'.$$

Ligningen blir dermed:

$$2x^2 - 4xy - y^2 - 4x + 10y - 13 =$$

$$3x'^2 - 2y'^2 - \frac{18}{\sqrt5}x' + \frac{16}{\sqrt5}y' - 13 = 0$$

Dette kan omskrives til:

$$3\left(x'^2 - \frac{6}{\sqrt5}x'\right) - 2\left(y'^2 - \frac{8}{\sqrt5}y'\right) = 13$$

$$3\left(x'^2 - \frac{6}{\sqrt5}x' + \left(\frac{3}{\sqrt5}\right)^2\right) - 2\left(y'^2 - \frac{8}{\sqrt5} + \left(\frac{4}{\sqrt5}\right)^2\right) = 13 + \frac{27}{5} - \frac{32}{5}$$

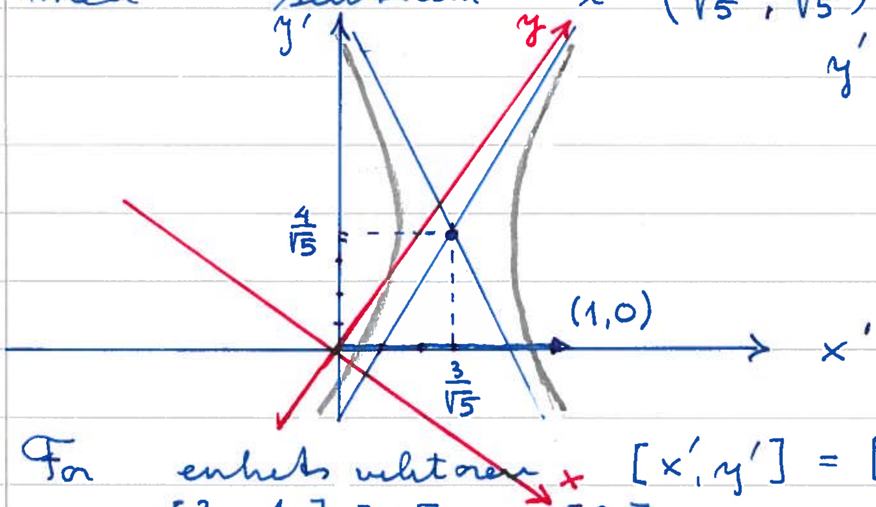
eller

$$3\left(x' - \frac{3}{\sqrt5}\right)^2 - 2\left(y' - \frac{4}{\sqrt5}\right)^2 = 12$$

$$\frac{\left(x' - \frac{3}{\sqrt5}\right)^2}{2^2} - \frac{\left(y' - \frac{4}{\sqrt5}\right)^2}{(\sqrt6)^2} = 1$$

Dette blir en hyperbel i $x'y'$-systemet med sentrum i $\left(\frac{3}{\sqrt5}, \frac{4}{\sqrt5}\right)$. Asymptoter

$$y' = \pm\sqrt{\frac{3}{2}}\left(x' - \frac{3}{\sqrt5}\right) + \frac{4}{\sqrt5}$$



For enhetsvektoren $[x', y'] = [1, 0]$ har vi:

$$\begin{bmatrix}\frac{2}{\sqrt5} & \frac{4}{\sqrt5}\\ -\frac{1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt5}\\ -\frac{1}{\sqrt5}\end{bmatrix}$$ i xy-systemet.

Vi har en dreining av koordinat-systemet som er s.a. $\begin{bmatrix}1\\0\end{bmatrix}$ i $x'y'$-systemet ligger i 4. kvadrant i xy-systemet.