Kap. 6 Supp. ex.

2) $A = I_n - \frac{2}{\|x\|^2} x x^T$

Vis at A er ortogonal og symmetrisk

Symmetrisk: $A^T = I_n^T - \frac{2}{\|x\|^2} (x x^T)^T$

$= I_n - \frac{2}{\|x\|^2} (x^{TT} x^T) = I_n - \frac{2}{\|x\|^2} x x^T = A$

$A^T = A$

Ortogonal: Må vise at $A^T A = I_n$

$A^T A = A^2 = \left(I_n - \frac{2}{\|x\|^2} x x^T\right)^2 =$

$\left(I_n^2 - \frac{4}{\|x\|^2} x x^T + \frac{4}{\|x\|^4} (x x^T)^2\right) = I_n - \frac{4}{\|x\|^2} x x^T + \frac{4}{\|x\|^4} (x x^T)(x x^T)$

$x^T x = \|x\|^2$

$= I_n - \frac{4}{\|x\|^2} x x^T + \frac{4}{\|x\|^4} \|x\|^2 \cdot x x^T$

$= I_n$ $\quad A^T A = I_n$

3 Ser på ligningssystemet $Ax = 0$

$$A = \begin{bmatrix} r_1 \\ \hline r_2 \\ \hline \vdots \\ \hline r_n \end{bmatrix}, \quad Ax \text{ er da: } \begin{bmatrix} \langle r_1^T, x \rangle \\ \langle r_2^T, x \rangle \\ \vdots \\ \langle r_n^T, x \rangle \end{bmatrix}$$

der $\langle \cdot, \cdot \rangle$ er det Euclidske indreproduktet.

$$\begin{bmatrix} \langle r_1^T, x \rangle \\ \vdots \\ \langle r_n^T, x \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \iff \begin{bmatrix} r_1^T \text{ er ortogonal på } x \\ r_2^T \text{ er ortogonal på } x \\ \vdots \\ r_n^T \text{ er ortogonal på } x. \end{bmatrix}$$

Så $\boxed{Ax = 0} \iff$ x er ortogonal på radvektorene i A.

Kap. 8 Supp. ex.

13 $L : M_{2\times2} \longrightarrow M_{2\times2}$

$A \longmapsto A^T$

Finn $[L]_B$, der B er standardbasisen på $M_{2\times2}$

$$[L]_B = \left[[L(b_1)]_B \,\middle|\, [L(b_2)]_B \,\middle|\, [L(b_3)]_B \,\middle|\, [L(b_4)]_B \right]$$

der

$$b_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; b_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; b_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \; b_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[L]_B = \left[[b_1]_B \,\middle|\, [b_3]_B \,\middle|\, [b_2]_B \,\middle|\, [b_4]_B \right] = \underline{\underline{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}}$$

17

$T: V \to V$

$$[T(x)]_B = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_2 \\ x_1 - x_3 \end{bmatrix} \text{ hvis } [x]_B = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\boxed{\text{Finn } [T]_B}$

Vi vet at $[T]_B = \left[\, [T(b_1)]_B \mid [T(b_2)]_B \mid [T(b_3)]_B \,\right]$

des $B = \{b_1, b_2, b_3\}$.

Vi vet også at $[b_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $[b_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ og

$[b_3]_B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Så $[T(b_1)]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $[T(b_2)]_B = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ og

$[T(b_3)]_B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\Rightarrow [T]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Kap. 9.1

3) a) Løs systemet

$$y_1' = 4y_1 + y_3$$

$$y_2' = -2y_1 + y_2$$

$$y_3' = -2y_1 + y_3$$

Skriver på matriseform:

$$y' = Ay \text{ , der } \quad A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Diagonalisering:

$$\det(A - \lambda I) = (4-\lambda)(1-\lambda)^2 + 1 \cdot 2 \cdot (1-\lambda)$$

$$= (1-\lambda)\left(2 + (4-\lambda)(1-\lambda)\right) = (1-\lambda)\left(\lambda^2 - 5\lambda + 6\right)$$

$$\lambda_1 = 1 , \quad \lambda_2 = \frac{5 + \sqrt{25-24}}{2} = 3 , \quad \lambda_3 = \frac{5 - \sqrt{25-24}}{2} = 2 .$$

$\boxed{\text{tilhørende egenvektor } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$

Egenvektor for λ_2:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \qquad \left. \begin{matrix} v_1 = -v_3 \\ v_1 = -v_2 \end{matrix} \right\} \Longrightarrow \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Egenvektor for λ_3:

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \qquad \left.\begin{matrix} v_1 = -2v_3 \\ v_1 = -2v_2 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \qquad P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Antar $Pu = y$ og får

$$Pu' = APu \quad \text{eller} \quad u' = P^{-1}APu$$

$$\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} u_1 \\ 2u_2 \\ 3u_3 \end{bmatrix} \qquad \begin{matrix} u_1 = C_1 e^{x} \\ u_2 = C_2 e^{2x} \\ u_3 = C_3 e^{3x} \end{matrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = P \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_2 + u_3 \\ u_1 - 2u_2 - u_3 \\ -2u_2 - u_3 \end{bmatrix} = \begin{bmatrix} C_2 e^{2x} + C_3 e^{3x} \\ C_1 e^{x} - 2C_2 e^{2x} - C_3 e^{3x} \\ -2C_2 e^{2x} - C_3 e^{3x} \end{bmatrix}$$

---

b) $$\left.\begin{matrix} y_1(0) = C_2 + C_3 = -1 \\ y_2(0) = C_1 - 2C_2 - C_3 = 1 \\ y_3(0) = -2C_2 - C_3 = 0 \end{matrix}\right\} \begin{matrix} -3 = -2 \\ C_1 = 1 \\ -C_2 = -1,\ C_2 = 1 \end{matrix}$$

Solution: $$\begin{bmatrix} e^{2x} - 2e^{3x} \\ e^{x} - 2e^{2x} + 2e^{3x} \\ -2e^{2x} + 2e^{3x} \end{bmatrix}$$

7

$$y'' - y' - 6y = 0$$

$$y_1 = y \quad , \quad y_2 = y'$$

$$y_1' = y_2$$

$$y_2' = y'' = y' + 6y = y_2 + 6y_1$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad , \quad A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix}$$

Diagonalise:

Egenverdier:

$$\lambda(\lambda - 1) - 6 = 0$$

$$\lambda^2 - \lambda - 6 = 0 \qquad \lambda_1 = 3 \ , \ \lambda_2 = -2$$

Egenvektorer: λ_1

$$\begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \ \Big\} \rightsquigarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

λ_2

$$\begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \ \Big\} \longrightarrow \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \qquad P^{-1}AP = D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$$

Løses som i forrige oppgave:

$$u_1 = c_1 e^{3x} \qquad y_1 = c_1 e^{3x} + c_2 e^{-2x}$$

$$u_2 = c_2 e^{-2x} \qquad y_2 \ 3c_1 e^{3x} - 2c_2 e^{-2x}$$

$$\boxed{y = y_1 = c_1 e^{3x} + c_2 e^{-2x}}$$