Løsningsforslag eksamen 24.nov. 2000

Oppg 1

a)
$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -4 & -1 & -9 \\ 0 & -5 & 0 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

$\Longrightarrow$ rang $A = 3$

rang A + nullitet $A = 4$ $\Longrightarrow$ nullitet $A = 1$

$$A \cdot \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \underline{0} \iff x = -w,\ y = -w \text{ og } z = -5w$$

$$A \begin{bmatrix} -w \\ -w \\ -5w \\ w \end{bmatrix} = \begin{bmatrix} -w - 2w + 3w \\ -3w - 2w + 5w \\ -2w + w + w \end{bmatrix} = \underline{0}$$

Basis for nullrommet til A: $\left\{ \underline{v} = \begin{bmatrix} 1 \\ 1 \\ 5 \\ -1 \end{bmatrix} \right\}$

Basis for radrommet til A:

$\{ [1,0,0,1],\ [0,1,0,1],\ [0,0,1,5] \}$

Basis for kolonnerommet til A:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$$

b)

A $n \times n$-matrise, inverterbar

B $n \times m$-matrise

$\Longrightarrow AB$ $n \times m$-matrise

rang(AB) + nullitet$(AB) = m$

rang(B) + nullitet$(B) = m$

$\underline{x} \in$ nullrommet til $AB \iff AB\underline{x} = \underline{0}$

$\iff A^{-1}AB\underline{x} = A^{-1}\underline{0} = \underline{0} \iff B\underline{x} = \underline{0}$

$\iff \underline{x} \in$ nullrommet til B

$\Longrightarrow$ nullitet(AB) = nullitet(B)

$\Longrightarrow$ rang(AB) = rang(B)

Oppg 2

a) $A = \begin{bmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{bmatrix}$

Karak. polynom: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 0.7 & -0.3 \\ -0.3 & \lambda - 0.7 \end{vmatrix}$

$= (\lambda - 0.7)^2 - 0.09 = \lambda^2 - 1.4\lambda + 0.4$

$= (\lambda - 1)(\lambda - 0.4)$

Egenverdier: $\lambda_1 = 1$ og $\lambda_2 = 0.4$

Egenvektorer:

$\lambda_1 = 1$: $0.3x - 0.3y = 0 \Rightarrow x = y$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0.4$ $-0.3x - 0.3y = 0 \Rightarrow y = -x$ $\quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} = -\frac{1}{2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$D = P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 0.4 \end{bmatrix}$

b) 6.800.000 stemmer ved hvert valg.

x_0 = rep. stemmer i 1960 (= 4.828.000)
y_0 = dem. —— " —— (= 1.972.000)

x_i = rep. stemmer ved valget i $1960+4i$, $i = 0,1,2,\ldots$
y_i = dem. —— " ——

$\underline{v_i} = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ $\quad i = 0,1,2,\ldots$

$\Rightarrow$ $\underline{v_i} = A\underline{v_{i-1}}$, der A fra a)

Valg resultatet i 2000 er $\underline{v_{10}}$

Siden $A = PDP^{-1}$, er $\underline{v_{10}} = A^{10}\underline{v_0} = PD^{10}P^{-1}\underline{v_0}$

$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & (0.4)^n \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$

$= \begin{bmatrix} \frac{1}{2}(1+(0.4)^n) & \frac{1}{2}(1-(0.4)^n) \\ \frac{1}{2}(1-(0.4)^n) & \frac{1}{2}(1+(0.4)^n) \end{bmatrix}$

$\Rightarrow \underline{v_{10}} \simeq \begin{bmatrix} 3.400.150 \\ 3.399.850 \end{bmatrix}$

Den rep. kandidaten vinner valget i 2000 med 300 stemmer.

Når $n \to \infty$, så vil $(0,4)^n \to 0$

Dvs. at

$$A^n \longrightarrow \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Dvs. at

$$\underline{v}_n \longrightarrow \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1/2(x_0+y_0) \\ 1/2(x_0+y_0) \end{bmatrix}$$

De to partiene vil få nøyaktig like mange stemmer.

