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MIDTERM EXAM MA1202:
Lineær algebra med anvendelser

Monday 9.mars
Time: 15.15-16.45
Examination aids: Code D; Specified simple calculator (HP30S eller Citizen SR-270X).

Problem 1 Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & -1 \\ 1 & 2 & 3 & 0 & 3 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

A and R are row equivalent. (You may assume this.)

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

c) What is the rank of A? In an indirect way, find the dimension of the null space of A.

We can regard the equation $Ax = 0$ as a system consisting of 4 equations in 5 unknowns. How many linearly independent equations can you choose from this system? How many parameters are there in the general solution?

d) Find a basis for the null space of A.

Problem 2 Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix}2\\0\\2\\0\end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix}3\\1\\3\\1\end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix}2\\2\\0\\0\end{bmatrix} \quad \mathbf{u}_5 = \begin{bmatrix}1\\-1\\3\\1\end{bmatrix}.$$

a) Find a basis for $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$.
(Hint: Look at the columns of A in problem 1.)

b) Find the coordinates of $\mathbf{u}_5$ in the basis described in part a).

c) Is the set $\{\mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$ linearly independent?

Problem 3

a) Use the Gram–Schmidt process to find an orthonormal basis $S = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ for the column space of A in problem 1.

b) Let $\mathbf{b} = (2, 0, 0, 0) \in \mathbb{R}^4$. Calculate the orthogonal projection $\mathbf{p} = \text{Proj}_{\text{Col}(A)}\mathbf{b}$ of $\mathbf{b}$ onto the column space of A.

What is the coordinates of $\mathbf{p}$ in the basis S?

c) (Hard?) Let

$$M = [\mathbf{u}_1|\mathbf{u}_2|\mathbf{u}_4] \quad \text{and} \quad \mathbf{p} = \text{Proj}_{\text{Col}(M)}\mathbf{b},$$

the projection of $\mathbf{b}$ onto the column space of M. ($\mathbf{u}_i$ is defined in problem 2.)

The least squares solution of $M\mathbf{x} = \mathbf{b}$ is – as you know – solutions of the system $M\mathbf{x} = \mathbf{p}$. Now we will examine this system.

In vector form, we write it as

$$x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_4 = \mathbf{p},$$

where $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_4$ are the column vectors of M and $\mathbf{x} = (x_1, x_2, x_3)$.

Rewrite this using cordinate vectors in the basis $S = \{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ that you found in problem 3.

How can you use a QR factorisation of M to find this equation?

What is the least squares solution of the inconsistent system $M\mathbf{x} = \mathbf{b}$?