# Løsningsforslag øving 13

a) $T(k_1v_1 + k_2v_2 + k_3v_3)$

$= k_1Tv_1 + k_2Tv_2 + k_3Tv_3$

$= k_1(v_2 - v_3) + k_2(-v_1 + v_3) + k_3(v_1 - v_2)$

$= (k_3 - k_2)v_1 + (k_1 - k_3)v_2 + (k_2 - k_1)v_3$

b) $[Tv_1]_B = [v_2 - v_3]_B = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$[Tv_2]_B = [-v_1 + v_3]_B = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$[Tv_3]_B = [v_1 - v_2]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$[T]_{BB} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$

c) Når $[v]_B = [k_1, k_2, k_3]$, må

$v = k_1 v_1 + k_2 v_2 + k_3 v_3$.

Ved a) får vi

$$\underline{[Tv]_B} = [(k_3 - k_2)v_1 + (k_1 - k_3)v_2 + (k_2 - k_1)v_3]_B$$

$$= \underline{\underline{\begin{bmatrix} k_3 - k_2 \\ k_1 - k_3 \\ k_2 - k_1 \end{bmatrix}}}$$

Alternativ

$$\underline{[Tv]_B} = [T]_{BB}[v]_B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \underline{\underline{\begin{bmatrix} k_3 - k_2 \\ k_1 - k_3 \\ k_2 - k_1 \end{bmatrix}}}$$

d) $\mathcal{B} = \{u_1, u_2, u_3\}$

$$[Id]_{\mathcal{S}\mathcal{B}} = [(u_1)_{\mathcal{S}} | (u_2)_{\mathcal{S}} | (u_3)_{\mathcal{S}}] = [u_1 | u_2 | u_3]$$

$$= \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \end{bmatrix}$$

$$[Id]_{\mathcal{B}\mathcal{S}} \cdot [Id]_{\mathcal{S}\mathcal{B}} = [Id]_{\mathcal{B}\mathcal{B}} = I,$$

Só

$$[Id]_{\mathcal{B}\mathcal{S}} = [Id]_{\mathcal{S}\mathcal{B}}^{-1} = [Id]_{\mathcal{S}\mathcal{B}}^{T},$$

sendo $[Id]_{\mathcal{S}\mathcal{B}}$ e ortogonal.

$$\rightsquigarrow [Id]_{\mathcal{B}\mathcal{S}} = \begin{bmatrix} \frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & 0 & \frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}.$$

e) $[T] = [T]_{\mathcal{S}\mathcal{S}} = \cancel{[T]_{\mathcal{S}\mathcal{B}}[Id]_{\mathcal{S}\mathcal{B}}}$

$$= [T]_{\mathcal{S}\mathcal{B}}[Id]_{\mathcal{B}\mathcal{S}}$$

$$= \left[ \underset{\substack{\| \\ (Tv_1)}}{(v_2 - v_3)_{\mathcal{S}}} \;\middle|\; \underset{\substack{\| \\ (Tv_2)}}{(-v_1 + v_3)_{\mathcal{S}}} \;\middle|\; \underset{\substack{\| \\ (Tv_3)}}{(v_1 - v_2)_{\mathcal{S}}} \right] [Id]_{\mathcal{B},\mathcal{S}}$$

$$= \begin{bmatrix} -\tfrac{1}{2}\sqrt{2} & -\tfrac{1}{2}\sqrt{2} & \sqrt{2} \\ -1 & 1 & 0 \\ \tfrac{1}{2}\sqrt{2} & -\tfrac{1}{2}\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \tfrac{1}{2}\sqrt{2} & 0 & \tfrac{1}{2}\sqrt{2} \\ -\tfrac{1}{2}\sqrt{2} & 0 & \tfrac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sqrt{2} & -1 \\ -\sqrt{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Evt. kan en bruke formulen

$[T]_{\mathcal{S}\mathcal{S}} = [Id]_{\mathcal{S}\mathcal{B}}[T]_{\mathcal{B}\mathcal{B}}[Id]_{\mathcal{B}\mathcal{S}}$

f) La $\mathcal{B}' = \{w_1, w_2, w_3\}$.

Vi skal finne $[T]_{\mathcal{B}'\mathcal{B}'}$.

<u>1. forsøk</u> (evt. 1. alternativ)

$$[T]_{\mathcal{B}'\mathcal{B}'} = [Id]_{\mathcal{B}'\mathcal{S}} [T]_{\mathcal{S}\mathcal{S}} [Id]_{\mathcal{S}\mathcal{B}'}.$$

- $[T]_{\mathcal{S}\mathcal{S}} = [T]$ i oppg. e
- $[Id]_{\mathcal{S}\mathcal{B}'} = \left[ [w_1]_{\mathcal{S}} | \dots | [w_3]_{\mathcal{S}} \right] = [w_1 | w_2 | w_3]$

Denne matrisen er <u>unitær</u>, så

- $[Id]_{\mathcal{B}'\mathcal{S}} = \left([Id]_{\mathcal{S}\mathcal{B}'}\right)^{-1} = \left([Id]_{\mathcal{S}\mathcal{B}'}\right)^{H}$ (konjugert-transponert)

$$[T]_{\mathcal{B}'\mathcal{B}'} = [w_1|w_2|w_3]^H \begin{bmatrix} 0 & \sqrt{2} & -1 \\ -\sqrt{2} & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} [w_1|w_2|w_3]$$

$$= \dots \text{mange regning} \dots = \begin{bmatrix} \sqrt{3}i & 0 & 0 \\ 0 & -\sqrt{3}i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2. forsøk (evt. 2. alternativ.)

Vi kan sjekke følgende

- $w_1, w_2, w_3$ er egenvektorer for $T$:

$$Tw_1 = \sqrt{3}i\, w_1 \quad , Tw_2 = -\sqrt{3}i\, w_2 \, , \quad Tw_3 = 0 w_3$$

- $\{w_1, w_2, w_3\}$ er ortonormal

$$w_i \cdot w_j = \begin{cases} 0 & i = j \\ 1 & i \neq j \end{cases}$$ (sjekk evt. dette)

Dersom $U = [w_1 | w_2 | w_3]$, så er

$$U^H [T] U = \begin{bmatrix} \sqrt{3}i & 0 & 0 \\ 0 & -\sqrt{3}i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Da gjenstår bare følgende observasjoner:

$$U = [Id]_{\mathcal{S}\mathcal{B}'} \quad , U^H = [Id]_{\mathcal{B}'\mathcal{S}} \, ,$$

og vi vet at

$$[T]_{\mathcal{B}'\mathcal{B}'} = [Id]_{\mathcal{B}'\mathcal{S}} [T]_{\mathcal{S}\mathcal{S}} [Id]_{\mathcal{S}\mathcal{B}'} = \begin{bmatrix} \sqrt{3}i & 0 & 0 \\ 0 & -\sqrt{3}i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

g) Obs: $U^* = U^H$ ($-^*$, $-^H$ betyr det samme)

Av diskusjonen over, ser vi at vi kan velge $U = [w_1 | w_2 | w_3]$.

h) Av diskusjonen over ser vi at

$\{w_1, w_2, w_3\}$ er egen~~verdier~~ vektorer tilhørende

egenverdiene $\{\sqrt{3}i, -\sqrt{3}i, 0\}$.