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Oppg. 3

$$M_{22} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

a) $B = \left\{ \underline{v}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \underline{v}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \underline{v}_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \underline{v}_4 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \right\}$

1. B gen. M_{22}:

$$(b+c)\underline{v}_1 + (a-b-c)\underline{v}_2 + (b+2c+d-a)\underline{v}_3 + (b+c+d-a)\underline{v}_4 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

2. $$x\underline{v}_1 + y\underline{v}_2 + z\underline{v}_3 + w\underline{v}_4 = \begin{bmatrix} x+y & x-z+w \\ z-w & y+w \end{bmatrix} = \underline{0}$$

$$\iff x = y = z = w = 0$$

$\therefore$ B lin. uavh.

$\Longrightarrow$ B er en basis for M_{22}.

(Kan også anta at kjent at rang $M_{22} = 4$. Da er det nok å vise enten 1. el. 2.)

b)

$T: M_{22} \longrightarrow M_{22}$ def. ved at

$$T(A) = A + A^T$$

T er lin. operator:

La $A, B \in M_{22}$ og $k \in \mathbb{R}$

$$T(A+B) = (A+B) + (A+B)^T$$
$$= A + B + A^T + B^T$$
$$= T(A) + T(B)$$

$$T(kA) = (kA) + (kA)^T = kA + kA^T = kT(A)$$

$\Longrightarrow$ T er lin. operator på M_{22}

Matrisa til T m.h.p. $\mathcal{B}$:

$$T(\underline{v}_1) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = 2\underline{v}_1 + \underline{v}_3$$

$$T(\underline{v}_2) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2\underline{v}_2$$

$$T(\underline{v}_3) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$T(\underline{v}_4) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 2\underline{v}_3 + 2\underline{v}_4$$

$$[T]_{\mathcal{B}} = \Big[[T(\underline{v}_1)]_{\mathcal{B}} \;\vdots\; [T(\underline{v}_2)]_{\mathcal{B}} \;\vdots\; [T(\underline{v}_3)]_{\mathcal{B}} \;\vdots\; [T(\underline{v}_4)]_{\mathcal{B}} \Big]$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

c)

$\Rightarrow$: Anta at $B \in R(T)$

Dvs at det finnes $A \in M_{22}$ slik at $T(A) = B$.

$$\Longrightarrow \quad B = A + A^T$$

Da er

$$B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = B$$

$$\Longrightarrow B \in \{ B \in M_{22} \mid B^T = B \}$$

$\Leftarrow$: Anta at $B^T = B$.

Da er

$$T(\tfrac{1}{2}B) = (\tfrac{1}{2}B) + (\tfrac{1}{2}B)^T$$
$$= \tfrac{1}{2}B + \tfrac{1}{2}B^T = \tfrac{1}{2}B + \tfrac{1}{2}B = B$$

$$\Longrightarrow B \in R(T).$$

d) $A \in M_{22}$ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$T(A) = \begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix}$$

$T(A) = \underline{0} \iff a = 0 = d$ og $c = -b$

$\iff A \in \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \middle| b \in \mathbb{R} \right\}$

$\Rightarrow \operatorname{Ker} T = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \middle| b \in \mathbb{R} \right\}$

(Bruk evt. $[T]_B$)

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Oppg. 4

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{bmatrix}$$

(A hermitisk $\Rightarrow$ har kun reelle egenverdier)

a) karak. polynom: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & 0 & 0 \\ 0 & \lambda - 2 & -i \\ 0 & i & \lambda - 2 \end{vmatrix}$

$= (\lambda - 2)\left[(\lambda-2)^2 + i^2\right] = (\lambda - 2)\left[\lambda^2 - 4\lambda + 4 - 1\right]$

$= (\lambda - 2)\left[\lambda^2 - 4\lambda + 3\right] = (\lambda - 2)(\lambda - 1)(\lambda - 3)$

Egenverdier: $\lambda_1 = 1$, $\lambda_2 = 2$ og $\lambda_3 = 3$

$\lambda_1 = 1$: $\left.\begin{aligned} -x &= 0 \\ -y - iz &= 0 \\ iy - z &= 0 \end{aligned}\right\} \Rightarrow x = 0,\ y = -iz$

$\underline{v}_1 = \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix}$

$\lambda_2 = 2$: $\left.\begin{aligned} -iz &= 0 \\ iy &= 0 \end{aligned}\right\} \Rightarrow y = z = 0$ $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_3 = 3$: $\left.\begin{aligned} x &= 0 \\ y - iz &= 0 \\ iy + z &= 0 \end{aligned}\right\} \Rightarrow x = 0\ \ y = iz$

$\underline{v}_3 = \begin{bmatrix} 0 \\ i \\ 1 \end{bmatrix}$

La $P = \begin{bmatrix} 0 & 1 & 0 \\ -i & 0 & i \\ 1 & 0 & 1 \end{bmatrix}$

P er nå inverterbar og

$$P^{-1}AP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Kontroll:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ -i & 0 & i & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2i & 0 & 1 & i \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2}i & \frac{1}{2} \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{1}{2}i & \frac{1}{2} \end{array}\right] \Rightarrow P^{-1} = \begin{bmatrix} 0 & \frac{1}{2}i & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & \frac{1}{2}i & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & i \\ 0 & -i & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -i & 0 & i \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2}i & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2}i & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 \\ -i & 0 & 3i \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

b)

$$\begin{aligned} x' &= 2x \\ y' &= 2y + iz \\ z' &= -iy + 2z \end{aligned}$$

$$\underline{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \underline{w}' = A\underline{w}$$

fra a)

Substituer $\underline{w} = P\underline{u}$ fra a)

$$\Rightarrow P\underline{u}' = AP\underline{u} \Rightarrow \underline{u}' = P^{-1}AP\underline{u} = D\underline{u}$$

Dette gir at

$$\underline{u} = \begin{bmatrix} c_1 e^{t} \\ c_2 e^{2t} \\ c_3 e^{3t} \end{bmatrix}$$

$$\Rightarrow \underline{w} = P\underline{u} = \begin{bmatrix} c_2 e^{2t} \\ -ic_1 e^{t} + ic_3 e^{3t} \\ c_1 e^{t} + c_3 e^{3t} \end{bmatrix}$$

$$\begin{aligned} x(0) &= c_2 e^0 = c_2 = 4 \\ y(0) &= -ic_1 e^0 + ic_3 e^0 = -ic_1 + ic_3 = 0 \\ z(0) &= c_1 e^0 + c_3 e^0 = c_1 + c_3 = 6 \end{aligned}$$

$$\Rightarrow c_1 = c_3 = 3 \text{ og } c_2 = 4$$

Dvs. at

$$\begin{aligned} x(t) &= 4e^{2t} \\ y(t) &= -3ie^{t} + 3ie^{3t} \\ z(t) &= 3e^{t} + 3e^{3t} \end{aligned}$$

Kontroll:

$$x'(t) = 2\cdot 4e^{2t} = 2x(t)$$

$$y'(t) = -3ie^{t} + 9ie^{3t}$$

$$z'(t) = 3e^{t} + 9e^{3t}$$

$$2y(t) + iz(t) = -6ie^{t} + 6ie^{3t} + 3ie^{t} + 3ie^{3t}$$
$$= -3ie^{t} + 9ie^{3t} = y'(t)$$

$$-iy(t) + 2z(t) = -3e^{t} + 3e^{3t} + 6e^{t} + 6e^{3t}$$
$$= 3e^{t} + 9e^{3t} = z'(t)$$

OK

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Oppg. 5

a)

(*) $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx \quad , f, g \in P_2$

1) $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx = \int_0^1 g(x)f(x)\,dx = \langle g, f\rangle$ $\quad f, g \in P_2$

2) $\langle f+g, h\rangle = \int_0^1 (f(x)+g(x))h(x)\,dx$

$= \int_0^1 f(x)h(x)\,dx + \int_0^1 g(x)h(x)\,dx \qquad f, g, h \in P_2$

$= \langle f, h\rangle + \langle g, h\rangle \qquad ((f+g)(x) = f(x) + g(x))$

3) $k \in \mathbb{R}, \quad f, g \in P_2 \qquad ((kf)(x) = kf(x))$

$\langle kf, g\rangle = \int_0^1 kf(x)g(x)\,dx = k\int_0^1 f(x)g(x)\,dx$

$= k\langle f, g\rangle$

4) $f \in P_2$

$\langle f, f\rangle = \int_0^1 f(x)^2\,dx$

$f(x)^2 \geq 0 \ \forall x \implies \int_0^1 f(x)^2\,dx \geq 0$

$\implies \langle f, f\rangle \geq 0$

$\langle f, f\rangle = 0 \iff \int_0^1 f(x)^2\,dx = 0 \iff f(x)^2 = 0 \ \forall$

$\iff f(x) = 0 \ \forall x \iff f = 0$

$\rightarrow$ (*) defineres et indreprodukt på P_2

2) La $u_2 = p_2 - \frac{\langle p_2, u_1\rangle}{\|u_1\|^2} u_1$

$$\langle p_2, u_1\rangle = \int_0^1 (1+x^2)x\,dx = \int_0^1 (x+x^3)dx$$

$$= 1/2 + 1/4 = 3/4$$

$$\Rightarrow u_2(x) = (1+x^2) - \frac{3/4}{1/3}x = 1 - \frac{9}{4}x + x^2$$

$$\|u_2\|^2 = \int_0^1 (1-\tfrac{9}{4}x+x^2)^2 dx = \int_0^1 (1 - 9/2x + \tfrac{113}{16}x^2 - 9/2x^3 + x^4)dx$$

$$= 1 - 9/2 \cdot 1/2 + 113/16 \cdot 1/3 - 9/2 \cdot 1/4 + 1/5 = \frac{43}{240}$$

$\{u_1, u_2\}$ er en ortogonal basis for V, men ikke ortonormal.

La $v_1 = \frac{1}{\|u_1\|}u_1$. Dvs. $v_1(x) = \sqrt{3}\,x$

La $v_2 = \frac{1}{\|u_2\|}u_2$. Dvs. $v_2(x) = \sqrt{\frac{15}{43}}(4-9x+4x^2)$

Da er $\{v_1, v_2\}$ er ortonormal basis for V.

Kontroll:

$$\langle v_1, v_2\rangle = \int_0^1 \sqrt{\frac{45}{43}}(4x-9x^2+4x^3)dx$$

$$= \sqrt{\frac{45}{43}}(4 \cdot 1/2 - 9 \cdot 1/3 + 4 \cdot 1/4) = \sqrt{\frac{45}{43}}(2-3+1) = 0$$ OK.

b) $V = \{p \in P_2 \mid p(x) = a + bx + ax^2\}$

1) $p, q \in V$ o: $p(x) = a + bx + ax^2$
$q(x) = c + dx + cx^2$

$$(p+q)(x) = p(x) + q(x)$$
$$= (a+c) + (b+d)x + (a+c)x^2 \in V$$
$$\Rightarrow p+q \in V$$

2) $k \in \mathbb{R}$, $p \in V$ o: $p(x) = a + bx + ax^2$

$$(kp)(x) = kp(x) = (ka) + (kb)x + (ka)x^2$$
$$\Rightarrow kp \in V$$

$\Rightarrow V$ er et underrom av P_2

c) La $p_1(x) = x$, $p_2(x) = 1 + x^2$

Da er $B = \{p_1, p_2\}$ en basis for V, men den er ikke ortonormal

Bruker Gram-Schmidt metoden (V en end. dim.)

1) La først $u_1 = p_1$
$$\|u_1\|^2 = \int_0^1 x^2 dx = 1/3$$

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d)

Den beste tilnærmingen til $f(x) = 1 + x$ med et polynom i V vil være $\text{proj}_V f$

Siden $\{v_1, v_2\}$ fra c) er en ortonormal basis for V, er

$$\text{proj}_V f = \langle f, v_1 \rangle v_1 + \langle f, v_2 \rangle v_2$$

$$\langle f, v_1 \rangle = \int_0^1 (1+x)\sqrt{3}\,x\,dx = \sqrt{3}\int_0^1 (x + x^2)\,dx$$

$$= \sqrt{3}\left(\tfrac{1}{2} + \tfrac{1}{3}\right) = \frac{5}{2\sqrt{3}}$$

$$\langle f, v_2 \rangle = \int_0^1 (1+x)\sqrt{\frac{15}{43}}(4 - 9x + 4x^2)\,dx$$

$$= \sqrt{\frac{15}{43}}\int_0^1 (4 - 5x - 5x^2 + 4x^3)\,dx$$

$$= \sqrt{\frac{15}{43}}\left(4 - \tfrac{5}{2} - \tfrac{5}{3} + 1\right) = \sqrt{\frac{15}{43}} \cdot \frac{5}{6}$$

Dvs. at

$$(\text{proj}_V f)(x) = \frac{5}{2\sqrt{3}} \cdot \sqrt{3}\,x + \sqrt{\frac{15}{43}} \cdot \frac{5}{6} \cdot \sqrt{\frac{15}{43}}(4 - 9x + 4x^2)$$

$$= \frac{5}{2}x + \frac{25}{86}(4 - 9x + 4x^2) = \frac{5}{43}(10 - x + 10x^2)$$